α-odd continued fractions

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Regular continued fractions

Standard algorithm

Let $x > 0$, then a_0 is the largest integer less than x.

$x = a_0 + \dfrac{1}{r_1}$ for $1 < r_1$. Let a_1 be the largest integer less that r_1.

$x = a_0 + \dfrac{1}{a_1 + \dfrac{1}{r_2}}$ for $1 < r_2$. Continue this process,

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}} = [a_0; a_1, a_2, \dots] \, .$$

Facts and tricks

- This process terminates if and only if x is rational.
- $n = n - 1 + \frac{1}{1}$, so we require that the last digit is greater than 1.
- Then every real number has a unique continued fraction expansion.
- $\frac{1}{[a_0; a_1, a_2, \dots]} = \dfrac{1}{a_0 + [0; a_1, a_2, \dots]} = [0; a_0, a_1, a_2, \dots]$.
- $\frac{1}{[0; a_1, a_2, \dots]} = a_1 + \dfrac{1}{a_2 + [0; a_3, \dots]} = [a_0; a_1, a_2, \dots]$.

Regular continued fraction Gauss map

Define $T : [0,1) \to [0,1)$ by

$$T(x) = \begin{cases} \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} \frac{1}{x} - k & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k} \right] \\ 0 & \text{if } x = 0 \end{cases}.$$

Note that $T([0; a_1, a_2, a_3, \ldots]) = [0; a_2, a_3, \ldots]$

Natural Extension of the regular Gauss map

Define $\overline{T} : [0,1)^2 \to [0,1)^2$ by

$$\overline{T}(x,y) = \begin{cases} \left(\frac{1}{x} - k, \frac{1}{y+k}\right) & \text{for } x \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \\ (0,y) & \text{if } x = 0 \end{cases}.$$

$\overline{T}(([0; a_1, a_2, \dots], [0; b_1, b_2, \dots])) = ([0; a_2, a_3, \dots], [0; a_1, b_1, b_2, \dots])$.

Facts about the regular Gauss map

- The dynamical system $([0,1), T, \mathcal{B}_{[0,1)}, \mu)$ is ergodic for $d\mu = \frac{1}{\log 2}\frac{dx}{x+1}$
- The dynamical system $([0,1)^2, \overline{T}, \mathcal{B}_{[0,1)^2}, \nu)$ is ergodic for $d\nu = \frac{1}{\log 2}\frac{dxdy}{(1+xy)^2}$

Nearest integer continued fractions

In the Euclidean algorithm definition of continued fractions, we could take the nearest integer instead of the floor.

$T_{NICF} : \left[-\frac{1}{2}, \frac{1}{2}\right) \to \left[-\frac{1}{2}, \frac{1}{2}\right)$ by

$$x \mapsto \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + \frac{1}{2} \right\rfloor = \frac{1}{|x|} - k \text{ for } |x| \in \left(\frac{1}{k - \frac{1}{2}}, \frac{1}{k + \frac{1}{2}} \right)$$

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For example,

$$\cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{3}}}} = \cfrac{1}{3 - \cfrac{1}{4 + \cfrac{1}{3}}}$$

Nakada α-continued fractions

Nakada α-continued fractions

Nakada (1981) introduced the α continued fractions.

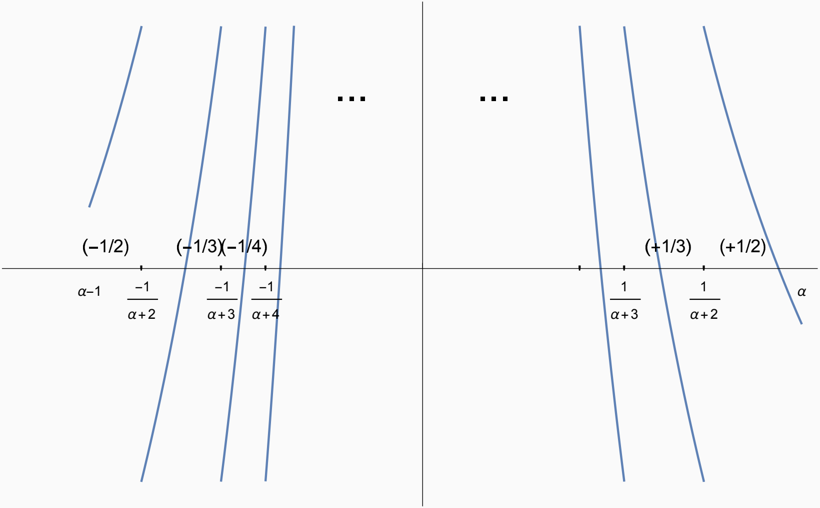
Define f_α on $[\alpha - 1, \alpha)$ to be

$$f_\alpha(x) = \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + 1 - \alpha \right\rfloor = \frac{1}{|x|} - k \text{ for } |x| \in \left(\frac{1}{k+\alpha}, \frac{1}{k-1+\alpha} \right]$$

Regular continued fractions when $\alpha = 1$

Nearest integer continued fractions when $\alpha = \frac{1}{2}$

Nakada α-Gauss map



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Then the first digit of x is $(\text{sign}(x)/k)$ $\epsilon_i = \text{sign}\, f_\alpha^{i-1}(x), a_i = k$ for $|f_\alpha^{i-1}(x)| \in \left(\frac{1}{k+\alpha}, \frac{1}{k-1+\alpha} \right]$

$$x = \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \cfrac{\epsilon_3}{\ddots}}} = [(\epsilon_1/a_1)(\epsilon_2/a_2), \dots]_\alpha$$

Natural Extension

Define $\overline{f_\alpha} : \Omega_\alpha \to \Omega_\alpha$ by

$$(x, y) \mapsto \left(\frac{1}{|x|} - a_1, \frac{1}{a_1 + \epsilon y} \right) \text{ for } |x| \in \left(\frac{1}{a_1 - 1 + \alpha}, \frac{1}{a_1 + \alpha} \right)$$

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$$\left(\frac{\epsilon_0}{a_0 + \frac{\epsilon_1}{a_1 + \dots}}, \frac{1}{a_{-1} + \frac{\epsilon_{-1}}{a_{-2} + \dots}} \right) \mapsto \left(\frac{\epsilon_1}{a_1 + \frac{\epsilon_2}{a_2 + \dots}}, \frac{1}{a_0 + \frac{\epsilon_0}{a_{-1} + \dots}} \right)$$

What is Ω_α?

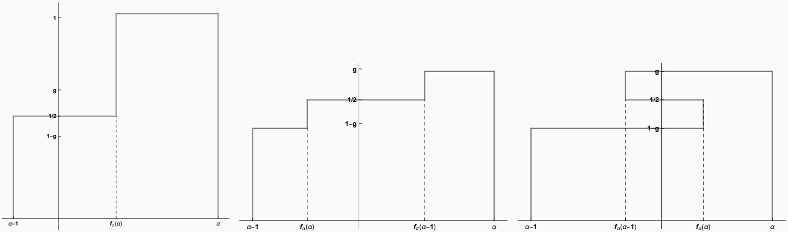
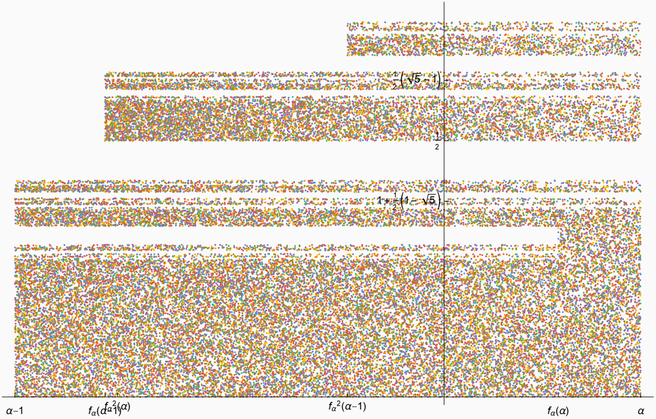


Figure 1: $g < \alpha \leq 1$ $\frac{1}{2} \leq \alpha \leq g$ $\sqrt{2}-1 \leq \alpha \leq \frac{1}{2}$
The natural extension domain Ω_α

$\alpha < \sqrt{2} - 1$



Odd integer continued fractions

Odd integer algorithm

What if instead of rounding down in the Euclidean algorithm, we found the nearest odd integer?

1. How do we deal with even integers?
2. How do we get all positive real numbers?

Odd integers, an example

Let $x = \frac{36}{13}$.

- Since $x \in (2, 3)$, $x = 3 - r_1$ for $r_1 \in (0, 1)$.

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- $x = 3 - \dfrac{1}{\frac{13}{3}} = 3 - \dfrac{1}{5 - r_2}$ for $r_2 \in (0, 1)$.

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- $x = 3 - \cfrac{1}{\frac{13}{3}} = 3 - \cfrac{1}{5 - r_2}$ for $r_2 \in (0,1)$. Here, $r_2 = \frac{2}{3}$.
- $x = 3 - \cfrac{1}{5 - \cfrac{1}{\frac{3}{2}}} = 3 - \cfrac{1}{5 - \cfrac{1}{1 + r_3}}$ for $r_3 \in (0,1)$.

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- $x = 3 - \dfrac{1}{5 - \dfrac{1}{1 + \dfrac{1}{1+1}}} = 3 - \dfrac{1}{5 - \dfrac{1}{1 + \dfrac{1}{3-1}}}$

Another example

$$G = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} = \cfrac{1}{1 - \cfrac{1}{3 - \cfrac{1}{3 - \dots}}}$$

Odd Continued fractions

Rieger (1980) defined a new type of continued fractions

$$x = a_0 + \cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \cfrac{\epsilon_3}{\ddots}}} = a_0 + \langle\!\langle(\epsilon_1/a_1)(\epsilon_2/a_2), \ldots\rangle\!\rangle_o$$

where the a_i are odd, $\epsilon_i = \pm 1$, and $a_i + \epsilon_{i+1} \geq 2$. That is, you never have a 1 followed by subtraction.

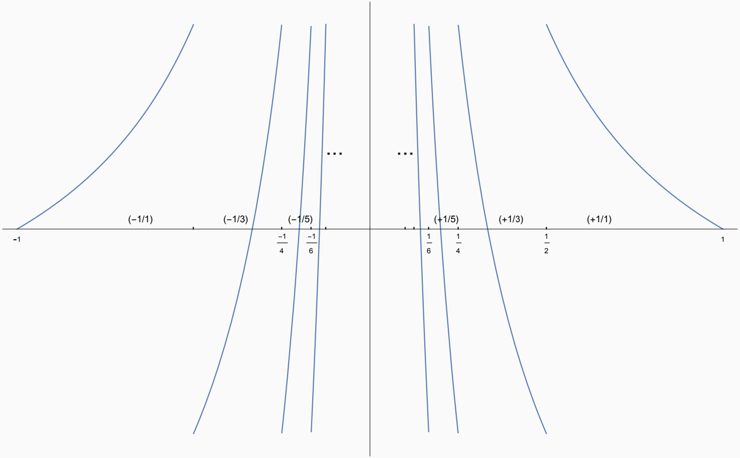
The extended odd Gauss Map

Define T_o on $[-1, 1)$

$$T_o(x) = \frac{1}{|x|} - (2k+1) \text{ for } |x| \in \left(\frac{1}{2k+2}, \frac{1}{2k}\right]$$

Then the first digit of x is $(\text{sign}(x)/(2k+1))$.
$\epsilon_i = \text{sign } T_0^{i-1}(x), a_i = 2k+1$ for $|T_o^{i-1}(x)| \in \left(\frac{1}{2k+2}, \frac{1}{2k}\right]$

Odd Gauss Map



$$T_o\big(\langle\!\langle(\epsilon_1/a_1)(\epsilon_2/a_2),\ldots\rangle\!\rangle_o\big) = \langle\!\langle(\epsilon_2/a_2)(\epsilon_3/a_3),\ldots\rangle\!\rangle_o.$$

Grotesque Continued Fractions

Rieger (1980) defined the grostesque continued fractions

$$y = \cfrac{\epsilon_0}{a_0 + \cfrac{\epsilon_{-1}}{a_{-1} + \cfrac{\epsilon_{-2}}{\ddots}}} = \langle\!\langle(\epsilon_1/a_1)(\epsilon_2/a_2), \ldots\rangle\!\rangle_{grot}$$

where the a_i are odd, $\epsilon_i = \pm 1$, and $a_i + \epsilon_i \geq 2$.

- $\frac{-1}{1+z}$ is only allowed in the odd continued fractions
- $\frac{1}{1-z}$ is only permitted in grotesque continued fractions

Grotesque Gauss map

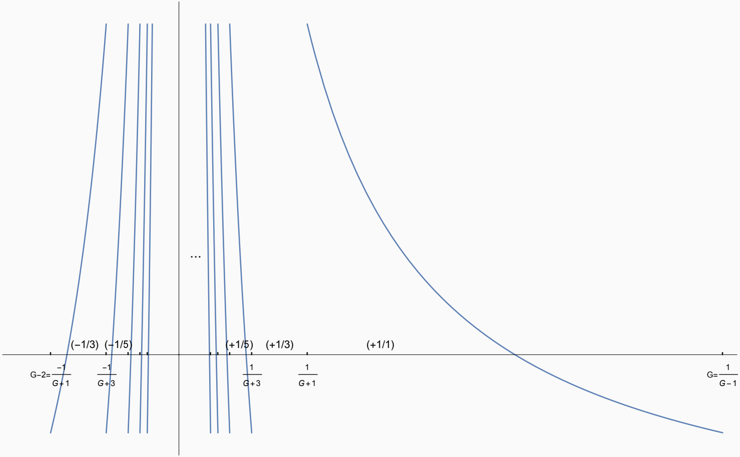
Define $T_{grot} : [G-2, G) \to [G-2, G)$ where $G = \frac{1+\sqrt{5}}{2}$ by

$$T_{grot}(x) = \frac{1}{|x|} - (2k+1) \text{ for } |x| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G} \right]$$

Then the first digit of x is $(\text{sign}(x)/(2k+1))$

$\epsilon_i = \text{sign } T_{grot}^{i-1}(x), a_i = 2k+1$ for $|T_{grot}^{i-1}(x)| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G} \right]$

Grotesque Gauss Map



$$T_{grot}\left(\langle\langle(\epsilon_1/a_1)(\epsilon_2/a_2),\ldots\rangle\rangle_{grot}\right)=\langle\langle(\epsilon_2/a_2)(\epsilon_3/a_3),\ldots\rangle\rangle_{grot}.$$

What are we doing?

The odd continued fractions and grotesque continued fractions are *dual continued fraction expansions*.

Goals:

1. Define a way to interpolate between the grotesque continued fractions and the (extended) odd continued fractions.
2. Find invariant measures and natural extensions of the relevant Gauss maps.
3. Extend these results to other dynamical systems that generate continued fractions with odd denominators.

α-expansions with odd partial quotients, (Boca-M, 2019)

Comparing Gauss maps

If we ignore the rules for generating the digits, both T_o and T_{grot} take

$$x \mapsto \frac{1}{|x|} - (2k+1)$$

$$\langle\!\langle (\epsilon_1/a_1)(\epsilon_2/a_2)\ldots\rangle\!\rangle \mapsto \langle\!\langle (\epsilon_2/a_2)(\epsilon_3/a_3)\ldots\rangle\!\rangle$$

$T_o : [-1,1) \to [-1,1)$
Intervals $|x| \in \left(\frac{1}{2k+2}, \frac{1}{2k}\right]$

$T_{grot} : [G-2, G) \to [G-2, G)$
Intervals $|x| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G}\right]$

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$T_o : [-1,1) \to [-1,1)$	$T_{grot} : [G-2, G) \to [G-2, G)$
Intervals $\lvert x\rvert \in \left(\frac{1}{2k+2}, \frac{1}{2k}\right]$	Intervals $\lvert x\rvert \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G}\right]$
$(\epsilon_i/1)$ always followed by $(+1/a_{i+1})$	Never $(-1/1)$.

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If we ignore the rules for generating the digits, both T_o and T_{grot} take

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$T_o : [-1,1) \to [-1,1)$	$T_{grot} : [G-2, G) \to [G-2, G)$
Intervals $\|x\| \in \left(\frac{1}{2k+2}, \frac{1}{2k}\right]$	Intervals $\|x\| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G}\right]$
$(\epsilon_i/1)$ always followed by $(+1/a_{i+1})$	Never $(-1/1)$.

Define a new map $\varphi_\alpha : [\alpha-2, \alpha) \to [\alpha-2, \alpha)$ by

$$\varphi_\alpha(x) = \frac{1}{|x|} - (2k+1) \text{ for } |x| \in \left(\frac{1}{2k+1+\alpha}, \frac{1}{2k-1+\alpha}\right]$$

α-odd Gauss map

We will consider $\alpha \in [g, G)$ for $g = \frac{-1+\sqrt{5}}{2} = G - 1$.

$\varphi_\alpha : [\alpha - 2, \alpha) \to [\alpha - 2, \alpha)$ by

$$\varphi_\alpha(x) = \frac{1}{|x|} - (2k+1) \text{ for } |x| \in \left(\frac{1}{2k+1+\alpha}, \frac{1}{2k-1+\alpha} \right]$$

Again, we define $\epsilon_i = \text{sign}(\varphi_\alpha^{i-1}(x)), a_i = 2k+1$ such that $|\varphi_\alpha^{i-1}(x)| \in \left(\frac{1}{2k+1+\alpha}, \frac{1}{2k-1+\alpha} \right]$

$\varphi_\alpha(\langle\langle(\epsilon_1/a_1)(\epsilon_2/a_2)(\epsilon_3/a_3)\ldots\rangle\rangle_\alpha) = \langle\langle(\epsilon_2/a_2)(\epsilon_3/a_3)\ldots\rangle\rangle_\alpha$

Digit restrictions

Let $d_\alpha(x) = 2\lfloor \frac{1}{2|x|} - \frac{1-\alpha}{2} \rfloor + 1 = a_1(x)$ emphasizing the α.

Lemma

1. When $g < \alpha \leq G$ we have $d_\alpha(\alpha) = 1$ and when $g \leq \alpha < G$ we have $d_\alpha(\alpha - 2) = 1$.
2. When $\alpha \in (g, 1) \cup (1, G)$ we have

$$\frac{1}{\varphi_\alpha(\alpha)} + \frac{1}{\varphi_\alpha(\alpha - 2)} = -2$$

Digit restrictions

3. When $g < \alpha \leq 1$ we have

$$\begin{aligned} -\frac{1}{3+\alpha} < \varphi_\alpha(\alpha-2) \leq 0 \leq \varphi_\alpha(\alpha) < \frac{1}{1+\alpha}, \\ d_\alpha(\varphi_\alpha(\alpha)) = d_\alpha(\varphi_\alpha(\alpha-2)) - 2 \\ \textit{and} \quad \varphi_\alpha^2(\alpha) = \varphi_\alpha^2(\alpha-2). \end{aligned} \tag{1}$$

4. When $1 \leq \alpha < G$ we have

$$\begin{aligned} -\frac{1}{1+\alpha} < \varphi_\alpha(\alpha) \leq 0 \leq \varphi_\alpha(\alpha-2) < \alpha, \\ d_\alpha(\varphi_\alpha(\alpha)) = d_\alpha(\varphi_\alpha(\alpha-2)) + 2 \\ \textit{and} \quad \varphi_\alpha^2(\alpha) = \varphi_\alpha^2(\alpha-2). \end{aligned} \tag{2}$$

Natural extensions

Natural extension of the α-odd Gauss map

We define $\overline{\varphi}_\alpha : \Omega_\alpha \to \Omega_\alpha$ by

$$\left(\cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \dots}}, \cfrac{1}{a_0 + \cfrac{\epsilon_0}{a_{-1} + \dots}} \right) \mapsto \left(\cfrac{\epsilon_2}{a_2 + \cfrac{\epsilon_3}{a_3 + \dots}}, \cfrac{1}{a_1 + \cfrac{\epsilon_1}{a_0 + \cfrac{\epsilon_0}{a_{-1} + \dots}}} \right)$$

Or

$$(x, y) \mapsto \left(\frac{1}{|x|} - (2k+1), \frac{1}{2k+1+\epsilon(x)y} \right)$$
$$\text{for } |x| \in \left(\frac{1}{2k+1+\alpha}, \frac{1}{2k-1+\alpha} \right], \epsilon(x) = \operatorname{sign}(x).$$

Natural extension of the grotesque Gauss map

We define $\overline{T}_{grot} : [G-2, G) \times [0,1) \to [G-2, G) \times [0,1)$ by

$$\left(\cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \dots}}, \cfrac{1}{a_0 + \cfrac{\epsilon_0}{a_{-1} + \dots}} \right) \mapsto \left(\cfrac{\epsilon_2}{a_2 + \cfrac{\epsilon_3}{a_3 + \dots}}, \cfrac{1}{a_1 + \cfrac{\epsilon_1}{a_0 + \cfrac{\epsilon_0}{a_{-1} + \dots}}} \right)$$

Or

$$(x, y) \mapsto \left(\frac{1}{|x|} - (2k+1), \frac{1}{2k+1+\epsilon(x)y} \right)$$
$$\text{for } |x| \in \left(\frac{1}{2k+1+G}, \frac{1}{2k-1+G} \right], \epsilon(x) = \text{sign}(x).$$

Natural extension of the extended odd Gauss map

We define $\overline{T}_0 : \big([-1,0) \times [0, 2-G)\big) \cup \big([0,1) \times [0,G)\big) \to \big([-1,0) \times [0, 2-G)\big) \cup \big([0,1) \times [0,G)\big)$ by

$$\left(\cfrac{\epsilon_1}{a_1 + \cfrac{\epsilon_2}{a_2 + \dots}}, \cfrac{1}{a_0 + \cfrac{\epsilon_0}{a_{-1} + \dots}} \right) \mapsto \left(\cfrac{\epsilon_2}{a_2 + \cfrac{\epsilon_3}{a_3 + \dots}}, \cfrac{1}{a_1 + \cfrac{\epsilon_1}{a_0 + \cfrac{\epsilon_0}{a_{-1} + \dots}}} \right)$$

Or

$$(x, y) \mapsto \left(\frac{1}{|x|} - (2k+1), \frac{1}{2k+1+\epsilon(x)y} \right)$$
$$\text{for } |x| \in \left(\frac{1}{2k+2}, \frac{1}{2k} \right], \epsilon(x) = \text{sign}(x).$$

Natural extension domains

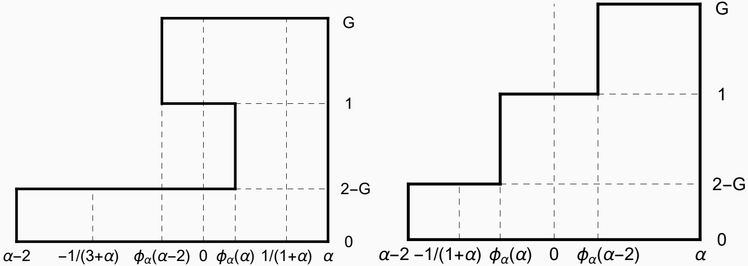

Figure 2: The natural extension domains Ω_α for $g \leq \alpha \leq 1$ (left) and $1 \leq \alpha \leq G$ (right)

Some ergodic theory results

(Boca-M, 2019)

- The measure $\mu_\alpha = \frac{1}{3\log G}\frac{dxdy}{(1+xy)^2}$ is Φ_α-invariant on Ω_α.
- $d\nu_\alpha = h_\alpha d\lambda$ is φ_α-invariant with h_α coming from integrating μ_α over the y-values of the natural extension domain.
- $(\Omega_\alpha, \mathcal{B}_{\Omega_\alpha}, \tilde{\mu}_\alpha, \Phi_\alpha)$ gives the natural extension of $(I_\alpha, \mathcal{B}_{I_\alpha}, \nu_\alpha, \varphi_\alpha)$
- $(I_\alpha, \mathcal{B}_{I_\alpha}, \nu_\alpha, \varphi_\alpha)$ is ergodic, exact, and mixing of all orders.

Standard consequences of ergodicity

Let $\frac{p_n}{q_n} = \langle\!\langle (\epsilon_1/a_1)\dots(\epsilon_n/a_n) \rangle\!\rangle_\alpha$,

For every $\alpha \in [g, G]$ and almost every $x \in I_\alpha$:

(i) $\lim_n \frac{1}{n} \log |q_n(x;\alpha)x - p_n(x;\alpha)| = -\frac{\pi^2}{18 \log G}$.

(ii) $\lim_n \frac{1}{n} \log q_n(x;\alpha) = \frac{\pi^2}{18 \log G}$.

(iii) $\lim_n \frac{1}{n} \log \left| x - \frac{p_n(x;\alpha)}{q_n(x;\alpha)} \right| = -\frac{\pi^2}{9 \log G}$.

For every $\alpha \in [g, G]$ the entropy of φ_α with respect to ν_α is

$$H(\alpha) = -2 \int_{\alpha-2}^{\alpha} \log|x|\, h_\alpha(x)\, dx = \frac{\pi^2}{9 \log G}.$$