Complexified continued fractions and complex Brjuno and Wilton functions

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Continued fractions and the Gauss map

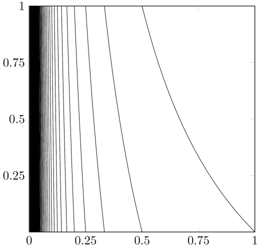
The Gauss map $A : (0,1) \mapsto [0,1]$ is

$$A(x) = \left\{\frac{1}{x}\right\} = \frac{1}{x} - \left[\frac{1}{x}\right].$$

Each $x \in \mathbb{R} \setminus \mathbb{Q}$ has an infinite continued fraction obtained by iterating A: $x_0 = x - [x]$, $a_0 = [x]$, then for $n \geq 0$: $x_{n+1} = A(x_n)$, $a_{n+1} = \left[\frac{1}{x_n}\right] \geq 1$, and

$$x = a_0 + x_0 = a_0 + \cfrac{1}{a_1 + x_1} = \ldots = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + x_n}}}$$

and we will write $x = [a_0; a_1, \ldots, a_n, \ldots]$.



Convergents, diophantine, Brjuno and Wilton numbers

- The *nth-convergent* is $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}$.
- Convergents are the *best rational approximations* of x: let
$$\beta_n = \Pi_{i=0}^{n} x_i = (-1)^n (q_n x - p_n) \quad \text{for } n \geq 0, \quad \text{and } \beta_{-1} = 1$$
then if $0 < q < q_{n+1}$, $q \neq q_n$, for all $p \in \mathbb{Z}$ one has $|qx - p| > \beta_n$.
- For all $x \in \mathbb{R} \setminus \mathbb{Q}$ and for all $n \geq 0$ one has $\beta_n \leq \left(\frac{\sqrt{5}-1}{2}\right)^n$ and $\frac{1}{2} \leq q_{n+1}\beta_n \leq 1$.
- *Diophantine numbers*: $q_{n+1} = \mathcal{O}(q_n^{1+\tau})$ for some $\tau \geq 0$, i.e. $a_{n+1} = \mathcal{O}(q_n^{\tau})$.
- *Brjuno numbers*: $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < +\infty$.
- *Wilton numbers*: $\sum_{n=0}^{\infty} (-1)^n \frac{\log q_{n+1}}{q_n} < +\infty$.
- Diophantine numbers are also Brjuno (and Wilton), but also "many" Liouville numbers are Brjuno numbers: for example $\sum_{n \geq 1} 10^{-n!}$.
- Brjuno numbers where introduced by Thomas Cherry in 1964 and conjectured to be a sufficient condition for the existence of Siegel disks in holomorphic dynamics. This was later proved by Brjuno.
- Wilton numbers appear in Wilton's work [Wil33] on the boundary behaviour of a "quasi" modular functions associated to the number of divisors function $d(n)$.

Brjuno numbers and the Brjuno function

- The *Brjuno function*
  $B : \mathbb{R} \setminus \mathbb{Q} \to (0, +\infty]$,

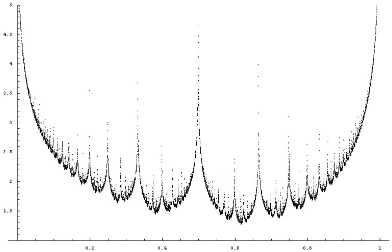
$$B(x) := \sum_{n=0}^{\infty} \beta_{n-1} \log x_n^{-1} < +\infty .$$

  was introduced by Yoccoz [Yoc88, Yoc95] around 1988.

- x is a Brjuno number if and only if $B(x) < +\infty$

- Indeed there exists $C > 0$ such that

$$\left| B(x) - \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} \right| \leq C .$$



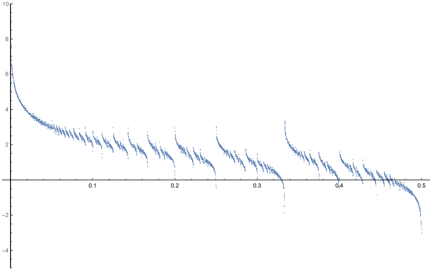
Wilton numbers and the Wilton function

- The *Wilton function* $W : \mathbb{R} \setminus \mathbb{Q} \to (0, +\infty]$,

$$W(x) := \sum_{n=0}^{\infty} (-1)^n \beta_{n-1} \log x_n^{-1} < +\infty .$$

  appears in the work of Wilton [Wil33] in 1933.
- x is a Wilton number if and only if $W(x) < +\infty$
- Indeed there exists $C > 0$ such that

$$\left| W(x) - \sum_{n=0}^{\infty} (-1)^n \frac{\log q_{n+1}}{q_n} \right| \leq C .$$



Yoccoz's theorems on the optimality of the Brjuno condition

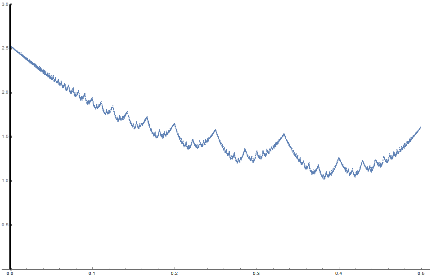
- Let $P_\lambda(z) = \lambda\left(z - \frac{z^2}{2}\right)$. *When is the origin stable?*
- Easy case: $|\lambda| < 1$. What about the indifferent case $\lambda = e^{2\pi i x}$, $x \in \mathbb{R} \setminus \mathbb{Q}$?
- *Miracle: stability is equivalent to analytic linearizability,* i.e. existence of an analytic map $H_\lambda(z) = z + \ldots$: $(H_\lambda^{-1} \circ P_\lambda \circ H_\lambda)(z) = \lambda z$.
- **Theorem(Yoccoz [Yoc88])** *The origin is stable if and only if x is a Brjuno number.*
- The radius of convergence $r_2(e^{2\pi i x})$ of the linearization tells how big the stability domain around the fixed point is.
- **Theorem(Yoccoz [Yoc88, Yoc95], Buff and Chéritat [BC04])** There exists a universal constant $C_1 > 0$ such that

$$B(x) - C_1 \leq -\log r_2(e^{2\pi i x}) \leq B(x) + C_1 \, .$$

  i.e. the function $x \mapsto B(x) + \log r_2(e^{2\pi i x})$ belongs to L^∞.
- This bound implies that the Brjuno function also gives a rather precise estimate of the size of the Siegel disks: as x approaches a rational number both $r_2(e^{2\pi i x})$ and $B(x)$ diverge but their difference stays uniformly bounded.

The Hölder interpolation conjecture (aka M-Moussa-Yoccoz conjecture)

- **Conjecture (Hölder interpolation) [MMY97]**The periodic function defined on the set of Brjuno numbers by $x \mapsto B(x) + \log r_2(e^{2\pi i x})$ extends to a 1/2–Hölder continuous function as x varies in $\mathbb{R}$.
- In 2006 Buff and Chéritat [BC06] proved that the function $x \mapsto B(x) + \log r_2(e^{2\pi i x})$ extends to a continuous function.
- In 2015 Chéritat and Cheraghi [CC15], using the renormalization invariant class of Inou and Shishikura, proved of the Hölder interpolation conjecture for rotation numbers of high type. However they form a set of zero measure.



John Raymond Wilton's 1933 theorem

An approximate functional equation with applications to a problem of Diophantine approximation.

By *J. R. Wilton* in Adelaide (South Australia).

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1. In recent papers, published at the same time, Chowla [1]) and Walfisz [2]) have proved a number of results as to the order of magnitude of $\sum_{n\leqq\omega} d(n)\cos 2\pi n x$ and kindred sums, where $d(n)$ is the number of divisors of the positive integer n, and x is (except in one case) irrational. In the present communication I obtain most of their theorems and some new results by a systematic application of an approximate functional equation for the sum considered; the method is that originated by Hardy and Littlewood nearly twenty years ago [3]), and employed by Oppenheim [4]) to prove some analogous theorems concerning $\sum_{n\leqq\omega} r(n)e^{i\pi n x}$, and more general sums [5]), where $r(n)$ is the number of ways of expressing n as the sum of two squares.

The condition for the convergence of the series (10) may be stated with precision:

(19$_{\text{III}}$) $\sum_{n\geqq 1}\frac{d(n)}{n}\cos 2\pi n x$ *converges if and only if* $\sum_{r\geqq 0} x x_1\cdots x_{r-1}\log^2\frac{1}{x_r}$ *is convergent* [9]);

(20$_{\text{III}}$) $\sum_{n\geqq 1}\frac{d(n)}{n}\sin 2\pi n x$ *converges if and only if* $\sum_{r\geqq 0}(-)^r x x_1\cdots x_{r-1}\log\frac{1}{x_r}$ *is convergent* [9]).

Wilton's theorem and the Brjuno function

7. In the same way as Theorem 3 is derived from Theorem 1, we derive from (2. 2) and (2. 21)

Theorem 4. *If x is irrational, $0 < x < 1$, $\omega > 1 + A$, and m is given by* (6. 1), *then*

$$\sum_{n \leqq \omega} \frac{d(n)}{n} \cos 2\pi n x = \mathfrak{F}(x) + o(1) + X_{m-1} \tag{7. 1$_{\mathrm{III}}$}$$

$$+\, x x_1 \cdots x_{m-1} \int_1^{\omega_m} \frac{\log t + 2\gamma}{t} \cos 2\pi x_m t \, dt, \tag{7. 11}$$

$$\sum_{n \leqq \omega} \frac{d(n)}{n} \sin 2\pi n x = \mathfrak{F}(x) + o(1) + \frac{1}{2}\pi X'_{m-1} \tag{7. 2$_{\mathrm{III}}$}$$

$$+\, (-)^m x x_1 \cdots x_{m-1} \log \frac{1}{x_m} \int_0^{2\pi x_m \omega_m} \frac{\sin t}{t} \, dt, \tag{7. 21}$$

where

$$X_m = \frac{1}{2}\left(\log^2 \frac{1}{x} + x \log^2 \frac{1}{x_1} + \cdots + x x_1 \cdots x_{m-1} \log^2 \frac{1}{x_m}\right) \tag{7. 31}$$

$$-\, (\log 2\pi - \gamma)\left(\log \frac{1}{x} + x \log \frac{1}{x_1} + \cdots + x x_1 \cdots x_{m-1} \log \frac{1}{x_m}\right),$$

$$X'_m = \log \frac{1}{x} - x \log \frac{1}{x_1} + \cdots + (-)^m x x_1 \cdots x_{m-1} \log \frac{1}{x_m}. \tag{7. 32}$$

Modular forms and integrals

- For $k \geq 2$ even integer, the Fourier expansion of the Eisenstein series E_k of weight k defined in the upper-half plane is

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

  where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

- For all $k \geq 4$, E_k is modular of weight k under the action of $SL_2(\mathbb{Z})$: if $\gamma \in SL_2(\mathbb{Z})$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then for all $z \in \mathbb{H}$

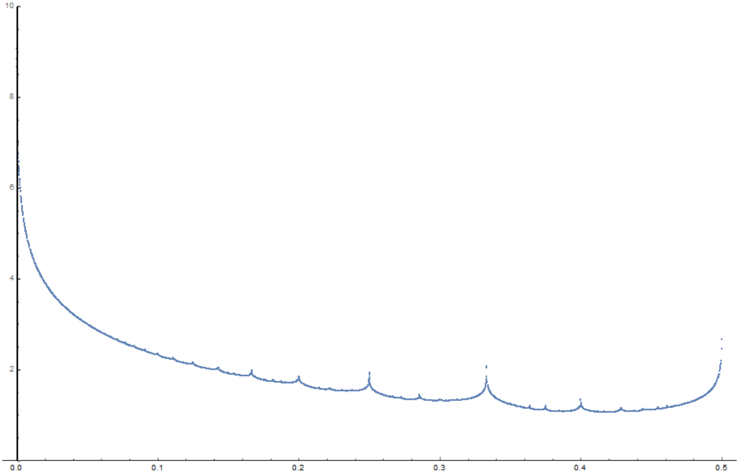
$$E_k(\gamma \cdot z) = (cz + d)^k E_k(z),$$

- E_2 is quasi-modular of weight 2: $E_2(\gamma \cdot z) = (cz + d)^2 E_2(z) - \frac{6}{\pi} ic(cz + d)$,
- The Eisenstein series can also be constructed starting from the series $\sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(m+nz)^k}$: using Lipschitz's summation formula one obtains the above Fourier expansions.

Petrykiewicz's work on differentiability of modular integrals and k-Brjuno functions

- Let k be a positive integer. The k-Brjuno function is defined by $B_k(x) = \sum_{n=0}^{\infty} \beta_{n-1}^k \log x_n^{-1}$, and converges at an irrational x if and only if the sum $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n^k}$ converges.
- I. Petrykiewicz in her PhD thesis discovered another surprising connection between B_k and modular integrals: for $k \geq 2$ even we "integrate" $(k+1)$ times $\mathrm{Im} E_k$ and get the continuous and 1-periodic function $F_k(x) = \sum_{n=1}^{\infty} \frac{\sigma_{k-1}(n)}{n^{k+1}} \sin(2\pi n x)$.
- Is it differentiable?
- In 2014 I. Petrykiewicz [Pet17] proved that if $B_2(x) < +\infty$ (and $\lim_{n\to\infty} \frac{\log q_{n+4}(x)}{q_n^2} = 0$), then F_2 is differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$, whereas if $B_2(x) = +\infty$ then F_2 is not differentiable at x.
- **Conjecture:** *For all $k \in 2\mathbb{N}$, F_k is differentiable at $x \in \mathbb{R} \setminus \mathbb{Q}$ if and only if $B_k(x) < +\infty$.*
- More connections with modular forms and analytic number theory are found in the works of Balazard, Bettin, Chamizo, Drappeau, Jaffard, Maier, Manin, Martin, Rassias, Rivoal, Roques, Seuret, etc.

The 2-Brjuno function



k-Brjuno and Wilton functions as $\mathrm{PGL}(2,\mathbb{Z})$-cohomology classes

- Let $k > 0$ be an integer. The k-Brjuno function satifies a functional equation under the action of $\mathrm{PGL}(2,\mathbb{Z})$:

$$B_k(x) = B_k(x+1)\,, \forall x \in \mathbb{R}\setminus\mathbb{Q}\,, \quad B_k(x) = -\log x + x^k B_k\left(\frac{1}{x}\right), \ \forall x \in \mathbb{R}\setminus\mathbb{Q} \cap (0,1)$$

- The Wilton function W verifies the same equation of B_1 but with a different sign:

$$W(x) = W(x+1)\,, \forall x \in \mathbb{R}\setminus\mathbb{Q}\,, \quad W(x) = -\log x - xW\left(\frac{1}{x}\right), \ \forall x \in \mathbb{R}\setminus\mathbb{Q} \cap (0,1)$$

- Thus the set of Wilton and k-Brjuno numbers are $\mathrm{PGL}(2,\mathbb{Z})$ invariant.
- One can compute W and B_k *exactly* for quadratic irrationals.
- Indeed one can prove (see [MMY06, MMY01]) that B_k and W are cocycles under the standard $\mathrm{PGL}(2,\mathbb{Z})$–action on $\mathbb{R}\setminus\mathbb{Q}$ lifted to functions using an appropriate *automorphic factor*, namely the multiplication by x^k on $(0,1)$.
- They are the cocycles with coboundary $x \mapsto -\log\{x\}$.
- All functions which will differ from B_k (or from W) by something whose coboundary is *more regular* than $x \mapsto -\log\{x\}$ will converge on the same set and might be considered representatives of the same *cohomology class*.
- For example Yoccoz's u function, which computes r_2, differs from B_1 by an L^∞ function.

L^p regularity of Brjuno-like and Wilton functions

- Let $k \geq 0$. We introduce the "automorphic Koopman operator" $(T_k f)(x) = x^k f \circ A$, $x \in (0,1)$, where A is the Gauss map.
- k-Brjuno and Wilton functions respectively solve

$$(1 - T_k)B_k = -\log x\,, \quad (1 + T_1)W(x) = -\log x\,,$$

- Using the absolutely continuous invariant probability measure $\frac{dx}{(1+x)\log 2}$ preserved by the Gauss map one shows [MMY97] that T_k has spectral radius bounded by g^k where $g = \frac{\sqrt{5}-1}{2}$.
- Indeed

$$\begin{aligned}\|T_k^m f\|_{L^p}^p &= \int |(T_k^m f)(x)|^p \frac{dx}{(1+x)\log 2} \\ &= \int \beta_{m-1}^{pk} |(f \circ A^m)(x)|^p \frac{dx}{(1+x)\log 2} \\ &\leq g^{pk(m-1)} \int |f(x)|^p \frac{dx}{(1+x)\log 2} = g^{pk(m-1)} \|f\|_{L^p}^p\end{aligned}$$

where the invariance of the measure by A allows one to replace $f \circ A^m$ with f.

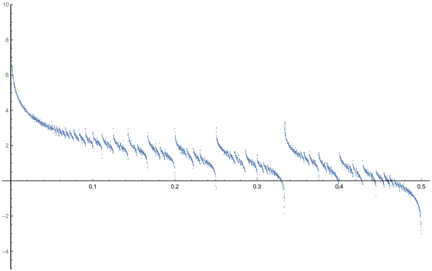
L^p regularity of Brjuno-like and Wilton functions

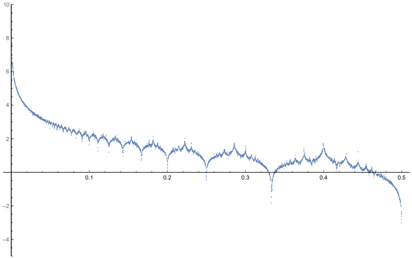
- The operator $(1 - T_k)$ is therefore invertible and its inverse is given by the norm convergent series $\sum_{k\geq 0} T_k^k$.
- For the Wilton function the bound on the spectral radius of T_1 implies that the operator $(1 + T_1)$ is also invertible and that it is given by the norm convergent series $\sum_{k\geq 0}(-1)^k T_1^k$.
- Thus the Brjuno function is simply obtained as $\sum_{k\geq 0} T_1^k(-\log\{\cdot\})$ and the Wilton function as $\sum_{k\geq 0}(-1)^k T_1^k(-\log\{\cdot\})$
- Since the function $x \mapsto -\log\{x\}$ belongs to L^p for all $1 \leq p < +\infty$ the Brjuno function $B \in \cap_{p\geq 1} L^p(\mathbb{T})$. Note that $B \notin L^\infty(\mathbb{T})$. The same holds for the Wilton function.

BMO regularity of k-Brjuno and Wilton functions

- Let $f \in L^1_{\text{loc}}(\mathbb{R})$ and let $f_I = \frac{1}{|I|}\int_I f(x)dx$ denote its mean value on an interval I.
- For an interval U, we say that a function $f \in \text{BMO}(U)$ (Bounded Mean Oscillation) if
$$\|f\|_{*,U} := \sup_{I\subset U} \frac{1}{|I|}\int_I |f(x) - f_I|dx < \infty.$$
- It's easy to check that $x \mapsto -\log|x| \in \text{BMO}([-1,1])$ but $x \mapsto -\log\{x\}$ does not.
- **Theorem [LMPS23]** *The k-Brjuno functions belong to BMO. However the Wilton function is not BMO, whereas the NICF Wilton function belongs to BMO.*
- Why does it matter? Well BMO is an important space in harmonic analysis, interpolating between $L^\infty([0,1])$ and $\cap_{p\geq 1} L^p([0,1])$.
- More importantly by Fefferman's duality theorem BMO is the dual of the Hardy space H^1, thus *one can add an L^∞ function to a BMO function so that the harmonic conjugate of the sum will also be L^∞*. In the case of the k-Brjuno functions we now know much more to be true.

The Wilton function vs. the NICF Wilton function





The complex k-Brjuno functions

Theorem [LMPS23]

- The complex k-Brjuno function $\mathcal{B}_k : \mathbb{H} \to \mathbb{C}$ is the analytic periodic function

$$\mathcal{B}_k(z) = -\sum_{p/q\in\mathbb{Q}} \det\begin{pmatrix} p' & p \\ q' & q \end{pmatrix}^{k+1} \cdot \left\{ \frac{1}{\pi}\left[(p'-q'z)^k \mathrm{Li}_2\left(\frac{p'-q'z}{qz-p}\right) - (q''z-p'')^k \mathrm{Li}_2\left(\frac{p''-q''z}{qz-p}\right.\right.\right.$$

$$+\sum_{n=0}^{k} \frac{\det\begin{pmatrix} p' & p \\ q' & q \end{pmatrix}^n}{n\pi q^n}\left[(p'-q'z)^{k-n}\left(-\delta_{n,0}\mathrm{Li}_2\left(-\frac{q'}{q}\right) + \delta_{n,1}\log\left(1+\frac{q'}{q}\right) - \sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{1}{(1+q'/q\right.\right.\right.$$

$$-(q''z-p'')^{k-n}\left(-\delta_{n,0}\mathrm{Li}_2\left(-\frac{q''}{q}\right) + \delta_{n,1}\log\left(1+\frac{q''}{q}\right) - \sum_{i=1}^{n-1}\frac{1}{i}\left(\frac{1}{(1+q''/q)^i} - 1\right)\right)\Bigg]\Bigg]\Bigg\},$$

  where $[\frac{p'}{q'}, \frac{p''}{q''}]$ is the Farey interval such that $\frac{p}{q} = \frac{p'+p''}{q'+q''}$ (with the convention $p' = p-1,\ q' = 1,\ p'' = 1,\ q'' = 0$ if $q = 1$) and $\mathrm{Li}_2(z)$ is the dilogarithm.

- The real part of $\mathrm{Re}\mathcal{B}_k$ is bounded on the upper half-plane and its non-tangential limit on $\mathbb{R}$ is continuous at all irrational points and has a decreasing jump of $\frac{\pi}{q^k}$ at each rational number $\frac{p}{q}$.

- The non-tangential limit of $\mathrm{Im}\mathcal{B}_k$ on $\mathbb{R}$ is the k-Brjuno function B_k.

The complex Wilton function

- **Theorem [LMPS23].** The complex Wilton function $\mathcal{W} : \mathbb{H} \to \mathbb{C}$ is the analytic periodic function

$$\begin{aligned}\mathcal{W}(z) = -\frac{1}{\pi} \sum_{p/q \in \mathbb{Q}} \Bigg\{ & \det\begin{pmatrix} p' & p \\ q' & q \end{pmatrix} (p' - q'z)\left[\mathrm{Li}_2\left(\frac{p' - q'z}{qz - p}\right) - \mathrm{Li}_2\left(-\frac{q'}{q}\right)\right] \\ & + \det\begin{pmatrix} p'' & p \\ q'' & q \end{pmatrix} (p'' - q''z)\left[\mathrm{Li}_2\left(\frac{p'' - q''z}{qz - p}\right) - \mathrm{Li}_2\left(-\frac{q''}{q}\right)\right] \\ & + \frac{1}{q} \log\left(\frac{q^2}{(q + q')(q + q'')}\right) \Bigg\}.\end{aligned}$$

- The above formula can be compared with the corresponding one for the complex Brjuno function [MMY01], which is also the case $k = 1$ in the previous slide:

$$\begin{aligned}\mathcal{B}(z) = -\frac{1}{\pi} \sum_{p/q \in \mathbb{Q}} \Bigg\{ & (p' - q'z)\left[\mathrm{Li}_2\left(\frac{p' - q'z}{qz - p}\right) - \mathrm{Li}_2\left(-\frac{q'}{q}\right)\right] \\ & + (p'' - q''z)\left[\mathrm{Li}_2\left(\frac{p'' - q''z}{qz - p}\right) - \mathrm{Li}_2\left(-\frac{q''}{q}\right)\right] + \frac{1}{q} \log \frac{q + q''}{q + q'} \Bigg\}.\end{aligned}$$

Hyperfunctions and complex extension of cocycles

- Let $A'([0,1])$ denote the (Fréchet space of) hyperfunctions u with support contained in $[0,1]$. These are linear functionals on functions analytic in a complex neighborhood of $[0,1]$.
- $A'([0,1])$ is canonically isomorphic to $\mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, the space of functions holomorphic on $\mathbb{C} \setminus [0,1]$ and vanishing at infinity.
- To construct the complex analytic extension $\mathcal{B}_f$ of the solutions B_f of the functional equations $(1-T)B_f = f$ the strategy is :
  - take the restriction of the periodic function f to the interval $[0,1]$;
  - consider its associated hyperfunction u_f and its holomorphic representative $\varphi_{u_f} \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$.
  - Recover a holomorphic *periodic* function on $\mathbb{H}$ by summing over integer translates:

$$\mathcal{B}_f(z) = \sum_{n\in\mathbb{Z}} \left(\sum_{\mathcal{M}} \varphi_{u_f} \right)(z-n)\,.$$

- The above series converges to the complex extension $\mathcal{B}_f$ of the real function B_f. The main difficulty (unless f belongs to some L^p space) would be to recover B_f as non–tangential limit of the imaginary part of $\mathcal{B}_f$ as $\mathfrak{Im}\, z \to 0$.

The dilogarithm and the Brjuno and Wilton functions

- To construct the complex Brjuno and Wilton functions one has to take

$$\varphi_0(z) = -\frac{1}{\pi}\mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{1}{\pi}\int_0^1 \frac{-\log t}{z-t}\,dt\,,$$

  where Li_2 is the dilogarithm $\mathrm{Li}_2(z) = \sum_{n=1}^{+\infty}\frac{z^n}{n^2}$ for $|z| \le 1$.

- $\mathrm{Li}_2\left(\frac{1}{z}\right)$ is the Cauchy–Hilbert transform of the real function

$$\varphi_0(t) = \begin{cases} -\log t & \text{if } t \in [0,1] \\ 0 & \text{elsewhere} \end{cases}$$

- Note also that $\mathrm{Im}\mathrm{Li}_2(t \pm i0) = \pm\pi\log t$, where $t \in [1,+\infty)$. Moreover $|\mathrm{Li}_2(z)| = \mathcal{O}(\log^2|z|)$ as$|z| \to +\infty$.

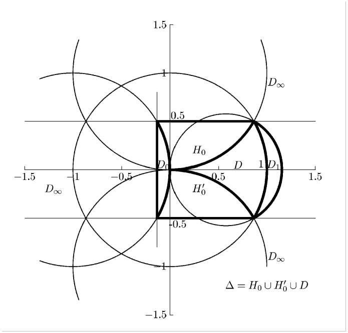
Complexification of the operator T_k

- T_k extends to $A'([0,1])$: using its isomorphism with $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ one obtains:
- $(T_1\varphi)(z) = -z\sum_{m=1}^{\infty}\left[\varphi\left(\frac{1}{z}-m\right)-\varphi(-m)\right]+\sum_{m=1}^{\infty}\varphi'(-m)\,.$
- $(T_k\varphi)(z) = -\sum_{m=1}^{\infty} z^k\left(\varphi\left(\frac{1}{z}-m\right)-\varphi(-m)\right)+\sum_{m=1}^{\infty}\sum_{n=1}^{k}\frac{z^{k-n}}{n!}\varphi^{(n)}(-m)\,.$
- Then $(1-T_k)^{-1}\varphi(z)=\sum_{r\geq 0}(T_k^r\varphi)(z)=\sum_{g\in\mathcal{M}}(L_{k,g}\varphi)(z)\,,$
- $\mathcal{M}$ is the free monoid of matrices g of $\mathrm{GL}(2,\mathbb{Z})$ of the form
  $g=g(m_1)\cdots g(m_r)\,,\ \ r\geq 0\,,\ m_i\geq 1\,,$ where $g(m)=\begin{pmatrix}0 & 1\\ 1 & m\end{pmatrix}$, $m\geq 1$
- $(L_g\varphi)(z)=(a-cz)\left[\varphi\left(\frac{dz-b}{a-cz}\right)-\varphi\left(-\frac{d}{c}\right)\right]-\det(g)c^{-1}\varphi'\left(-\frac{d}{c}\right)\,.$
- $L_{k,g}\varphi(z):=\det(g)^{k+1}\left[(a-cz)^k\varphi\left(\frac{dz-b}{a-cz}\right)-\sum_{n=0}^{k}(a-cz)^{k-n}\frac{\det(g)^n}{c^n n!}\varphi^{(n)}\left(-\frac{d}{c}\right)\right]$
- The series $\sum_{g\in\mathcal{M}}(L_g\varphi)(z)$ actually converges in $\mathcal{O}^1(\overline{\mathbb{C}}\setminus[0,1])$ to a function $\sum_{\mathcal{M}}\varphi$.

The complexified operator T_k

- One can prove that on functions belonging to $\mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$ and to the Hardy spaces $H^p(\mathbb{H}^{\pm})$ the complexified operator has the same spectral radius of the real version of T_k acting on the spaces $L^p(0,1)$.
- One can prove the following estimate: let $\rho > 0$, then for all $k \in \mathbb{N}$, there exists $\overline{C}_{\rho,k}$ such that, for all $r \geq 0$ and $\varphi \in \mathcal{O}^1(\overline{\mathbb{C}} \setminus [0,1])$, we have

$$\sup_{z \in V_\rho(D_\infty)} |(T_k^r \varphi)(z)| \leq \overline{C}_{\rho,k} \left(\frac{\sqrt{5}-1}{2} \right)^{rk} \sup_{z \in V_\rho(D_\infty)} |\varphi(z)|$$



Domains of complexification of the continued fraction

We consider a continued fraction on a compact subset of $\mathbb{C}$ which is a complex analog of regular continued fractions. At the beginning of this section we will first define some domains which will be important for defining our complex continued fraction algorithm which we introduce in the sequel.

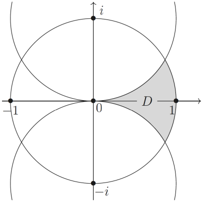
$$D = \{z \in \mathbb{C} \mid |z| \leq 1, |z-i| \geq 1, |z+i| \geq 1, \mathrm{Re}(z) > 0\}$$

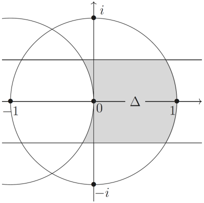
$$H_0 = \left\{z \in \mathbb{C} \mid |z-i| \leq 1, |z+1| \geq 1, |\mathrm{Im}(z)| \leq \frac{1}{2}\right\}$$

$$H_0' = \left\{z \in \mathbb{C} \mid |z+i| \leq 1, |z+1| \geq 1, |\mathrm{Im}(z)| \leq \frac{1}{2}\right\}$$

$$H = H_0 \cup H_0'$$

$$\Delta = D \cup H_0 \cup H_0' = \left\{z \in \mathbb{C} \mid |z| \leq 1, |z+1| \geq 1, |\mathrm{Im}(z)| \leq \frac{1}{2}\right\}$$





The complex continued fraction, the level one partition of D and the level two partition of D(1)

- When $z \in D$ then $1/z \in \bigcup_{n \in \mathbb{N}} (n + \Delta)$.
- If $1/z \in n + \Delta$, then we take $m_1 = n$ and we set $z_1 := 1/z - m_1$
- If $z_1 \in \Delta - D$, then we finish the process.
- If $z_1 \in D$, then we define m_2 by a integer n such that $1/z_1 \in n + \Delta$.
- We obtain a continued fraction expansion $\{m_i\}_{i=1}^r$ such that

  $$\frac{1}{z_i} = m_{i+1} + z_{i+1} \qquad \text{for all } 0 < i \leq r$$

  where $z_i \in D$ for $i < r$, and $z_r \in \Delta$.
- $D(m_1, \cdots, m_r)$ is the set of $z_0 \in D$ whose first r complex continued fraction entries equal $\{m_i\}_{i=1}^r$.

Regularity properties of k-Brjuno

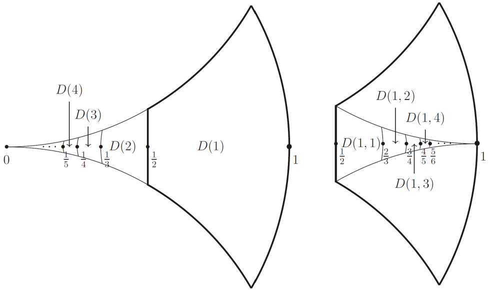


FIGURE 8. The sets $D(m_1, \cdots, m_r)$. The left figure is the partition of D by $D(n)$. The right figure is that of the sets $D(1, n)$ in $D(1)$

The complex continued fraction

- As in the real case, we define p_ℓ/q_ℓ by

$$\frac{p_\ell}{q_\ell} = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_\ell}}}}$$

  and $q_0 = p_{-1} = q_{-2} = 1$ and $p_0 = p_{-2} = q_{-1} = 0$.

- Then, if $z \in D(m_1, \cdots, m_\ell)$, for all $0 < i < \ell$

$$z_0 = \frac{p_{i-1} z_i + p_i}{q_{i-1} z_i + q_i} \quad \text{and} \quad z_i = -\frac{q_i z_0 - p_i}{q_{i-1} z_0 - p_{i-1}}.$$

  thus $\prod_{i=0}^{\ell}(-z_i) = q_\ell z_0 - p_\ell$.

- We have $p_\ell - q_\ell z_0 = \frac{(-1)^{\ell+1}}{q_\ell z_{\ell+1} + q_{\ell+1}}$, thus $|p_\ell - q_\ell z_0| < \frac{1}{q_{\ell+1}}$.
- We set $H(m_1, \cdots, m_\ell) = D(m_1, \cdots, m_\ell) \setminus \mathrm{int}\left(\bigcup_{m_{\ell+1} \geq 1} D(m_1, \cdots, m_{\ell+1})\right)$

The partition of the strip $|\mathrm{Im} z| < \frac{1}{2}$

- The sets $H(m_1,\cdots,m_r)$ give a partition of D as in the Figure.
- Then we have

$$\left\{z : |\mathrm{Im} z| \le 1/2\right\} = \mathbb{R}\setminus\mathbb{Q} \sqcup$$

$$\bigcup_{n\in\mathbb{Z}} \bigcup_{r\ge 0} \bigcup_{m_1,\cdots,m_r\ge 1} \left[H(m_1,\cdots,m_r)+n\right]$$

where the sets on the r.h.s. have disjoint interiors.

- Each set $H(m_1,\cdots,m_r)+n$ meets $\mathbb{R}$ i a unique point which is rational.

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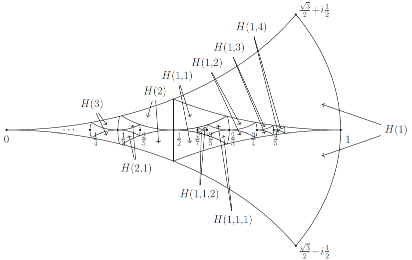

Figure 9. The partition of D by the sets $H(m_1,\cdots,m_r)$

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Boundary behaviour of the imaginary part of the complex Wilton function

- For $n \geq 0$, $m_1, \cdots, m_n \geq 1$ and $z_0 \in H(m_1, \cdots, m_n)$, we have

$$\mathrm{Im}\mathcal{W}(z_0) = -W_{\text{finite}}\left(\frac{p_n}{q_n}\right) + (-1)^n (p_{n-1} - q_{n-1}\mathrm{Re}(z_0))\mathrm{Im}\bar{\varphi}_1(z_n + 1) \\ + r_n(z_0),$$

  with $|r_n(z_0)| \lesssim \frac{\log q_n}{q_n} |z_n| \log^2(1 + |z_n|^{-1})$. and

$$\bar{\varphi}_1(z) = -\frac{1}{\pi}\left[z\mathrm{Li}_2 \frac{z}{1-z} + \mathrm{Li}_2 \frac{1}{z-1}\right] - z\frac{\pi}{12} - \frac{\log 2}{\pi}.$$

- For any Wilton number x and any $R > 0$, we have

$$\lim_{u \to 0,\ u \in U_R} \mathrm{Im}\mathcal{W}(u + x) = -W(x).$$

- Let x be an irrational Diophantine number and $0 < r < 1/2$ such that

$$\liminf_{q \to \infty} \|qx\|_{\mathbb{Z}}\ q^{1/r - 1} = \infty,$$

  where $\|\cdot\|_{\mathbb{Z}}$ denotes the distance from the nearest integer. Then,

$$\lim_{u \to 0,\ u \in \widetilde{U}_r} \mathrm{Im}\mathcal{W}(u + x) = -W(x).$$

Real and imaginary part of the complex Brjuno function almost on the boundary (T. Carletti)

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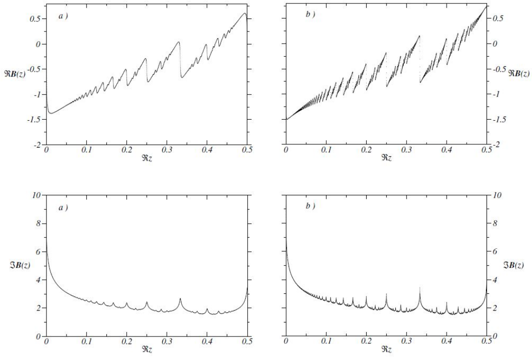

FIGURE 1. Plot of $\mathbf{B}(z)$ vs $\Re z$ at $\Im z$ fixed. The top line contains $\Re\mathbf{B}$ whereas on the bottom line we plot $\Im\mathbf{B}$. a) is for $\Im z = 10^{-3}$, whereas b) is for $\Im z = 10^{-4}$. Each plot has 10000 points $\Re z$ uniformly distributed in $[0, 1/2]$. $k_1 = 80$, $k_2 = 20$, $N_{max} = 151$.

Action of the operator T_k on Hölder functions: modular smoothing

- Let $(T_{k,e}f)(x) = x^k f \circ A_{1/2}(x)$, $\forall x \in (0, 1/2)$, act on even periodic functions.
- **Theorem (M, Moussa, Yoccoz)** When $k = 1$ the exponent $1/2$ plays here a crucial role: if η denotes the Hölder exponent of f then:
  - if $\eta > 1/2$ then $B_{1,f}$ is $1/2$–Hölder continuous;
  - if $\eta < 1/2$ then $B_{1,f}$ is also η–Hölder continuous;
  - if $\eta = 1/2$ then $B_{1,f}$ admits $x^{1/2} \log x$ as continuity modulus.
- The distinguished role played by the exponent $1/2$ is perhaps the most convincing theoretical argument supporting the Hölder interpolation conjecture.
- This result makes the numerical results of Buric, Percival and Vivaldi [BPV90] less mysterious.