Approximations of the Lagrange and Markov spectra

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Diophantine approximations (I)

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Definition

The *Lagrange spectrum* $L \subset \mathbb{R}$ is $L := \{l(\alpha) < \infty : \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$,

$$l(\alpha) := \limsup_{p,q \to \infty} \frac{1}{|q(q\alpha - p)|}$$

Diophantine approximations (II)

Given $h(x,y) = ax^2 + bxy + cy^2$ a real, indefinite, binary quadratic form with positive discriminant $\Delta(h) := b^2 - 4ac > 0$, let

$$m(h) := \sup_{(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\sqrt{\Delta(h)}}{|h(p,q)|}$$

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Definition

The *Markov spectrum* $M \subset \mathbb{R}$ is $M := \{m(h) < \infty : h \text{ as above } \}$.

Beginning of L and M (I)

Hurwitz (1890): $\sqrt{5} = \min L$ because

$$\#\left\{\frac{p}{q} \in \mathbb{Q} : |\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}\right\} = \infty, \quad \forall\, \alpha \in \mathbb{R} \setminus \mathbb{Q},$$

and

$$\#\left\{\frac{p}{q} \in \mathbb{Q} : |\frac{1+\sqrt{5}}{2} - \frac{p}{q}| < \frac{1}{(\sqrt{5}+\varepsilon)q^2}\right\} < \infty, \quad \forall\, \varepsilon > 0.$$

Beginning of L and M (II)

Markov (1880) : $L \cap [\sqrt{5}, 3) = M \cap [\sqrt{5}, 3) =$

$$\left\{\sqrt{5} < \sqrt{8} < \frac{\sqrt{221}}{5} < \dots\right\} = \left\{\sqrt{9 - \frac{4}{z_n^2}} : n \in \mathbb{N}\right\}$$

where $x_n \leq y_n \leq z_n$, $(x_n, y_n, z_n) \in \mathbb{N}^3$ is a *Markov triple*, i.e.,

$$x_n^2 + y_n^2 + z_n^2 = 3x_n y_n z_n$$

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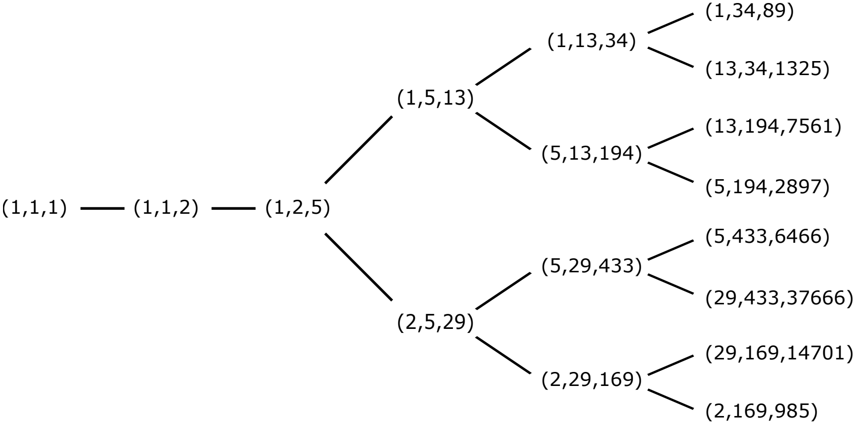
Some related topics

- *Markov uniqueness conjecture*: Bombieri, Aigner, ...;
- $\mathbb{Z}$*-pts of M.-H. var.*: Zagier, Baragar, Gamburd-Magee-Ronan;
- *Geod. of hyperb. surf.*: McShane-Rivin, Mirzakhani, ...;
- *Dynamics on character varieties*: Goldman, Cantat, ...;
- *Markov expanders*: Bourgain-Gamburd-Sarnak, ...

Markov's tree

All Markov triples are deduced from $(1,1,1)$ via *Vieta's involutions* $(x,y,z) \mapsto (3yz-x,y,z)$, etc. This leads to *Markov's tree*:



L and M after Perron (I)

Let $\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$ be the *shift dynamics* on $\Sigma = (\mathbb{N}^*)^{\mathbb{Z}}$, and consider the *height function* $f : \Sigma \to \mathbb{R}$,

$$f((a_n)_{n\in\mathbb{Z}}) := [a_0; a_1, \dots] + [0; a_{-1}, \dots]$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots}} + \cfrac{1}{a_{-1} + \cfrac{1}{\ddots}}$$

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Perron proved in 1921 that

$$L = \{\limsup_{n\to\infty} f(\sigma^n(a)) < \infty : a \in \Sigma\}$$

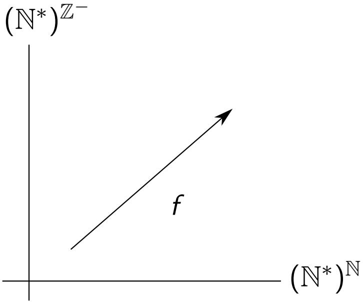
and

$$M = \{\sup_{n\in\mathbb{Z}} f(\sigma^n(a)) < \infty : a \in \Sigma\}$$

Perron's description of L and M

Remark

The key fact behind Perron's characterization of L is the identity $\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{(x_n+y_n)q_n^2}$ for $\alpha := [a_0; a_1, a_2, \dots,]$, $\frac{p_n}{q_n} := [a_0; a_1, \dots, a_n]$, $x_n := [0; a_{n+1}, \dots]$, $y_n := [0; a_n, \dots, a_1]$.



L and M after Perron (II)

This dynamical characterisation of L and M gives access to several results:

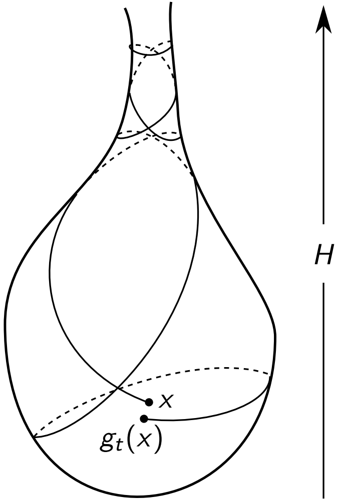
- Perron also showed in 1921 that $(\sqrt{12}, \sqrt{13}) \cap M = \emptyset$, $\sqrt{12}, \sqrt{13} \in L$, ...
- $L = \overline{\{\sup\limits_{n\in\mathbb{Z}} f(\sigma^n x) : x \text{ per.}\}} \subset M = \overline{\{\sup\limits_{n\in\mathbb{Z}} f(\sigma^n x) : x \text{ ev. per.}\}}$ are *closed* subsets of the real line,
- etc.

L, M and the modular surface

The relation between continued fractions and geodesics on the modular surface $\mathbb{H}/SL(2,\mathbb{Z})$ says that L and M correspond to heights of excursions of geodesics into the cusp of $\mathbb{H}/SL(2,\mathbb{Z})$.

Movie by Pierre Arnoux and Edmund Harriss.

Impressionistic picture of the modular surface



Ending of L and M

The works of Hall (1947), ..., Freiman (1975) give that the largest half-line of the form $[c, \infty)$ contained in $L \subset M$ is

$$\left[\frac{2221564096 + 283748\sqrt{462}}{491993569}, \infty\right)$$

This half-line is called *Hall's ray* in the literature and its left endpoint is called *Freiman's constant* $c_F = 4.5278\ldots$.

Intermediate portion of L and M (I)

We saw that L and M *coincide* before 3 and after c_F:

$$L \cap [\sqrt{5}, 3] = M \cap [\sqrt{5}, 3]$$

and

$$L \cap [c_F, \infty) = M \cap [c_F, \infty) = [c_F, \infty).$$

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However, Freiman (1968), Flahive (1977) and M.-Moreira (2018) proved that $M \setminus L$ has a rich structure near 3.11, 3.29 and 3.7, and $0.531 < \dim(M \setminus L) < 0.987$.

Intermediate portion of L and M (II)

Moreira (2016) showed that

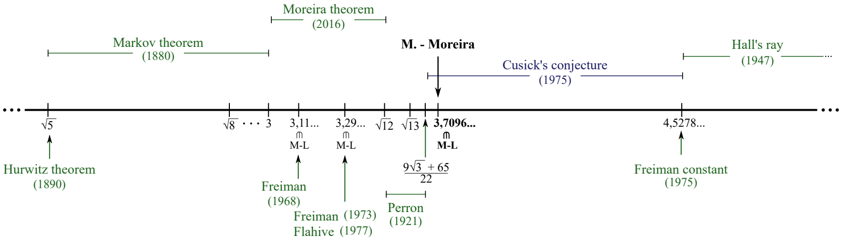
$$dim(L \cap (-\infty, t)) = dim(M \cap (-\infty, t))$$

for all $t \in \mathbb{R}$. Hence, $M \setminus L$ doesn't create "jumps in dimension" between L and M.

Moreira also proved that $d(t) := dim(L \cap (-\infty, t))$ is a continuous *non-Hölder* function of t such that

$$d(3+\varepsilon) > 0 \quad \forall \varepsilon > 0 \quad \text{and} \quad d(\sqrt{12}) = 1.$$

Global view of the Lagrange and Markov spectra



Fine details of the intermediate portion of L and M ?

Despite all recent progress, many basic problems are still open: e.g., Berstein conjectured that $[4.1, 4.52] \subset L \subset M$ and a folkloric question (cf. Cusick-Flahive) whether $\mathrm{int}(L \cap [3, \sqrt{12}]) \neq \emptyset$.

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Remark

This relates to sumsets / projections of certain Cantor sets: e.g., $\mathrm{int}(L \cap [3, \sqrt{12}]) \neq \emptyset$ is expected "as" Marstrand's theorem "predicts" that $\mathrm{int}(C(2) + C(2)) \neq \emptyset$ for the "nonlinear" Cantor set $C(2) := \{[0; \gamma] : \gamma \in \{1, 2\}^{\mathbb{N}}\}$ with $\dim(C(2)) > 1/2$.

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By "analogy" with the case of the famous Mandelbrot set, one could hope to build strategies to these kind of questions by the inspection of *rigorous drawings* of L and M.

Markov values at periodic orbits (I)

We know that $L = \overline{\{\sup\limits_{n\in\mathbb{Z}} f(\sigma^n x) : x \text{ periodic}\}}$ (and similarly for M).

This suggests to try to draw L by computing Markov values $m(x) := \sup\limits_{n\in\mathbb{Z}} f(\sigma^n x)$ at certain periodic words $x \in (\mathbb{N}^*)^{\mathbb{Z}}$.

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Unfortunately, the natural bound on the complexity of the resulting algorithm is *large*. Indeed:

- $\sqrt{5} \leq \ell = \limsup\limits_{j\to\infty} f(\sigma^n x) \leq \sqrt{21}$ implies $x \in \{1,2,3,4\}^{\mathbb{Z}}$;

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- given m, there are h_m and a sequence $j_i \to \infty$ such that $f(\sigma^j x) < \ell + 1/m \; \forall \; j \geq h_m$ and $f(\sigma^{j_i} x) \to \ell$;

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- given N, there is $S \in \{1,2,3,4\}^{2N+1}$ such that for infinitely many j_i's one has $S = (x_{j_i - N}, \dots, x_{j_i}, \dots, x_{j_i+N})$;

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- moreover, S can be connected to itself using factors of sizes $2N+1$ of words with Markov values $\leq t + 1/m$.

Markov values at periodic orbits (II)

Hence, there exists $1 \leq k \leq 4^{2N+1}$ and a factor $(a_0, \dots, a_{2N+k+1})$ of size $2N+k+1$ of a word with Markov value $\leq t + 1/m$ such that $(a_0, \dots, a_{2N}) = S = (a_k, \dots, a_{2N+k+1})$. In particular, since $|[0; z_1, \dots, z_n, z_{n+1}, \dots] - [0; z_1, \dots, z_n, w_{n+1}, \dots]| < \frac{1}{2^{n-1}}$,

$$\theta = \overline{(a_0, \dots, a_{k-1})} \in \bigcup_{1 \leq s \leq 4^{2N+1}} \{1,2,3,4\}^s$$

is a periodic word with Markov value $|m(\theta) - \ell| < \frac{1}{m} + \frac{1}{2^{N-2}}$.

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$$\theta = \overline{(a_0, \dots, a_{k-1})} \in \bigcup_{1 \leq s \leq 4^{2N+1}} \{1, 2, 3, 4\}^s$$

is a periodic word with Markov value $|m(\theta) - \ell| < \frac{1}{m} + \frac{1}{2^{N-2}}$.

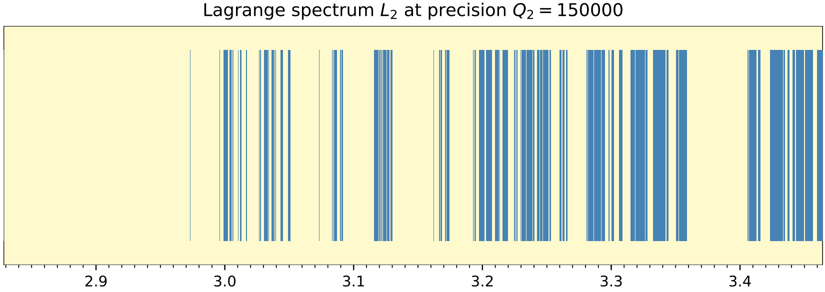
In summary, if we compute the Markov values of $\sim 4^{Q^4}$ periodic words of lengths $\leq Q^4$, then we obtain a $1/Q$-dense subset of L.

Main result

Theorem (Delecroix–M.–Moreira)

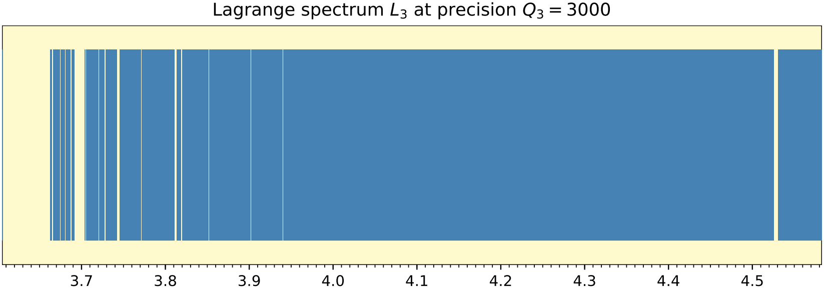
There is an algorithm providing finite sets $1/Q$-close (in Hausdorff topology) to L and M after time $O(Q^{2.367})$.

An approximation of $L_2 = L \cap [\sqrt{5}, \sqrt{12}]$ given by this algorithm:



Some initial remarks about the algorithm

The algorithm was implement in Sage by Delecroix and it is available at https:// plmlab.math.cnrs.fr/delecroix/lagrange



Our approx. of $L_K = \{\limsup f(\sigma^n x) : x \in \{1, \dots, K\}^{\mathbb{Z}}\}$ are $\frac{1}{250}$-close to L (which is not enough to tackle Berstein's conj.).

Some “fake news” (I)

As a warmup, let us describe a simplified version of the algorithm with a slightly worse polynomial complexity.

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Given $x = (x_m)_{m\in\mathbb{Z}} \in (\mathbb{N}^*)^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, let

$$\lambda_n(x) := f(\sigma^n(x)) := [x_n; x_{n+1}, \dots] + [0; x_{n-1}, \dots]$$

In order to describe $L \cap [0, R]$, it suffices to study the values of f along the orbits of the restriction of σ to $\{1, \dots, K = \lfloor R \rfloor\}^{\mathbb{Z}}$.

Fix Q and take n s.t. $b = (b_{-n}, \dots, b_0, \dots, b_n) \in \{1, \dots, K\}^{2n+1}$ generates a cylinder

$$\vec{b} := \{a \in \{1, \dots, K\}^{\mathbb{Z}} : a_j = b_j \, \forall \, |j| \leq n\}$$

with $\sup\limits_{a\in\vec{b}} \lambda_0(a) - \inf\limits_{a\in\vec{b}} \lambda_0(a) < 1/Q$.

Some "fake news" (II)

Consider the graph $\widetilde{G}_{K,Q}$ with set of vertices $\{1,\dots,K\}^{2n}$ and edges $u\to v$ when $u=(u_{-n},\dots,u_{n-1})$ and $v=(v_{-n},\dots,v_{n-1})$ satisfy $v_j=u_{j+1}\ \forall -n\leq j\leq n-2$. We equip the edges $u\to v$ of this *de Bruijn* graph with weights

$$w(u_{-n},\dots,v_n):=\frac{\sup\limits_{a\in\vec{b}}\lambda_0(a)+\inf\limits_{a\in\vec{b}}\lambda_0(a)}{2}.$$

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A *Lagrange edge* e is an edge belonging to cycle γ such that $w(e)$ is maximal among all edges in γ.

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A *Lagrange edge* e is an edge belonging to cycle γ such that $w(e)$ is maximal among all edges in γ.

The heart of the matter is the fact that the set of weights of Lagrange edges of $\widetilde{G}_{K,Q}$ is $1/Q$-close to $L \cap [0, R]$.

Genuine algorithm (I)

Even though de Bruijn graphs are pleasant, well-known objects, their usage in the previous construction is *suboptimal*: roughly speaking, the *combinatorial size* $2n+1$ of a word $(b_{-n},\dots,b_n)$ loses track of the *geometry* of $C(K)=\{[0;x]:x\in\{1,\dots,K\}^{\mathbb{N}}\}$.

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For this reason, we introduce the notion of *geometric size* of $b \in \{1, \dots, K\}^+ = \bigcup\limits_{n \in \mathbb{N}} \{1, \dots, K\}^n$, namely,

$$\mathrm{diam}(b) := \mathrm{diameter}\ \{[0; x] : x = b \cdots \in \{1, \dots, K\}^{\mathbb{N}}\}$$

and we consider

$$C_{K,Q} := \{b \in \{1, \dots, K\}^+ : \mathrm{diam}(b) \leq \frac{1}{Q} < \mathrm{diam}(b')\}.$$

Genuine algorithm (II)

Very roughly speaking, our idea is to build a graph $G_{K,Q}$ (playing the role of $\widetilde{G}_{K,Q}$) based on the set $C_{K,Q}$ (instead of $\{1, \ldots, K\}^n$).

Remark

The precise definition of $G_{K,Q}$ is somewhat involved. In particular, even though $C_{K,Q}$ serves to define the vertices and edges of $G_{K,Q}$, it is *not* the vertex set of this graph. Also, $G_{K,Q}$ has *two* types of edges (called "prolongation" and "shift").

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In any event, it takes time $O(m)$ to determine if an edge e of $G_{K,Q}$ is Lagrange, where $m := \#\text{edges of } G_{K,Q}$: indeed, it suffices to perform a depth-first search on the edges with weight $\leq w(e)$ to try to connect the endpoints of e.

Genuine algorithm (III)

It follows that it takes time $O(m^2)$ to compute the set of weights of Lagrange edges of $G_{K,Q}$.

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Actually, if we order the edges $e_1, \dots, e_m$ so that $w(e_i) \leq w(e_{i+1})$ and introduce the graphs $G^{(k)}$ obtained from $\{e_1, \dots, e_k\}$ after identifying vertices in the same strong connected component and removing loops, then $G^{(k)}$ is derived from $G^{(k-1)}$ by adding e_k and describing new connected components, and the Lagrange edges e_k are those creating cycles when added to $G^{(k-1)}$.

Genuine algorithm (III)

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Thus, we can employ methods of online cycle detection and maintenance of strongly connected components to compute all Lagrange edges of $G_{K,Q}$ in time $O(m^{3/2})$.

Furthermore, it is not difficult to check that the set of weights of Lagrange edges of $G_{K,Q}$ is $1/Q$-close to $L \cap [0, R]$.

Genuine algorithm (IV)

Hence, our task is to determine $m = \#\text{edges of } G_{K,Q}$.

A quick inspection of the definitions reveals that $m = O(\#C_{K,Q})^2$, so that our algorithm runs in time $O(\#C_{K,Q})^3$.

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At this point, we recall Bowen's equation

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where $\Lambda(\vec{b}) \sim 1/\mathrm{diam}(\vec{b}) \sim Q$ is the maximal derivative of the $|\vec{b}|$-iterate of the restriction of the Gauss map to the interval $\{[0; x] : x = b \cdots \in \{1, \dots, K\}^{\mathbb{N}}\}$.

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Consequently, $\# C_{K,Q} \sim Q^{\dim(C(K))}$ and the running time of our algorithm is $O(Q^{3\dim(C(K))})$. Since $\dim(C(4)) < 0.789$, our main theorem is proved.

Thank you! Merci! Obrigado!