Hotspot lemmas for noncompact spaces

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Theorem A real number x is normal to base b if, and only if, there exists a positive number C such that

$$\limsup_{n \to \infty} \frac{A_b(B, n, x)}{n} \leq C b^{-|B|}$$

for all blocks B of integers $0, 1, \ldots, b-1$.

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This result and its generalizations and extensions are among the fundamental tools used to study normal numbers in various numeration systems.

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$$\limsup_{n} \sum_{i} a_{i,n} \leq \sum_{i} \limsup_{n} a_{i,n}?$$

Yes, if the sums are finite. Counterexample for infinite sums:

Let $a_{i,i} = 1$ and $a_{i,n} = 0$ if $i \neq n$. Then $\limsup_n \sum_i a_{i,n} = 1$ and $\sum_i \limsup_n a_{i,n} = 0$.

The setup

Let X be a set with a collection of subsets $\mathcal{C} = \{C_m\}$ which form a *semi-σ-algebra*: that is $\mathcal{C}$ contains X and $\emptyset$, $\mathcal{C}$ is closed under finite intersection, and for any $A \in \mathcal{C}$ there is a countable disjoint collection of sets $\{C_k\} \subseteq \mathcal{C}$ such that $X \setminus A = \bigcup_k C_k$. Endow X with the topology and Borel σ-algebra generated by $\mathcal{C}$. Let μ be a probability measure and let $T : X \to X$ be a continuous map which preserves μ and is ergodic with respect to μ.

The *Birkhoff mean* of a measurable function f with respect to a point $x_0 \in X$ is given by

$$S_N(x_0, f) = \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x_0).$$

We define the sets

$$A_\ell(f, \delta) = \left\{ x \in X : \left| S_\ell(x, f) - \int f \, d\mu \right| > \delta \right\}.$$

The setup

We define an analog of the Hausdorff measure $H(\cdot)$ for a set E with respect to this family to be $\inf\{\sum \mu(C_i)\}$, where the infimum is taken over coverings (finite or countable) of E. We say that the measures μ and H are *coordinated* if any μ-measurable set is H-measurable.

An almost correct theorem

Theorem 1. *Let* x_0 *be a point such that* $x_0 \in X$. *If for an arbitrary set* I *from the family* $\{C_m\}$,

$$\limsup_{\nu\to\infty} \frac{S_\nu(x_0,\chi_I)}{\nu} \le \varphi(\mu(I))$$

and for any $\delta > 0$ *we have* $H_\varphi(A_l(T,\chi_I,\delta)) \to 0$ *as* $l \to \infty$, *then for an arbitrary set* V *from* Γ *the following asymptotic relation is valid:*

$$\lim_{\nu\to\infty} \frac{S_\nu(x_0,\chi_V)}{\nu} = \mu(V). \tag{3}$$

A counterexample

Theorems 1, 4, and 5 in Moshchevitin and Shkredov (2003) are incorrect as stated. A counterexample to all three is given by considering the space $X = \mathbb{N}^{\mathbb{N}}$ with the shift map T and the family $\mathcal{C}$ given by the cylinder sets $[\xi] = \{x \in X : x|_{\{1,\cdots,|\xi|\}} = \xi\}$ for $\xi \in \mathbb{N}^{<\infty}$. Consider the point $x_0 = (1, 2, 3, 4, \cdots)$. Then for any $I \in \mathcal{C}$

$$\limsup_{N\to\infty} \frac{S_N(x_0, \chi_I)}{N} = 0$$

since for a fixed $\xi \in \mathbb{N}^{<\infty}$, if $M = \max_{1\leq i\leq |\xi|} \xi_i$, then for $n > M$ we have $T^n \notin [\xi]$. For any probability measure μ on X and function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we have $0 \leq \varphi(\mu(I))$. Thus x_0 satisfies the assumptions in each theorem. However, for any probability measure μ on X, there must be some $a \in \mathbb{N}$ such that $\mu[a] > 0$. Thus,

$$\lim_{N\to\infty} \frac{S_N(x_0, \chi_{[a]})}{N} \neq \mu[a].$$

Tightness

The behavior of this counterexample, where the mass of the orbit escapes to infinity, is the only obstruction to the theorems. To correct these theorems we add a tightness condition which prevents this escape. We say a set of probability measures M is *tight* if for every $\epsilon > 0$ there is a compact set K such that for every $\mu \in M$, $\mu(X \setminus K) < \epsilon$. We define the empirical probability measures for $x \in X$ by $\mathcal{E}(x, n) = \sum_{i=0}^{n-1} \delta_{T^i x}/n$. Note that $S_N(x_0, f) = \int f \, d\mathcal{E}(x_0, N)$.

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It was suggested to the authors independently by N. G. Moshchevitin and I. D. Shkredov, J. Vandehey, and the anonymous referee of this article that rather than adding a tightness condition one may instead enlarge the class of cylinder sets on which one tests the hypotheses of the lemma.

Corrections of Theorem 1 in in Moshchevitin and Shkredov (2003)

Theorem Let $x_0 \in X$ be such that the set of probability measures $\{\mathcal{E}(x_0, n)\}_{n=1}^{\infty}$ is tight and let $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be monotone increasing. If for an arbitrary set I from the family $\mathcal{C}$

$$\limsup_{N\to\infty} \frac{S_N(x_0, \chi_I)}{N} \leq \varphi(\mu(I))$$

and for any $\delta > 0$ we have $\lim_{\ell\to\infty} H_\varphi(A_\ell(\chi_I, \delta)) = 0$, then for a Borel set B, we have

$$\lim_{N\to\infty} \frac{S_N(x_0, \chi_B)}{N} = \mu(B).$$

Corrections of Theorem 1 in Moshchevitin and Shkredov (2003)

Theorem Let $x_0 \in X$ and $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be monotone increasing. If for an arbitrary set $I = \bigcup_k C_k$ which is a countable union of cylinder sets $C_k \in \mathcal{C}$

$$\limsup_{N\to\infty} \frac{S_N(x_0, \chi_I)}{N} \leq \sum_k \varphi(\mu(C_k))$$

and for any $\delta > 0$ we have $\lim_{\ell\to\infty} H_\varphi(A_\ell(\chi_I, \delta)) = 0$, then for a Borel set B, we have

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Correction of Theorem 4 in Moshchevitin and Shkredov (2003) and Theorem 7 in Shkredov (2012)

Theorem Let T be the continued fraction map on $X = (0,1) \setminus \mathbb{Q}$ with the Gauss measure

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} \, dx.$$

Let $x \in X$ be such that $\{\mathcal{E}(x,n)\}_{n=0}^{\infty}$ is tight (with respect to the subspace topology on X). Let $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ satisfy $\psi(t) = O(t^{-\eta})$ as $t \to 0$ for any $\eta > 0$. If for any cylinder set I

$$\limsup_{N \to \infty} \frac{S_N(x, \chi_I)}{N} \leq \mu(I)\psi(\mu(I))$$

then for any Borel set B

$$\lim_{N \to \infty} \frac{S_N(x, \chi_I)}{N} = \mu(I).$$

Correction of Theorem 4 in Moshchevitin and Shkredov (2003) and Theorem 7 in Shkredov (2012)

To state the alternative version of Theorem 3 we need a new definition. For a countable alphabet A, an *extended cylinder set* of rank n is a set $C \subseteq A^{\mathbb{N}}$ of the form

$$\left\{x \in A^{\mathbb{N}} : \; x_1 \in S_1, \cdots, x_n \in S_n\right\}$$

where $S_1, \cdots, S_n \subseteq A$ are finite or co-finite. Note the extended cylinder sets form a semi-algebra whereas the cylinder sets only form a semi-σ-algebra.

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Corrections of Theorem 5 in in Moshchevitin and Shkredov (2003)

Theorem Let $p = (p_a)_{a=1}^{\infty}$ be a probability vector. Suppose for some $\eta_0 > 0$ the series $\sum_{a=1}^{\infty} p_a^{1-\eta_0}$ converges. Consider the system (X, T, μ) where $X = \mathbb{N}^{\mathbb{N}}$, T is the right shift, and μ is the Bernoulli measure given by sampling each digit i.i.d. according to p. That is $\mu[a_1, \cdots, a_n] = \prod_{i=1}^{n} p_{a_i}$. Let $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function such that $\varphi(t) = O(t^{1-\eta})$ as $t \to 0$ for some $\eta \in (0, 1)$. Suppose $x \in X$ is such that $\{\mathcal{E}(x, n)\}_{n=0}^{\infty}$ is tight. If for any cylinder set I

$$\limsup_{N\to\infty} \frac{S_N(x, \chi_I)}{N} \leq \varphi(\mu(I)),$$

then for any Borel set B

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$$\limsup_{N\to\infty} \frac{S_N(x, \chi_I)}{N} \leq \varphi(\mu(I)),$$

then for any Borel set B

$$\lim_{N\to\infty} \frac{S_N(x, \chi_B)}{N} = \mu(B).$$