Measure theoretic entropy of random substitutions

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13th April 2021

Based on joint work with P. Gohlke, D. Rust and T. Samuel.

Notation: symbolic dynamics

- $\mathcal{A} = \{a_1, \dots, a_N\}$ – *finite alphabet*
- $\mathcal{A}^+ = \{w_1 \dots w_n \colon n \in \mathbb{N}, w_i \in \mathcal{A}\}$
- $\mathcal{A}^\mathbb{Z} = \{\cdots x_{-2}x_{-1} \cdot x_0x_1x_2 \cdots : x_i \in \mathcal{A}\}$ – (two-sided) *full shift* on $\mathcal{A}$
- $S \colon \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$, $x_i \mapsto x_{i+1}$ – *left shift* map
- $X \subseteq \mathcal{A}^\mathbb{Z}$ closed and S-invariant – *subshift*
- We study the dynamical system (X, S)

Deterministic substitutions

Example (Period doubling substitution)

$$\theta\colon \begin{cases} a \to ab \\ b \to aa \end{cases}$$

$$a \mapsto ab \mapsto abaa \mapsto abaaabab \mapsto \cdots$$
$$b \mapsto aa \mapsto abab \mapsto abaaabaa \mapsto \cdots$$

We say that a word $u \in \mathcal{A}^+$ is (θ-)legal if there exists $a_i \in \mathcal{A}$ and $m \in \mathbb{N}$ such that u is a subword of $\theta^m(a_i)$.

$$X_\theta = \left\{ x \in \mathcal{A}^{\mathbb{Z}} \colon \text{every subword of } x \text{ is } \theta\text{-legal} \right\}$$

Properties of X_ϑ

Cantor set or finite, minimal, uniquely ergodic, zero topological entropy.

These properties hold whenever θ is *primitive* (definition to come).

Random substitutions

"Letters have choices for how they are substituted"

Example (Random period doubling substitution)

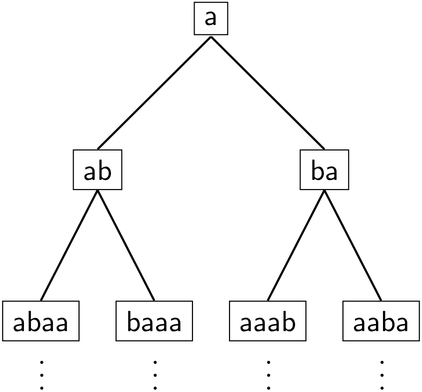
$$\vartheta\colon \begin{cases} a \to \begin{cases} ab \text{ with probability } p \\ ba \text{ with probability } 1-p \end{cases} \\ b \to aa \text{ with probability } 1 \end{cases}$$

Choices are independent for each letter.

$$a \mapsto ab \mapsto baaa \mapsto aaabbaba \mapsto \cdots$$
$$a \mapsto ba \mapsto aaab \mapsto babaabaa \mapsto \cdots$$

We write $\vartheta(a) = \{ab, ba\}$ to denote the set of all *realisations* of the letter a under ϑ.

Random substitutions



Random substitution subshifts

We can associate a subshift to a random substitution in a similar manner to deterministic substitutions.

A word $u \in \mathcal{A}^+$ is called (ϑ-)legal if there exist $a_i \in \mathcal{A}$ and $m \in \mathbb{N}$, and a realisation v of $\vartheta^m(a_i)$, such that u is a subword of v.

We write $\mathcal{L}_\vartheta = \{u \in \mathcal{A}^+ \colon u \text{ is } \vartheta\text{-legal}\}$.

$$X_\vartheta = \left\{ x \in \mathcal{A}^{\mathbb{Z}} \colon \text{ every subword of } x \text{ is } \vartheta\text{-legal} \right\}$$

The subshift X_ϑ is independent of the choice of probabilities.

The matrix of a random substitution

Define a matrix M_ϑ by

$$M_{i,j} = \textit{expected} \text{ number of occurrences of letter } a_i \text{ in } \vartheta(a_j)$$

The random period doubling substitution

$$\vartheta\colon \begin{cases} a \to \begin{cases} ab \text{ with probability } p \\ ba \text{ with probability } 1-p \end{cases} \\ b \to aa \text{ with probability } 1 \end{cases}$$

has matrix

$$M_\vartheta = \begin{pmatrix} p + (1-p) & 2 \\ p + (1-p) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

Notice that the matrix M_ϑ is independent of the probability p. This is because ϑ is *compatible*.

Compatible random substitutions

Definition

We say that a random substitution ϑ is *compatible* iff for all $a_i \in \mathcal{A}$ and $u, v \in \vartheta(a_i)$, we have $|u|_{a_j} = |v|_{a_j}$ for all $a_j \in \mathcal{A}$.

- For every $u \in \mathcal{L}_\vartheta$, every realisation of $\vartheta(u)$ has the same length. We write $|\vartheta(u)|$ to denote this common value.
- Guarantees entries of M_ϑ are independent of the probabilities.
- Letter frequencies exist uniformly. If ϑ is *primitive*, these are encoded in the right Perron–Frobenius eigenvector of M_ϑ.
- Word frequencies don't necessarily exist.

Primitive random substitutions

Definition

We say that ϑ is *primitive* iff M_ϑ is a primitive matrix, i.e. there is a $k \in \mathbb{N}$ such that M_ϑ^k is positive.

We let λ denote the Perron–Frobenius eigenvalue of M_ϑ and R denote the corresponding right eigenvector, normalised such that $\|\mathrm{R}\|_1 = 1$.

- λ is the expansion factor
- R encodes letter frequencies

For random period doubling, $\lambda = 2$ and $\mathrm{R} = (\frac{2}{3}, \frac{1}{3})^\top$.

Properties of primitive random substitutions

Theorem (Rust, Spindeler, 18')

Properties of X_ϑ for primitive ϑ:

- *Cantor set or finite*
- *Topologically transitive*
- *Either no periodic points or periodic points are dense*
- *Uncountably many minimal components*
- *Uncountably many invariant probability measures*
- *Canonical measure μ_{P} induced by production probabilities (shown to be ergodic by [Gohlke, Spindeler, 20'])*

Under mild assumptions:

- *Positive topological entropy*

Topological entropy

It was shown by [Gohlke, 20'] that topological entropy coincides with the notion of *inflation word entropy*.
Given a primitive and compatible random substitution ϑ over the alphabet $\mathcal{A} = \{a_1, \ldots, a_N\}$ and $n \in \mathbb{N}$, let q_n be the vector defined by

$$q_{n,i} = \log(\#\vartheta^n(a_i)).$$

Theorem (Gohlke, 20')

$$h_{top} = \lim_{n \to \infty} \frac{1}{\lambda^n} \mathrm{q}_n \cdot \mathrm{R}$$

If ϑ satisfies the *identical set condition* or *disjoint set condition* (we'll define later), then a closed form expression for the topological entropy can be obtained via inflation word entropy.

Measures arising from random substitutions

For a primitive and compatible random substitution ϑ and $u \in \mathcal{L}_\vartheta$, we define the *expected frequency* of u by

$$\mathrm{freq}(u) = \lim_{k\to\infty} \frac{\mathbb{E}[|\vartheta^k(a_i)|_u]}{|\vartheta^k(a_i)|},$$

where this limit is independent of the choice of $a_i \in \mathcal{A}$.

Define a measure μ_ϑ on X_ϑ by $\mu_\vartheta(\varnothing) = 0$, $\mu_\vartheta(X_\vartheta) = 1$ and $\mu_\vartheta([u]) = \mathrm{freq}(u)$ for all $u \in \mathcal{L}_\vartheta$. We call μ_ϑ the *frequency measure* corresponding to the random substitution ϑ.

Theorem (Gohlke, Spindeler, 20')

The measure theoretic dynamical system $(X_\vartheta, \mathcal{B}(X_\vartheta), \mu_\vartheta, S)$ is ergodic.

Sometimes we will write μ_{P} to emphasise the dependence on the production probabilities P.

Measure theoretic entropy

Given a primitive random substitution ϑ, the entropy of a shift invariant measure μ is given by

$$h(\mu) = \lim_{n \to \infty} -\frac{1}{n} \sum_{u \in \mathcal{L}_{\vartheta}^{n}} \mu([u]) \log \mu([u]).$$

We introduce a measure theoretic variant of inflation word entropy for primitive and compatible random substitutions.

Measure theoretic inflation word entropy

Given a random substitution ϑ and $n \in \mathbb{N}$, let

$$s_n(\vartheta) = -\sum_{a \in \mathcal{A}} R_a \sum_{t \in \vartheta^n(a)} \mathbb{P}[\vartheta^n(a) = t] \log\left(\mathbb{P}[\vartheta^n(a) = t]\right)$$

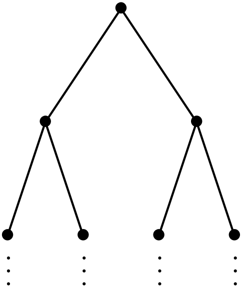
Theorem (Gohlke, M., Rust, Samuel, *in preparation*)

Let ϑ be a primitive and compatible random substitution. Then

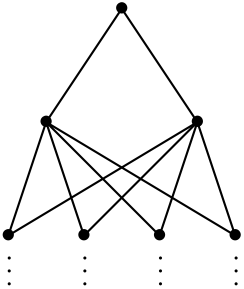
$$\frac{1}{\lambda} s_1(\vartheta) \leq h(\mu_\vartheta) \leq \frac{1}{\lambda - 1} s_1(\vartheta).$$

- *The upper bound is attained iff for all $a \in \mathcal{A}$, $k \in \mathbb{N}$ and $u \neq v \in \vartheta(a)$, we have $\vartheta^k(u) \cap \vartheta^k(v) = \varnothing$ (Disjoint Set Condition)*
- *The lower bound is attained iff for all $a \in \mathcal{A}$, $k \in \mathbb{N}$ and $u, v \in \vartheta(a)$, we have $\vartheta^k(u) = \vartheta^k(v)$ (Identical Set Condition) and ϑ has uniform production probabilities*

Disjoint/identical set condition



Disjoint set condition



Identical set condition

Examples

The random period doubling substitution

$$\vartheta\colon \begin{cases} a \to \begin{cases} ab \text{ with probability } p \\ ba \text{ with probability } 1-p \end{cases} \\ b \to aa \text{ with probability } 1 \end{cases}$$

satisfies the disjoint set condition. Since $\lambda = 2$ and $\mathrm{R} = (\frac{2}{3}, \frac{1}{3})^\top$, we have

$$h(\mu_\vartheta) = \frac{1}{\lambda - 1} s_1(\vartheta) = -\frac{2}{3} \left(p \log p + (1-p) \log(1-p) \right).$$

Examples

The random substitution defined by

$$\vartheta\colon \begin{cases} a \to \begin{cases} ab \text{ with probability } p \\ ba \text{ with probability } 1-p \end{cases} \\ b \to \begin{cases} ab \text{ with probability } p \\ ba \text{ with probability } 1-p \end{cases} \end{cases}$$

satisfies the identical set condition and has uniform production probabilities. We have $\lambda = 2$ and $\mathrm{R} = (\frac{1}{2}, \frac{1}{2})^{\top}$, so

$$h(\mu_{\vartheta}) = \frac{1}{\lambda} s_1(\vartheta) = -\frac{1}{2}\left(p \log p + (1-p)\log(1-p)\right).$$

What if ϑ doesn't satisfy ISC or DSC?

By considering higher powers of ϑ, we can obtain improved bounds.

Theorem (Gohlke, M., Rust, Samuel, *in preparation*)

Let ϑ be a primitive and compatible random substitution. Then, for all $n \in \mathbb{N}$, the entropy of μ_ϑ satisfies

$$\frac{1}{\lambda^n} s_n(\vartheta) \leq h(\mu_\vartheta) \leq \frac{1}{\lambda^n - 1} s_n(\vartheta),$$

where the sequence of lower bounds is nondecreasing in n. In particular,

$$\lim_{n \to \infty} \frac{1}{\lambda^n} s_n(\vartheta)$$

exists, and

$$h(\mu_\vartheta) = \lim_{n \to \infty} \frac{1}{\lambda^n} s_n(\vartheta) = \sup_{n \in \mathbb{N}} \frac{1}{\lambda^n} s_n(\vartheta).$$

Continuity of entropy

Corollary

$h(\mu_{\mathrm{P}})$ is continuous with respect to the production probabilities P.

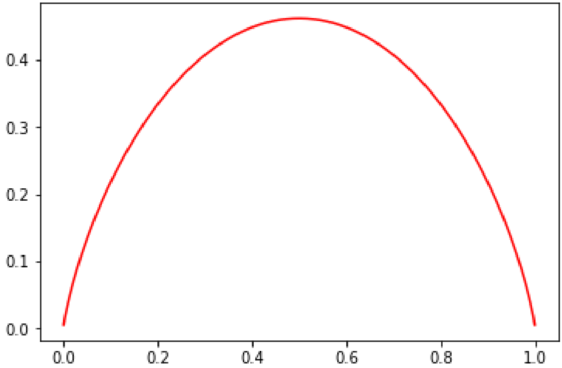


Figure: Measure theoretic entropy for random period doubling

The key lemma

We want to relate measures of cylinder sets via the action of the substitution.

For random period doubling, observe that the word $u = babab$ appears as a subword of two images of aaa under ϑ:

$$a\underbrace{b\,ab\,ab}_{u} \quad \underbrace{ba\,ba\,b}_{u}a$$

The image of no other three letter word contains $babab$ as a subword. Hence, for all $k \geq 2$ and $a_i \in \mathcal{A}$ we have

$$\mathbb{E}\left[|\vartheta^k(a_i)|_{babab}\right] = \mathbb{E}\left[|\vartheta^{k-1}(a_i)|_{aaa}\right]\,(p^3 + (1-p)^3).$$

The key lemma

It follows that

$$\mu([babab]) = \lim_{k\to\infty} \frac{\mathbb{E}[|\vartheta^k(a_i)|_{babab}]}{|\vartheta^k(a_i)|} = \frac{1}{2}(p^3 + (1-p)^3)\,\mu([aaa]).$$

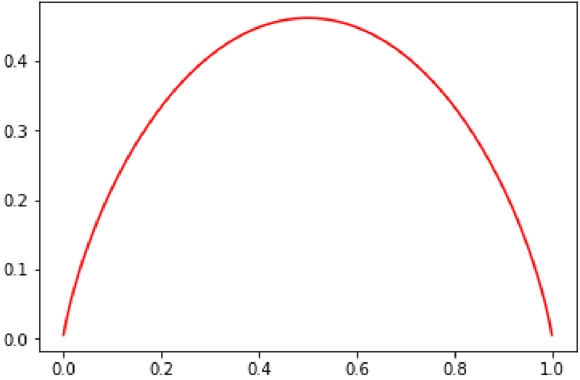
Key Lemma

Let ϑ be a primitive and compatible random substitution. Let $n \in \mathbb{N}$, and let $\ell \in \mathbb{N}$ be an integer such that every $v \in \mathcal{L}_\vartheta^\ell$ satisfies $|\vartheta(v)| \geq n + |\vartheta(v_1)|$. Then for every $u \in \mathcal{L}_\vartheta^n$ we have

$$\mu_\vartheta([u]) = \frac{1}{\lambda} \sum_{v \in \mathcal{L}_\vartheta^\ell} \mu_\vartheta([v]) \sum_{j=1}^{|\vartheta(v_1)|} \mathbb{P}\left[\vartheta(v)_{[j,j+n-1]} = u\right].$$

Recall that for random period doubling

$$h(\mu_p) = -\frac{2}{3}(p \log p + (1-p)\log(1-p))$$



Frequency measures of maximal entropy

$h(\mu_p)$ is maximised when $p = \frac{1}{2}$:

$$h(\mu_{\frac{1}{2}}) = \frac{2}{3} \log 2.$$

It was shown by [Baake, Spindeler, Strungaru, 17'] that

$$h_{\text{top}} = \frac{2}{3} \log 2,$$

so $\mu_{\frac{1}{2}}$ is a measure of maximal entropy.

Theorem (Gohlke, M., Rust, Samuel, *in preparation*)

Let ϑ be a primitive and compatible random substitution satisfying either the disjoint set condition or the identical set condition. If $\mathbb{P}[\vartheta(a) = s] = 1/\#\vartheta(a)$ for all $a \in \mathcal{A}$ and $s \in \vartheta(a)$, then μ_ϑ is a measure of maximal entropy.

Random Fibonacci with uniform probabilities

$$\vartheta\colon \begin{cases} a \mapsto \begin{cases} ab \text{ with probability } \frac{1}{2} \\ ba \text{ with probability } \frac{1}{2} \end{cases} \\ b \mapsto a \text{ with probability } 1 \end{cases}$$

Random Fibonacci does not satisfy (ISC) or (DSC).

A computer-assisted calculation gives

$$0.3908 < h(\mu_\vartheta) < 0.4140.$$

It is well-known that

$$h_{\text{top}} = \sum_{m=2}^{\infty} \frac{\log(m)}{\tau^{m+2}} \approx 0.444399,$$

so μ_ϑ is NOT a measure of maximal entropy.

Random Fibonacci with uniform probabilities

We can obtain frequency measures of greater entropy by considering higher powers of the substitution. Consider the square of random Fibonacci, with uniform probabilities:

$$\varphi\colon \begin{cases} a \mapsto \begin{cases} aab \text{ with probability } \frac{1}{3} \\ aba \text{ with probability } \frac{1}{3} \\ baa \text{ with probability } \frac{1}{3} \end{cases} \\ b \mapsto \begin{cases} ab \text{ with probability } \frac{1}{2} \\ ba \text{ with probability } \frac{1}{2} \end{cases} \end{cases}$$

We have

$$0.4177 < h(\mu_\varphi) < 0.4424,$$

so

$$h(\mu_\vartheta) < h(\mu_\varphi) < h_{\mathrm{top}}$$

Measures of maximal entropy

Theorem (Gohlke, M., Rust, Samuel, *in preparation*)

Let X be a subshift of a primitive and compatible random substitution. Then there exists a sequence of frequency measures $(\mu_n)_n$ such that μ_n converges weakly to a measure of maximal entropy μ for the subshift X.

Intrinsic ergodicity

For a class of random substitutions satisfying the disjoint set condition, the frequency measure of maximal entropy is the unique measure of maximal entropy.

Theorem (Gohlke, M., Rust, Samuel, *in preparation*)

Let ϑ be a primitive, compatible and recognisable random substitution of constant length L. Further, assume that at least one of the following holds:

- *$\vartheta(a)$ has the same cardinality for all $a \in \mathcal{A}$;*
- *the second eigenvalue of M_ϑ is strictly less than one in modulus.*

Then, the subshift X_ϑ is intrinsically ergodic; that is, it has a unique measure of maximal entropy.

A random substitution with intrinsically ergodic subshift

Let $0 < p, q < 1$, and let ϑ be the random substitution defined by

$$\vartheta\colon \begin{cases} a \mapsto \begin{cases} ababbb \text{ with probability } p \\ abbabb \text{ with probability } 1-p \end{cases} \\ b \mapsto \begin{cases} babaaa \text{ with probability } q \\ baabaa \text{ with probability } 1-q. \end{cases} \end{cases}$$

Then

$$h(\mu_{\frac{1}{2},\frac{1}{2}}) = h_{\mathrm{top}} = \frac{1}{5}\log 2,$$

and $\mu_{\frac{1}{2},\frac{1}{2}}$ is the unique measure of maximal entropy for the subshift X_ϑ.

Intrinsic ergodicity theorem: sketch of proof

Lemma

If ϑ satisfies the conditions of the theorem, then there is a constant $c > 0$ such that

$$\mu([w]) \geq \mu([v]) \frac{c^{|v|}}{|w| e^{|w| h}}$$

for every $v \in \mathcal{L}_\vartheta$, $m \in \mathbb{N}$ and $w \in \vartheta^m(v)$.

Using the constant length property and recognisability, we can split X_ϑ into disjoint subspaces

$$X_{m,k} = S^k(\vartheta^m(X_\vartheta)),$$

each of which is S^{L^m}-invariant. We consider the conditional measures

$$\mu_{m,k} = \frac{1}{\mu(X_{m,k})} \mu|_{X_{m,k}} = L^m \mu|_{X_{m,k}}.$$

We then apply similar arguments to Parry's proof for SFTs, applied to the dynamical system $(X_{m,k'}, \mu_{m,k'}, S^{L^m})$ for a suitable choice of k'.

The end

Thank you!