Multiplicative automatic sequences

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Disjointedness of additive and multiplicative structures

Theorem (Solymosi - 2009)

For any finite set $A \subset \mathbb{R}$,

$$\max |A \cdot A|, |A + A| \gg |A|^{4/3 - o(1)}.$$

Conjecture (Chowla)

Let $\lambda(n) = (-1)^k$, where k is the number of prime factors of n. Then for all $a_1 < a_2 < \ldots < a_m$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \lambda(n + a_1) \cdot \lambda(n + a_2) \cdots \lambda(n + a_m) = 0.$$

Sarnak Conjecture

The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is squarefree and} \\ & k \text{ is the number of prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

A dynamical system is said to be determinist, if its topological entropy is 0.

Conjecture (Sarnak - 2010)

For every complex sequence $\mathrm{u} = (u_n)_{n>0}$ that is obtained by a deterministic dynamical system,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} u_n \mu(n) = 0.$$

Multiplicative functions

Definition (Multiplicative function)

A function $f : \mathbb{N} \to \mathbb{C}$ is called *(completely) multiplicative* if $f(nm) = f(n)f(m)$ for all n, m that are coprime (for all n, m).

Examples: μ, λ

Definition (Dirichlet character)

We call $\chi : \mathbb{Z} \to \mathbb{C}$ a Dirichlet character (of modulus m) if

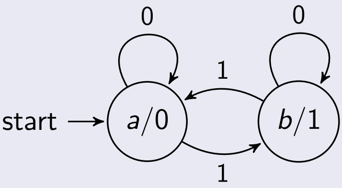
1. There exists $m > 0$ such that $\chi(n) = \chi(n + m)$ for all n.
2. If $\gcd(n, m) > 1$ then $\chi(n) = 0$; if $\gcd(n, m) = 1$ then $\chi(n) \neq 0$.
3. χ is completely multiplicative.

Deterministic Finite Automata

Definition (Automaton - DFA)

$$A = (Q, \Sigma = \{0, \ldots, k-1\}, \delta, q_0, \tau)$$

Example (Thue-Morse sequence)



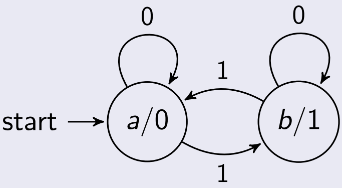
$$n = 22 = (10110)_2, \qquad u(22) = 1$$

$$\mathrm{u} = (u(n))_{n\geq 0} = 0110100110010110100101100110100 1\ldots$$

Different Points of View I

$$(u(n))_{n\geq 0} = 01101001100101101001011001101001\ldots$$

Automaton (Computer Science)



Substitution (Dynamics)

Coding of the fixpoint of a constant-length substitution:

$$a \to ab \qquad a \mapsto 0$$
$$b \to ba \qquad b \mapsto 1$$

Different Points of View II

$(u(n))_{n\geq 0} = 0110100110010110100101100110 1001\ldots$

Formal Power Series (Algebra)

Algebraicity over $\mathrm{F}_q(X)$.

$$t(X) := \sum_{n\geq 0} u(n)X^n$$

$$X + (1+X)^2 t(X) + (1+X)^3 t(X)^2 = 0$$

Finite Kernel

The λ-*kernel* of a sequence $a(n)$ is defined as

$$\{(a(n\lambda^k + r))_{n\geq 0} : k \geq 0, 0 \leq r < \lambda^k\}.$$

$a(n)$ is λ-automatic iff its λ-kernel is finite.

Being automatic in different bases

Question

Can a sequence be automatic in multiple bases?

Lemma

Let $\lambda, k \in \mathbb{N}$. A sequence is λ-automatic if and only if it is λ^k-automatic.

Proof works by considering the kernel.

Theorem (Cobham - 1972)

If a sequence $(a(n))_{n\geq 0}$ is both μ and λ automatic, where $\log(\mu)/\log(\lambda) \notin \mathbb{Q}$. Then $(a(n))_{n\geq 0}$ is eventually periodic.

Simple examples and Properties

Lemma

Let $(a(n))_{n\geq 0}$ be eventually periodic. Then it is λ-automatic for every $\lambda \in \mathbb{N}$.

Proof: Follows from considering the λ-kernel.

Lemma

Let $a_1(n), a_2(n)$ be, λ-automatic sequences, then so is $(a_1(n) \cdot a_2(n))$.

Proof: We look at the corresponding λ-kernels:

$$\begin{aligned}\{(a_1(n\lambda^k + r) \cdot a_2(n\lambda^k + r) : k \in \mathbb{N}, 0 \leq r < \lambda^k\}\\ \subset \{(a_1(n\lambda^k + r) : k \in \mathbb{N}, 0 \leq r < \lambda^k\}\\ \cdot \{(a_2(n\lambda^k + r) : k \in \mathbb{N}, 0 \leq r < \lambda^k\}.\end{aligned}$$

Disjointedness of automatic and multiplicative sequences

Theorem (M. - 2017)

Any automatic sequence is orthogonal to the Möbius function. (Also true for the associated dynamical system.) If the automatic sequence is primitive, then we also have a prime number theorem.

Theorem (Lemańczyk, M. - 2020)

Let a be a primitive automatic sequence. Then it is orthogonal to any bounded, aperiodic, multiplicative function $u : \mathbb{N} \to \mathbb{C}$, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} a(n)u(n) = 0.$$

Naive Question

Why not all bounded multiplicative functions?

Trivial counter-example: periodic sequences (e.g. Dirichlet characters).
Non-trivial counter-example: $a(n) = (-1)^{\nu_2(n)}$.

Definition (aperiodic sequence)

We call a sequence u aperiodic if for all $k, \ell \in \mathbb{N}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} u(kn + \ell) = 0.$$

Disjointedness of multiplicative sequences and algebraic generating series

Theorem (Bell, Bruin and Coons - 2012)

Let K be a field of characteristic 0, let $f : \mathbb{N} \to K$ be a multiplicative function, and its generating series $F(z) = \sum_{n \geq 1} f(n) z^n$ be algebraic over $K(z)$.
Then either f is finitely supported or there is a natural number k and a periodic multiplicative function $\chi : \mathbb{N} \to K$ such that $f(n) = n^k \chi(n)$ for all n.

BBC-Conjecture

Conjecture (Bell, Bruin and Coons - 2012)

For any multiplicative automatic sequence $a : \mathbb{N} \to \mathbb{C}$ there exists an eventually periodic function $f : \mathbb{N} \to \mathbb{C}$ such that $f(p) = a(p)$ for all primes p.

Theorem/Corollary (Klurman, Kurlberg; Konieczny - 2019)

The conjecture is true. Moreover, there exists h, λ such that a is λ-automatic and coincides with χ on integers that are coprime to $h\lambda$, where χ is either zero or a Dirichlet character.

- χ is a Dirichlet character: *dense case*
- $\chi = 0$: *sparse case*.

Result

Theorem (Konieczny, Lemańczyk, M. - 2020+)

A sequence $a : \mathbb{N} \to \mathbb{C}$ is multiplicative and automatic if and only if there exists a prime p such that a is p-automatic and of the form

$$a(n) = f_1(\nu_p(n)) \cdot f_2(n/p^{\nu_p(n)}), \tag{1}$$

where f_1 is eventually periodic and f_2 is multiplicative, eventually periodic and vanishes at all multiples of p.

Previous Results

- Schlage-Puchta (2003): A criterion for multiplicative sequences to not be automatic.
- Coons (2010): Non-automaticity of special multiplicative functions
- Li (2017): completely multiplicative automatic sequences, nonvanishing prime numbers
- Allouche, Goldmakher (2018): completely multiplicative, never vanishing automatic sequences
- Li (2019): characterizing completely multiplicative automatic sequences
- Klurman, Kurlberg; Konieczny (2019): showed a stronger version of BBC-conjecture

Simple example

Lemma

Let $(a(n))_{n\geq 0}$ be multiplicative and p-automatic. Then

$$a(n) = a(p^{\nu_p(n)}) \cdot a(n/p^{\nu_p(n)}),$$

where $\alpha \mapsto a(p^\alpha)$ is eventually periodic.

Proof: The first part follows by multiplicativity.
As the p-kernel is finite, there exists $k_1, k_2 \in \mathbb{N}$ such that $a(np^{k_1}) = a(np^{k_2})$ for all $n \in \mathbb{N}$.
Choose $n = p^\alpha$.

Corollary

Theorem 1 is true for eventually periodic multiplicative sequences (for every p).

f_1

Lemma

Let f_1 be eventually periodic with $f_1(0) = 1$. Then $a_1(n) = f_1(\nu_p(n))$ is p-automatic and multiplicative.

Proof: We consider again the p-kernel,

$$\begin{aligned}\{(f_1(\nu_p(np^k + r)))_{n\geq 0} : k \in \mathbb{N}, 0 \leq r < p^k\} \\ = \{f_1(\nu_p(n) + k)_{n\geq 0} : k \in \mathbb{N}\} \cup \{f_1(\nu_p(r))_{n\geq 0} : r \in \mathbb{N}\}\end{aligned}$$

Multiplicativity: If $(m, n) = 1$ then either $p \nmid m$ or $p \nmid n$. Thus, we have $\nu_p(mn) = \max(\nu_p(m), \nu_p(n))$ and $f_1(mn) = f_1(m)f_1(n)$.

f_2

Lemma

Let f_2 be multiplicative and eventually periodic. Then $a_2(n) = f_2(n/p^{\nu_p(n)})$ is p-automatic and multiplicative.

Proof: We consider once again the p-kernel:

$$\begin{aligned}
&\{(a_2(np^k + r))_{n\geq 0} : k \in \mathbb{N}, 0 \leq r < p^k\} \\
&\quad = \{(a_2(np^k))_{n\geq 0} : k \in \mathbb{N}\} \\
&\qquad \cup \{(a_2(np^k + r))_{n\geq 0} : k \in \mathbb{N}, 0 < r < p^k\} \\
&\quad = \{(a_2(n))_{n\geq 0}\} \cup \{(f_2(np^\ell + s))_{n\geq 0} : \ell \in \mathbb{N}, 0 < s < p^\ell\}.
\end{aligned}$$

Let $(m, n) = 1$. Then also $(m/p^{\nu_p(m)}, n/p^{\nu_p(n)}) = 1$. Thus, a_2 is also multiplicative.

Decomposing Dirichlet characters

Lemma

Let χ be a Dirichlet character of modulus $m = m_1 m_2$ where $(m_1, m_2) = 1$.Then $\chi = \chi_{m_1} \cdot \chi_{m_2}$, where $\chi_{m_i}(n)$ is a Dirichlet character of modulus m_i and $\chi_{m_i}(n) = \chi(n_i)$ with

$$n_i \equiv n \bmod m_i$$
$$n_i \equiv 1 \bmod m/m_i.$$

Corollary

Let χ be a Dirichlet character of modulus m. Then

$$\chi(n) = \prod_{p|m} \chi_{p^{\nu_p(m)}}(n).$$

Dense case

Assumption: $\nu_p(h\lambda) = 1$ for all $p \mid h\lambda$!
Thus, $\chi = \prod_{p \mid h\lambda} \chi_p$.

Proposition

Let $a(n)$ be a dense multiplicative automatic sequence. Then

$$a(n) = \prod_{p \mid h\lambda} \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}} \cdot \chi_p\left(\frac{n}{p^{\nu_p(n)}}\right),$$

where $\chi(\overline{p}) = \chi_{h\lambda/p}(p)$.

Dynamical systems

Dynamical System (X, T) related to u

$\mathrm{u} = (u_n)_{n\geq 0} \ldots$ bounded complex sequence

$T(\mathrm{u}) = (u_{n+1})_{n\geq 0} \ldots$ shift operator

$X = \overline{\{T^k(\mathrm{u}) : k \geq 0\}}$

Theorem (M., Yassawi; 2019)

Let a be a primitive λ-automatic sequence, which is not periodic. Then the continuous eigenvalues of (X, T) are isomorphic to $\mathbb{Z}(\lambda) \times \mathbb{Z}/h\mathbb{Z}$, where h is the height of a.

Intuition I

$$a(n) = \prod_{p \mid h\lambda} \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}} \cdot \chi_p\left(\frac{n}{p^{\nu_p(n)}}\right).$$

- The right hand side looks like a product of p-automatic sequences, where $p \mid h\lambda$.
- Thus we expect the continuous eigenvalues to be $\approx \mathbb{Z}(h\lambda)$.
- The continuous eigenvalues of $a(n)$ are only $\approx \mathbb{Z}(\lambda)$.
- Therefore, the contribution of $p \mid h$ should be trivial.

Intuition II

$$a(n) = \prod_{p \mid \lambda} \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}} \cdot \chi_p \left(\frac{n}{p^{\nu_p(n)}} \right).$$

If λ is composite, we can separate one contribution:

$$a(n) \cdot \frac{\chi(\overline{q})^{\nu_q(n)}}{a(q^{\nu_q(n)})} \cdot \chi_q^{-1} \left(\frac{n}{q^{\nu_q(n)}} \right) = \prod_{\substack{p \mid \lambda \\ p \neq q}} \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}} \cdot \chi_p \left(\frac{n}{p^{\nu_p(n)}} \right).$$

- Continuous eigenvalues of the left-hand side: $\approx \mathbb{Z}(\lambda)$.
- Continuous eigenvalues of the right-hand side: $\approx \mathbb{Z}(\lambda/q)$.

$$a(n) = \frac{a(p^{\nu_p(n)})}{\chi(\overline{p})^{\nu_p(n)}} \cdot \chi_p \left(\frac{n}{p^{\nu_p(n)}} \right).$$

Conclusion

- Capturing the independence of additive and multiplicative structures is hard.
- The intersection of automatic and multiplicative sequences is very special.
- Dynamics often gives you a good intuition.

Thank you!