Adding machine, Automata and Julia sets

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Introduction

Objective: study dynamical, spectral and probabilistic properties of stochastic adding machines. This work involves

- Julia sets
- Automata
- Topological Dynamics on Banach spaces
- Recurrence and transience of infinite state Markov chains.

Killeen and Taylor Stochastic adding machine

Let $N \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

$$N = \sum_{i=0}^{k(N)} \varepsilon_i(N) 2^i = \varepsilon_{k(N)} \cdots \varepsilon_{0(N)}$$

with $\varepsilon_i(N) = 0, 1$.

Example: $6 = 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 110_2$.

Let $N = \varepsilon_k \ldots \varepsilon_1 \varepsilon_0 \in \mathbb{N}$. The digits of $N+1$ in base 2 are given by the following algorithm

If $\varepsilon_0 = 0$, then:

$$\begin{array}{r|cccc} N = & \varepsilon_k & \ldots & \varepsilon_1 & 0 \\ & + & & & 1 \\ \hline N+1 = & \varepsilon_k & \ldots & \varepsilon_1 & 1 \end{array}$$

Stochastic adding machine in base 2

If $\varepsilon_{i-1}\ldots\varepsilon_0 = 1\ldots 1$ and $\varepsilon_i = 0$, then:

$$
\begin{array}{r|cccccccc}
N = & \varepsilon_k & \ldots & \varepsilon_{i+1} & {}^{1}0 & {}^{1}1 & \ldots & {}^{1}1 & 1 \\
 & + & & & & & & & 1 \\
\hline
N+1 = & \varepsilon_k & \ldots & \varepsilon_{i+1} & 1 & 0 & \ldots & 0 & 0
\end{array}
$$

0 transitions to 0 with probability $1-p$ (if we don't add the carry): $S(0,0)=1-p$.

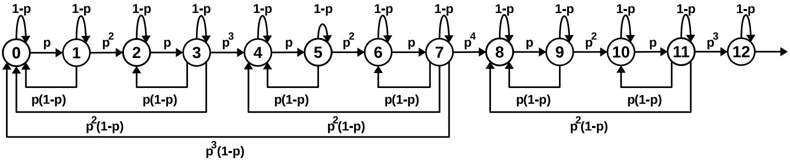
0 transitions to 1 with probability p (if we add the carry): $S(0,1)=p$.

1 transitions to 1 with probability $1-p$ (if we don´t add the carry): $S(1,1)=1-p$.

1 transitions to 0 with probability $p(1-p)$ (if we add the first carry and we don´t add the second one): $S(1,0)=p(1-p)$.

1 transitions to 2 with probability p^2 (if we add the first carry and also the second one): $S(1,2)=p^2$.

Transition graph of stochastic adding machine in base 2:



Transition operator S of the stochastic adding machine in base 2: where s_{ij} is the probability of transition from state i to state j.
Matrix S:

$$\begin{pmatrix}
1-p & p & 0 & 0 & 0 & 0 & 0 & 0\dots \\
p(1-p) & 1-p & p^2 & 0 & 0 & 0 & 0 & 0\dots \\
0 & 0 & 1-p & p & 0 & 0 & 0 & 0\dots \\
p^2(1-p) & 0 & p(1-p) & 1-p & p^3 & 0 & 0 & 0\dots \\
0 & 0 & 0 & 0 & 1-p & p & 0 & 0\dots \\
0 & 0 & 0 & 0 & p(1-p) & 1-p & p^2 & 0\dots \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & p\dots \\
p^3(1-p) & 0 & 0 & 0 & p^2(1-p) & 0 & p(1-p) & 1-p\dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

Remarks:

1. S is bi-stochastic, $S(X) \subset X$, where $X = l^\infty, l^p, p > 1$ or c_0;
2. stochastic adding machine has application on psychology.

Stochastic adding machine in base 2

Theorem:(Killeen, Taylor)(2000) Acting in $X = l^{\infty}(\mathbb{N})$,

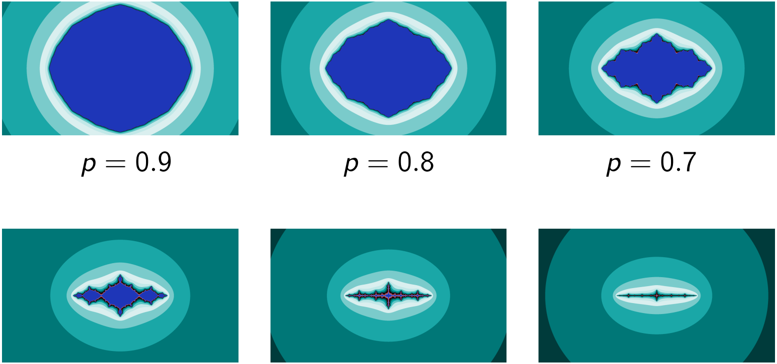
$$\sigma(S) = \sigma_{pt}(S) = \{z \in \mathbb{C} : (f^n(z))_{n\geq 0} \text{ is bounded}\},$$

where $f(z) = z^2 + c,\ c = \frac{p-1}{p^2},\ f^n = f \circ f \ldots \circ f.$

$$\sigma(S) = \{z \in \mathbb{C} : S - zI : X \to X \text{ is not bijective}\},$$

$$\sigma_{pt}(S) = \{z \in \mathbb{C} : S - zI \text{ is not injective}\}.$$

Stochastic adding machine in base 2



$p = 0.9$ $p = 0.8$ $p = 0.7$

$p = 0.6$ $p = 0.55$ $p = 0.51$

Questions

- What is the spectrum in other classical Banach spaces?
- What happens if we consider other bases instead of base 2?
- Is there exists a connection between topological properties of the spectrum, probabilistic properties of the Markov chain and also with dynamical properties of the transition matrix?
  Observe that if $p \geq 1/2$, the spectrum is connected
  If $0 < p < 1/2$, the spectrum is a Cantor set

Fibonacci base

Constant base $d \geq 2$

$$\sigma(S) = \sigma_{pt}(S) = \{z \in \mathbb{C} : (f^n(z))_{n \geq 0} \text{ is bounded}\},$$

$$f(z) = z^d + c, \; c = \frac{p-1}{p^{d/d-1}}.$$

Fibonacci base

$$F_0 = 1, \; F_1 = 2, \; F_n = F_{n-1} + F_{n-2}, \; \forall n \geq 2.$$

$$N = \sum_{i=0}^{k(N)} \varepsilon_i(N) F_i = \varepsilon_{k(N)} \dots \varepsilon_{0(N)}$$

with $\varepsilon_i(N) = 0, 1, \; \varepsilon_i \varepsilon_{i+1} \neq 11$ (greedy algorithm).

Fibonacci base

Exemple: $F_n : 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots$

$12 = 8 + 3 + 1 = F_0 + F_2 + F_4 = 10101$

Questions 1: How to define the adding machine?

Answer: by using automaton (Transducer): Classical

Questions 2: How to define the stochastic adding machine?

Answer: In collaboration with D. Smania, we defined a probabilistic transductor

Automaton: A finite automaton on A (finite alphabet): oriented graph set $Au = (Q, A, E, I, T)$ where

Q: finite set of states

$I \subset Q$: set of initial states

$T \subset Q$: set of terminal states.

$E \subset Q \times A \times Q$: set of edges.

The edges are labelled by the elements of A and

$(p, a, q) \in E$ is also denoted by $p \to q$.

$c = (p_0, a_0, p_1)(p_1, a_1, p_2) \dots (p_{n-1}, a_n, p_n)$ sequence of consecutive edges: finite path

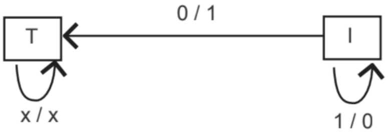
Trasductor

Transductor : finite automaton T on $A^\star \times B^\star$ labelled by elements of $A^\star \times B^\star$

$A^\star$ (resp. $B^\star$) : set of finite words on A (resp. B).

An edge of T from a state p to a state q labelled by $(u, v) \in A^\star \times B^\star$ will be denoted by $p \to q$. We say that T reads the word of input on $A^\star$ and transforms it to a word on $B^\star$.

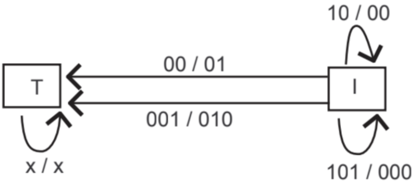
1. Transductor of base 2:



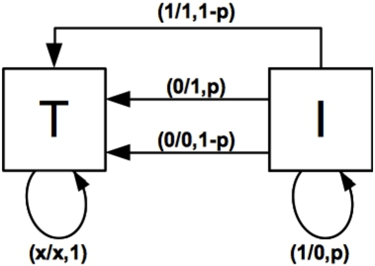
Example: $111 + 1 = 1000$

$$(0/1)\ (1/0)\ (1/0)\ (1/0) \rightarrow \ \text{Output:}\ 1000$$

2. Transductor of Fibonacci base:



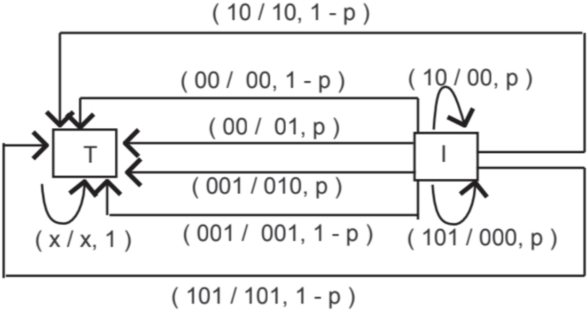
1. Probabilistic Transductor of base 2:



Example: $111 + 1 = 1000$ with probability p^4

$(0/0, 1)(0/1, p)\ (1/0, p)\ (1/0, p)\ (1/0, p)$
$\rightarrow 1000$ with probability p^4.

2. Probabilistic Transductor of Fibonacci base:



Fibonacci base:

Theorem:(Messaoudi, Smania)(2010) $\sigma_{pt}(S) \subset K \subset \sigma(S)$

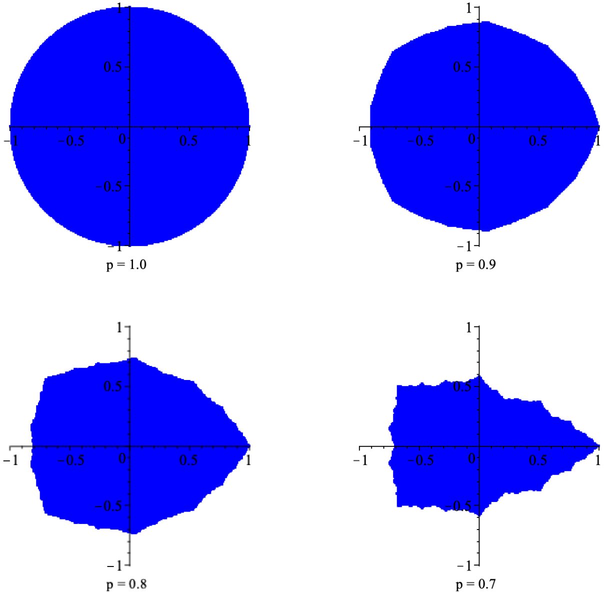
$$K = \{z \in \mathbb{C} : (f^n(z,z))_{n\geq 0} \text{ is bounded}\},$$

$$f : \mathbb{C}^2 \to \mathbb{C}^2,\ f(x,y) = (xy + c, x),\ \ c = \frac{p-1}{p^2}.$$

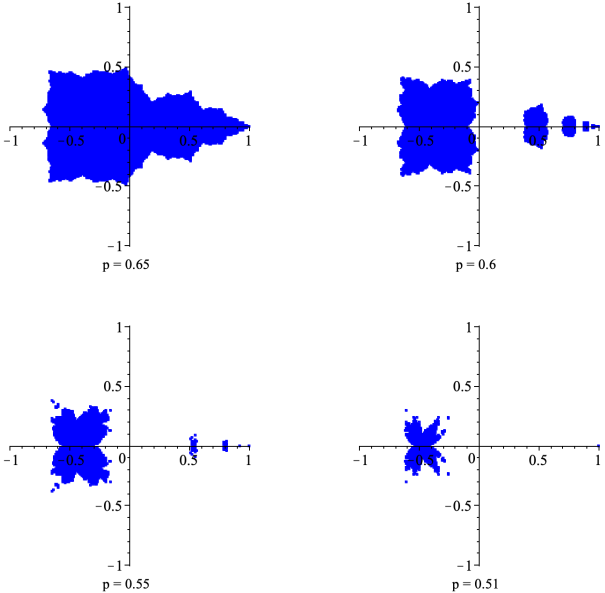
Remark: $K = \{z \in \mathbb{C} : (g_n(z))_{n\geq 0} \text{ is bounded}\}$, where $g_0(z) = z,\ g_1(z) = z,\ g_n(z) = g_{n-1}(z)g_{n-2}(z) + c,\ n \geq 2.$

Conjecture: $K = \sigma(S)$.

Images of K



Images of K



Idea of the proof

$\lambda \in \sigma_{pt}(S)$ implies $\exists v = (v_n)_{n\geq 0} \in l^\infty,\ Sv = \lambda v.$

We can prove that $v_n = q_n v_0, \forall n \geq 1,\ q_n = q_n(p, \lambda)$.

$$q_{F_n} = \frac{1}{p} q_{F_{n-1}} q_{F_{n-2}} - \frac{1-p}{p},\ n \geq 2.$$

and if $n = F_{n_1} + \cdots + F_{n_1}$

$$q_n = \prod_{i=1}^{k} q_{F_{n_i}}.$$

$\lambda \in \sigma_{pt}(S) \iff q_n$ bounded $\implies (q_{F_n}, q_{F_{n-1}})$ bounded .

Since $(q_{F_n}, q_{F_{n-1}}) = g^n(q_{F_1}, q_{F_0}),\ g(x, y) = (\frac{1}{p}xy - \frac{1-p}{p}, x)$,
we deduce that $\sigma_{pt}(S) \subset K$.

Questions

- Is $\sigma_{pt}(S) = K$?
  This is equivalent to prove that

$$(q_{F_n})_{n\geq 0} \text{ bounded implies } (q_n)_{n\geq 0} \text{ bounded }.$$

Conjecture: $K = \sigma(S)$.

Questions

- How about topological and dynamical properties of K?

$$K = \{z \in \mathbb{C} : (z, z) \in F(f)\},$$

  $F(f)$: Filled Julia set (in $\mathbb{C}^2$) of
  $f : \mathbb{C}^2 \to \mathbb{C}^2,\ f(x, y) = (xy + c, x),\ c = \frac{p-1}{p^2}$.
- How about topological and dynamical properties of F(f)?
  $F(f) = \{(x, y) \in \mathbb{C}^2 : f^n(x, y) \text{ bounded } \}$.

Properties of $F(f)$

$$F(f) = \{(x,y) \in \mathbb{C}^2 : (f^n(x,y))_{n\geq 0} \text{ is bounded}\},$$

$$f : \mathbb{C}^2 \to \mathbb{C}^2,\ f(x,y) = (xy + c, x),\ c = \frac{p-1}{p^2}.$$

Work of Julia sets in $\mathbb{C}^n$, $n \geq 2$ or in the projective spaces began very active in 80 with works of Hubbard, Fornaess-Sybony, Bedford, Smilie, Milnor among others.

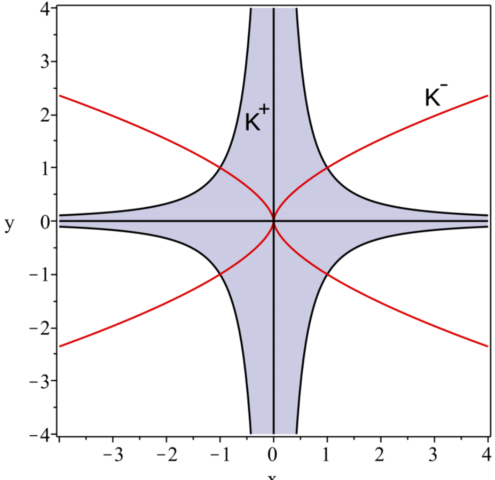
Several results for Hénon diffeomorphisms (and its generalizations).

Our map is not bijective. It appeared in the literature in a work of Guedj. Few properties of $F(f)$ are known.

If $c = 0$,

$$F(f) = \{(x, y) \in \mathbb{C}^2,\ |y| \leq |x|^{-\beta}\},$$

β: Golden number.



Proposition:(Bonnot, De Carvalho, Messaoudi):

- $F(f)$ is a closed non compact subset of $\mathbb{C}^2$.
- If $|c| < 1/4$, $\lambda(F(f)) = \infty$.

We also show the existence of a trapping bidisk for $F(f)$: in other words, a complex bidisk such that every point of $F(f)$ eventually enters the bidisk under forward iteration.

Conjecture: For sufficiently large $|c|, \lambda(F(f)) = 0$.

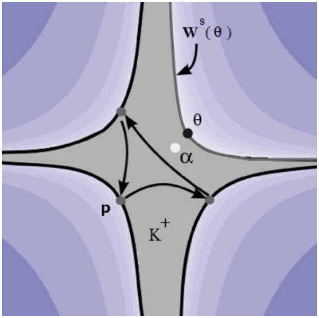
Theorem:(Bonnot, De Carvalho, Messaoudi): If $0 < c < 1/4$, then $F(f) \cap \mathbb{R}^2$ is a finite union of stable manifolds.

$$F(f) \cap \mathbb{R}^2 = W^s(\alpha) \cup W^s(\theta) \cup W^s(p) \cup W^s(f(p)) \cup W^s(f^2(p))$$

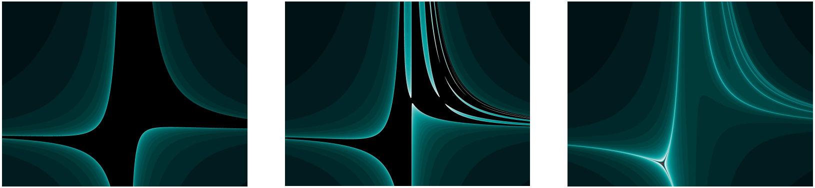
$\alpha =$ attractive fixed point, $\theta =$ saddle fixed point, $p = 3$ periodic point

Remark: The last result is true in the case $-1 < c < 0$ (see the work of Danilo Caprio).

Image for $c = 0.22$:



Properties of $F(f)$



a) c = - 0,8　　b) c = - 1　　c) c = - 2,1

Properties of K

$K = \{z \in \mathbb{C} : (z, z) \in F(f)\}$ and
$K = \{z \in \mathbb{C} : (g_n(z))_{n \geq 0} \text{ is bounded}\}$, where
$g_0(z) = z,\ g_1(z) = z^2,\ g_n(z) = g_{n-1}(z)g_{n-2}(z) + c,\ n \geq 2.$

Theorem:(A, B, M, S):

- K is a non empty compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected.
- $\exists 0 < a < b$ such that if $|c| > b$, then K is disconnected. If $|c| < a$, then K is a quasi disk.

Questions: How about other topological Properties?

Can we have a finite number of connected components?

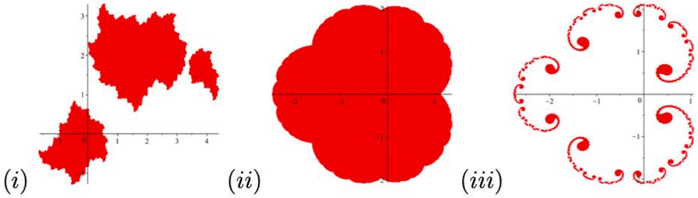
Big difficulty: K is not invariant by some map as classical Julia sets in $\mathbb{C}$.

Slices of K where

(i) $c = -0.72i$ and $y = 0.17 + 0.2i$,

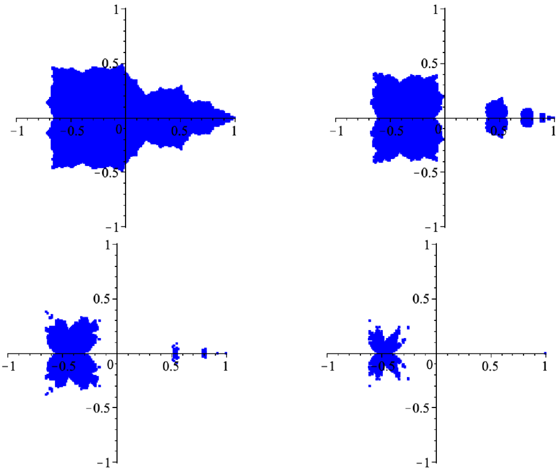
(ii) $c = 0.2$ and $y = 0.33$,

(iii) $c = 0.33$ and $y = 0.33$.



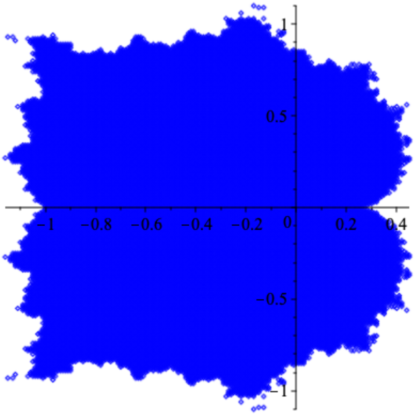
Idea of the proof: We used. Technique of classical Julia sets, Riemann Hurwitz Formula.

K is a quasi disk: Hubbard technique in $\mathbb{C}^2$.



Properties of K

$\{c \in \mathbb{C} : (f^n(0,0))_{n\geq 0} \text{ is bounded}\}$, $f(x,y) = (xy + c, x)$:



Remark: We can define a stochastic adding machine for a large class of recurrent sequences as done for Fibonacci base

For instance

If $F_n = aF_{n-1} + bF_{n-2},\ a \geq b \geq 1.$ The spectrum is connected to the map

$$f : \mathbb{C}^2 \to \mathbb{C}^2,\ f(x,y) = (x^a y^b + c, x),\ c = \tfrac{p-1}{p^2}.$$

If $F_n = a_1 F_{n-1} + a_2 F_{n-2} + \ldots + a_d F_{n-d},\ a_1 \geq a_2 \geq \ldots \geq a_d \geq 1,$

then the spectrum is connected to the map $f : \mathbb{C}^d \longrightarrow \mathbb{C}^d,$

$$f(x_1, \ldots, x_d) = (x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d} + c, x_1, x_2, \ldots, x_{d-1}).$$

Probabilistic properties

Case of base $d = 2$, the Markov chain is irreducible, aperiodic and null recurrent for all $0 < p < 1$.

Case $d = 2$, where the probability change at each stage.

Let $(p_i)_{i \geq 1}, \ 0 < p_i < 1$.

0 transitions to 0 with probability $1 - p_1$ if we don´t add the carry $S(0,0) = 1 - p_1$.

0 transitions to 1 with probability p_1 if we add the carry $S(0,1) = p_1$.

1 transitions to 1 with probability $1 - p_1$ if we don´t add the carry $S(1,1) = 1 - p_1$.

1 transitions to 0 with probability $p_1(1 - p_2)$ if we add the first carry and we don´t add the second one $S(1,0) = p_1(1 - p_2)$.

1 transitions to 2 with probability $p_1 p_2$ if we add the first carry and also the second one $S(1,2) = p_1 p_2$.

Theorem:(M, V, S)(2013):

- The Markov chain is null recurrent if $\prod_{i=1}^{\infty} p_i = 0$, otherwise it is transient.
- Acting in $X = l^{\infty}(\mathbb{N})$,

$$\sigma(S) = \sigma_{pt}(S) = \{z \in \mathbb{C} : (f_n \circ f_{n-1} \ldots \circ f_1(z))_{n\geq 0} \text{ is bounded}\},$$

where $f_i(z) = z^2 + c_i,\ c = \frac{p_i - 1}{p_i^2}$.

Remark: We obtain many results about the topology of $\sigma(S)$ which can be connected, have finite or infinite number of connected components or to be a Cantor.

$\exists 0 < a < 1$ such that if $p_i \geq a$ for all i, then $\sigma(S)$ is a quasi disk.

Questions

- Is there a difference of the spectrum between the null recurrent and transient cases?
- How about dynamical system associated to the transition matrix S?

Questions

Is there a difference of the spectrum between the null recurrent and transient cases?

Answer: If S is transient, $\sigma(S)$ can be connected or have a finite number of connected components.

The point spectrum of S acting in l^1 is empty.

If S is null recurrent, $\sigma(S)$ can also have infinite connected components. The point spectrum of S acting in l^1 is not empty.

How about dynamical system associated to the transition matrix S?

Questions

How about dynamical system associated to the transition matrix S?

Answer: If M is transient, there exists $\lambda_0 > 1$ such that for all $\lambda > \lambda_0$, the dynamical system $(\lambda S, l^1)$ is topologically transitive and Devaney chaotic.

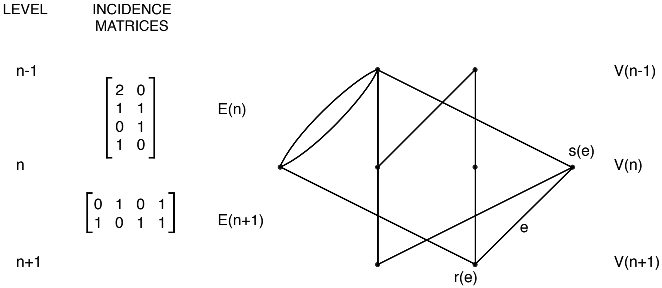
This is false if M is null recurrent.

Bratteli diagrams are important objects in the theories of operator algebras and dynamical systems.

It was originally defined in 1972 by O. Bratteli for classification of C^*-algebras and was connected with symbolic dynamical systems.

A *Bratteli diagram* is an infinite directed graph (V, E) where the vertex set V and the edge set E can be partitioned into finite sets, i.e $V = \cup_{k=0}^{\infty} V(k)$ and $E = \cup_{k=1}^{\infty} E(k)$, where $\# V(k) < \infty$ and $\# E(k) < \infty$ for every $k \geq 0$, such that there exist the source map $s : E \longrightarrow V$ and the range map $r : E \longrightarrow V$ such that s restricted to $E(k)$ is an onto map from $E(k)$ to $V(k-1)$ and r restricted to $E(k)$ is an onto map from $E(k)$ to $V(k)$ for every $k \geq 1$.

Bratteli diagrams



If $\#V(k-1) = l_{k-1}$ and $\#V(k) = l_k$, then $E(k)$ determines a $l_k \times l_{k-1}$ *incidence matrix* $M(k)$, where $M(k)_{i,j}$ is the number of edges going from vertex j in $V(k-1)$ to vertex i in $V(k)$.

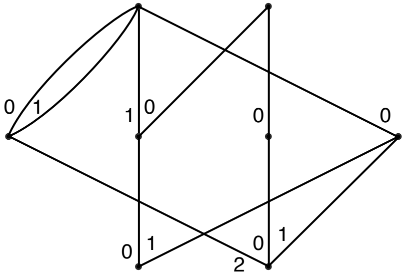
Definition

We say that (V, E) is a *simple Bratteli diagram* if for each nonnegative integer k, there exists an integer $n > k$ such that the product $M(n) \cdot M(n-1) \cdot \ldots \cdot M(k)$ have only non-zero entries.

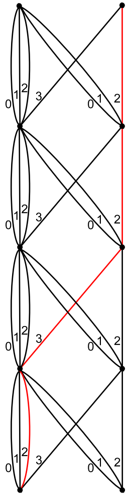
Definition

An ordered Bratteli diagram $(V, E, \geq)$ is a Bratteli diagram (V, E) together with a partial order $\geq$ on E such that edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$, in other words, we have a linear order on the set $r^{-1}(\{v\})$ for each $v \in V \setminus V(0)$.

Bratteli diagrams

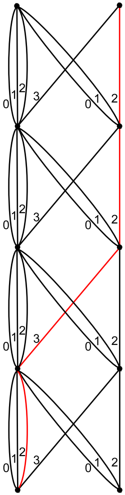


$$M=\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

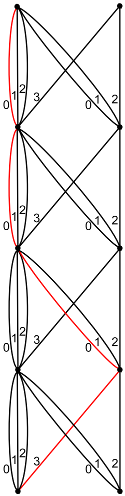


$X=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$

$$M = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

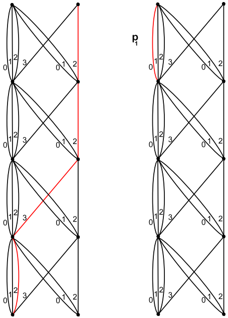


$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$



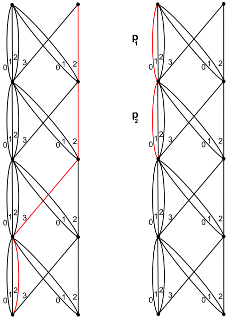
$V(x)=(0,0,0,3,\varepsilon_5,\varepsilon_6,\ldots)$

Stochastic Vershik map



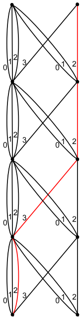
$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$

Stochastic Vershik map

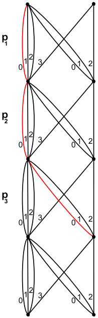


x=(2,2,3,2,ε_5,ε_6,...)

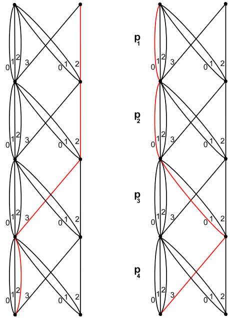
Stochastic Vershik map



$x = (2, 2, 3, 2, \varepsilon_5, \varepsilon_6, \ldots)$

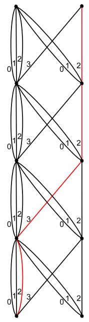


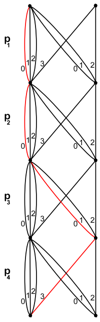
Stochastic Vershik map



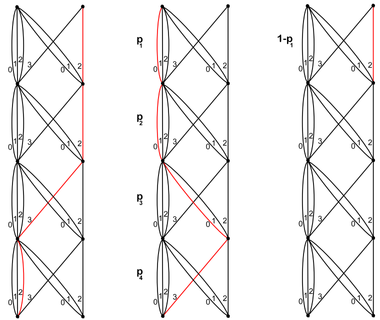
x=(2,2,3,2,ε_5,ε_6,...)

Stochastic Vershik map





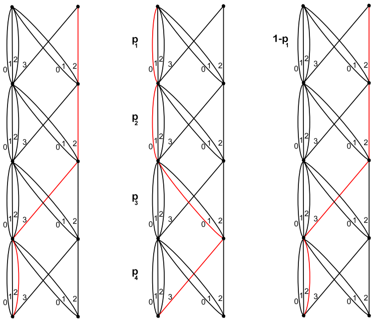
Stochastic Vershik map



x=(2,2,3,2,ε_5,ε_6,...)

x —> V(x)=(0,0,0,3,ε_5,ε_6,...)
with probability $p_1p_2p_3p_4$

Stochastic Vershik map

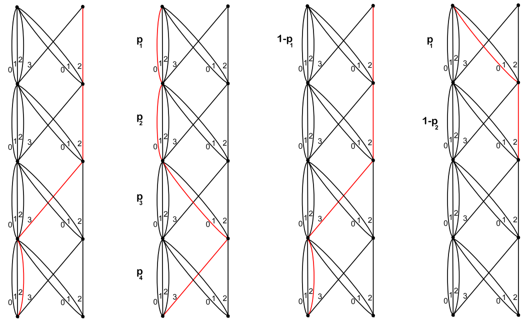


$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$

$x \rightarrow V(x)=(0,0,0,3,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1p_2p_3p_4$

$x \rightarrow x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $1-p_1$

Stochastic Vershik map

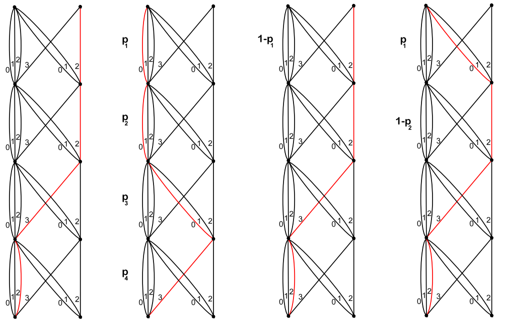


$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$

$x \longrightarrow V(x)=(0,0,0,3,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1p_2p_3p_4$

$x \longrightarrow x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $1-p_1$

Stochastic Vershik map



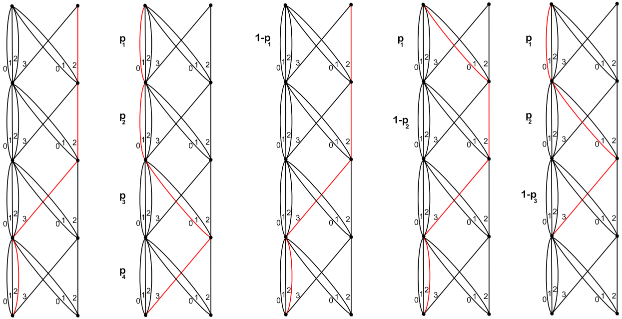
$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\dots)$

$x \longrightarrow V(x)=(0,0,0,3,\varepsilon_5,\varepsilon_6,\dots)$ with probability $p_1p_2p_3p_4$

$x \longrightarrow x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\dots)$ with probability $1-p_1$

$x \longrightarrow (0,2,3,2,\varepsilon_5,\varepsilon_6,\dots)$ with probability $p_1(1-p_2)$

Stochastic Vershik map



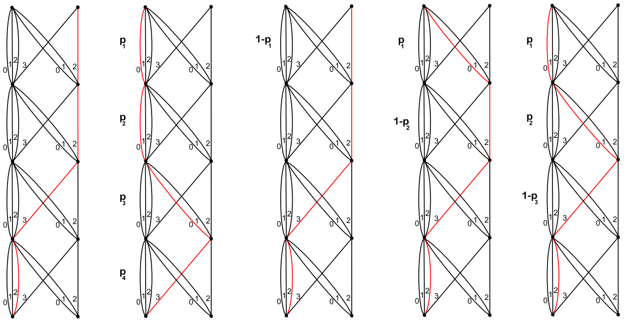
$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$

$x \longrightarrow V(x)=(0,0,0,3,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1p_2p_3p_4$

$x \longrightarrow x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $1-p_1$

$x \longrightarrow (0,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1(1-p_2)$

Stochastic Vershik map



$x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$

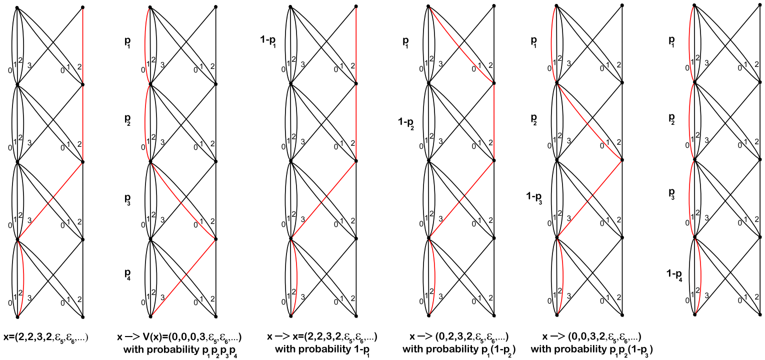
$x \longrightarrow V(x)=(0,0,0,3,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1p_2p_3p_4$

$x \longrightarrow x=(2,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $1-p_1$

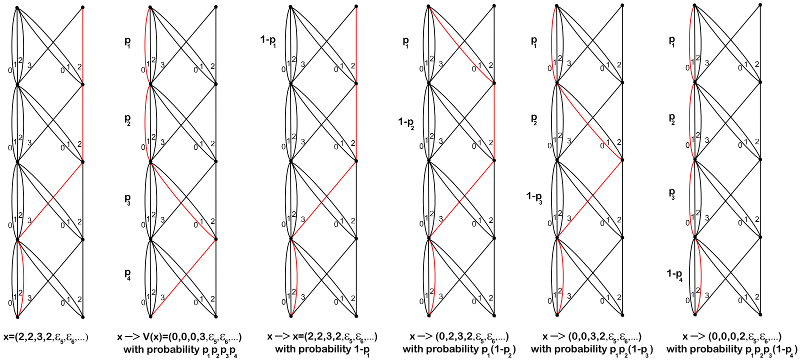
$x \longrightarrow (0,2,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1(1-p_2)$

$x \longrightarrow (0,0,3,2,\varepsilon_5,\varepsilon_6,\ldots)$ with probability $p_1p_2(1-p_3)$

Stochastic Vershik map



Stochastic Vershik map

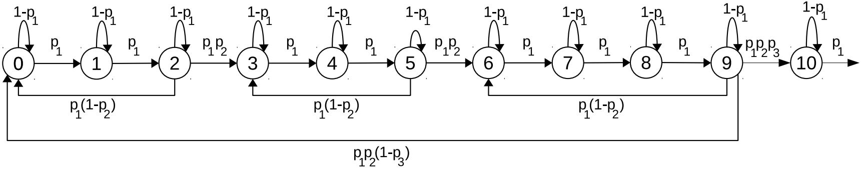


Thus, by definition of stochastic Vershik map, we have a Markov chain which the states set is $\mathbb{Z}_+$ and the transition operator $S = (S_{m,n})$, where

$$S_{m,n} \text{ is the probability that } V_B^m(x_0) \text{ goes to } V_B^n(x_0).$$

Example: $M = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$

Initial parts of the transition graph and operator:



$$\begin{pmatrix}
1-p_1 & p_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1-p_1 & p_1 & 0 & 0 & 0 & 0 & \cdots \\
p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1-p_1 & p_1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1-p_1 & p_1 & 0 & \cdots \\
0 & 0 & 0 & p_1(1-p_2) & 0 & 1-p_1 & p_1p_2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Probabilistic properties of stochastic Bratteli diagrams

Theorem

(Caprio, Messaoudi, Valle) Let $(p_i)_{i\geq 1}$ be a sequence of non-null probabilities such that $\#\{i : p_i < 1\} = \infty$. Thus,
a) every stochastic Bratteli diagram associated to $(p_i)_{i\geq 1}$ is an irreducible Markov chain;
b) the Markov chain is transient if and only if $\prod_{j=1}^{\infty} p_j > 0$.
c) if $\prod_{j=1}^{\infty} p_j = 0$, then the stochastic Bratteli diagram associated to $(p_i)_{i\geq 1}$ is null recurrent.

Theorem

(C, M, V) Let $B = (V, E, \geq)$ be a 2×2 stationary simple ordered Bratteli diagram with $a = c = 1$, $b > 0$ and $d = 0$. Then the BV stochastic adding machine associated to $(p_j)_{j\geq 1}$ is positive recurrent if p_j decreases to zero sufficiently fast as $j \to \infty$.

Spectral properties of stochastic Bratteli diagrams

Theorem

(C, M, V) Let S be the transition operator associated to the Bratteli diagram B and the stochastic Vershik map. Then, acting in $l^\infty(\mathbb{N})$, we have that the set of eigenvalues of S is

$$\sigma_{pt}(S) \subset$$
$$\mathcal{E} := \{z \in \mathbb{C} : (g_n \circ \ldots \circ g_0(z,z))_{n \geq 0} \text{ is bounded}\} \subset \sigma(S),$$

where $g_n : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are polynomials defined by

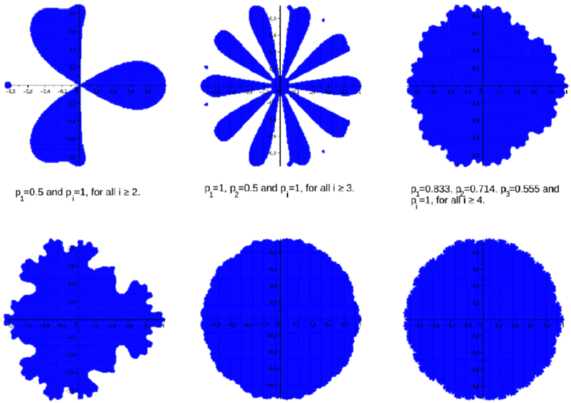
$$g_n(x,y) = \left(\frac{1}{p_{n+1}}x^a y^b - \frac{1-p_{n+1}}{p_{n+1}}, \frac{1}{p_{n+1}}x^c y^d - \frac{1-p_{n+1}}{p_{n+1}}\right), \text{ for all } n \geq 1.$$

Corollary: If $a+b=c+d$, then

$$\sigma_{pt}(S)=\{\lambda\in\mathbb{C}:(f_n\circ\ldots\circ f_0(\lambda))_{n\geq 0}\text{ is bounded}\},$$

where $f_0(x)=\frac{x-(1-p_1)}{p_1}$ and $f_n(x)=\frac{1}{p_{n+1}}x^{a+b}-\left(\frac{1}{p_{n+1}}-1\right)$, for all $n\geq 1$.

Example: $M = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$



p_1=0.5 and p_i=1, for all i ≥ 2.

p_1=1, p_2=0.5 and p_i=1, for all i ≥ 3.

p_1=0.833, p_2=0.714, p_3=0.555 and p_i=1, for all i ≥ 4.

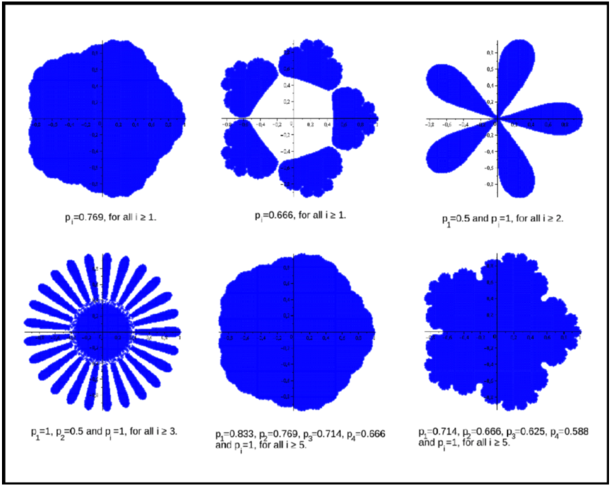
p_1=0.714, p_2=0.625, p_3=0.555 and p_i=1, for all i ≥ 4.

p_1=1, p_2=0.833, p_3=0.714, p_4=0.625 and p_i=1, for all i ≥ 5.

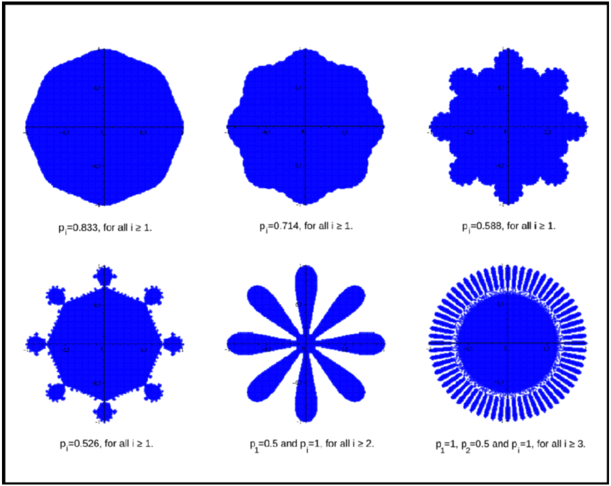
p_1=1, p_2=0.833, p_3=0.714, p_4=0.625, p_5=0.555 and p_i=1, for all i ≥ 6.

$\mathcal{E} = \{\lambda \in \mathbb{C} : (g_n \circ \ldots \circ g_0(\lambda, \lambda))_{n \geq 0} \text{ is bounded}\}$

Example: $M = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$



Exemplo: $M = \begin{pmatrix} 7 & 1 \\ 1 & 7 \end{pmatrix}$



Theorem

Suppose that $p_i = p$, for all $i \geq 1$. If $\det M = ad - bc < 0$ and $bc > \det M^2$, then the set $\mathcal{E}$ satisfies the following properties:

1. $\mathbb{C} \setminus \mathcal{E}$ *is a connected set.*
2. *If $p < \frac{1}{2}$, then $\mathcal{E}$ is not connected.*

OBRIGADO!