# Overlap Coincidences for general S-adic tilings

A joint work with Jörg Thuswaldner

December 2023, Yasushi Nagai (Shinshu University)

# Contents

# Analogies

| Symbolic | Geometric |
|---|---|
| Sequences (Words) | Tilings |
| The product topology on $\mathscr{A}^{\mathbb{N}}$ | Tiling metric |
| Shift $\sigma$ | Translation by $x \in \mathbb{R}^d$ |
| Subshift $X = \overline{\{\sigma^n(w) \mid n \in \mathbb{N}\}}$ | Continuous hull $X_{\mathscr{T}} = \overline{\{\mathscr{T} + x \mid x \in \mathbb{R}^d\}}$ |

# The definition of tilings

a tile: $S \subset \mathbb{R}^d$, compact, non-empty, $S^{\circ -} = S$

sometimes with a label $(S, l)$

a patch=a collection $\mathcal{P}$ of tiles such that

$$S, T \in \mathcal{P}, S \neq T \Rightarrow S^\circ \cap T^\circ = \emptyset$$

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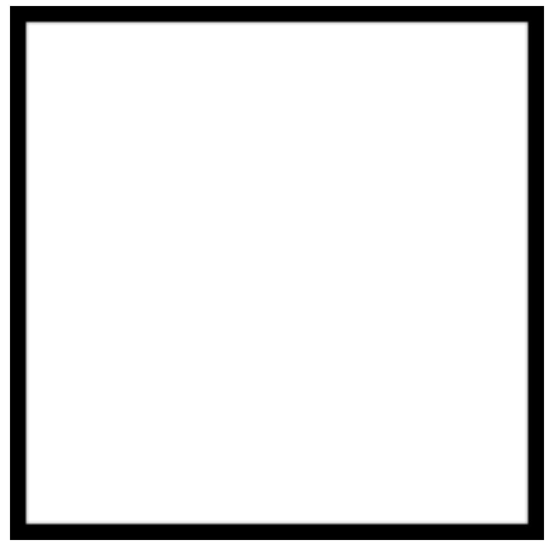
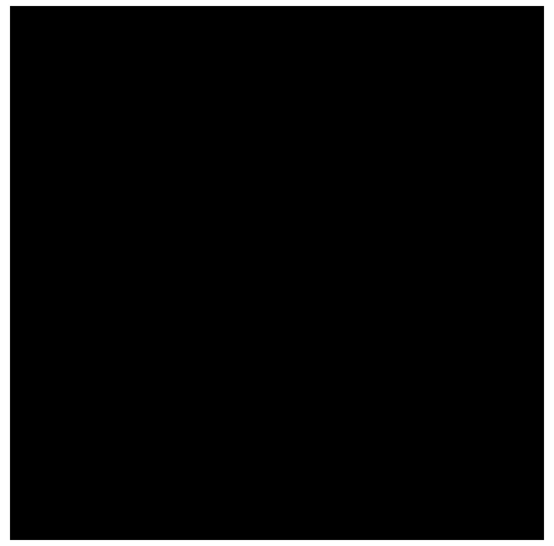
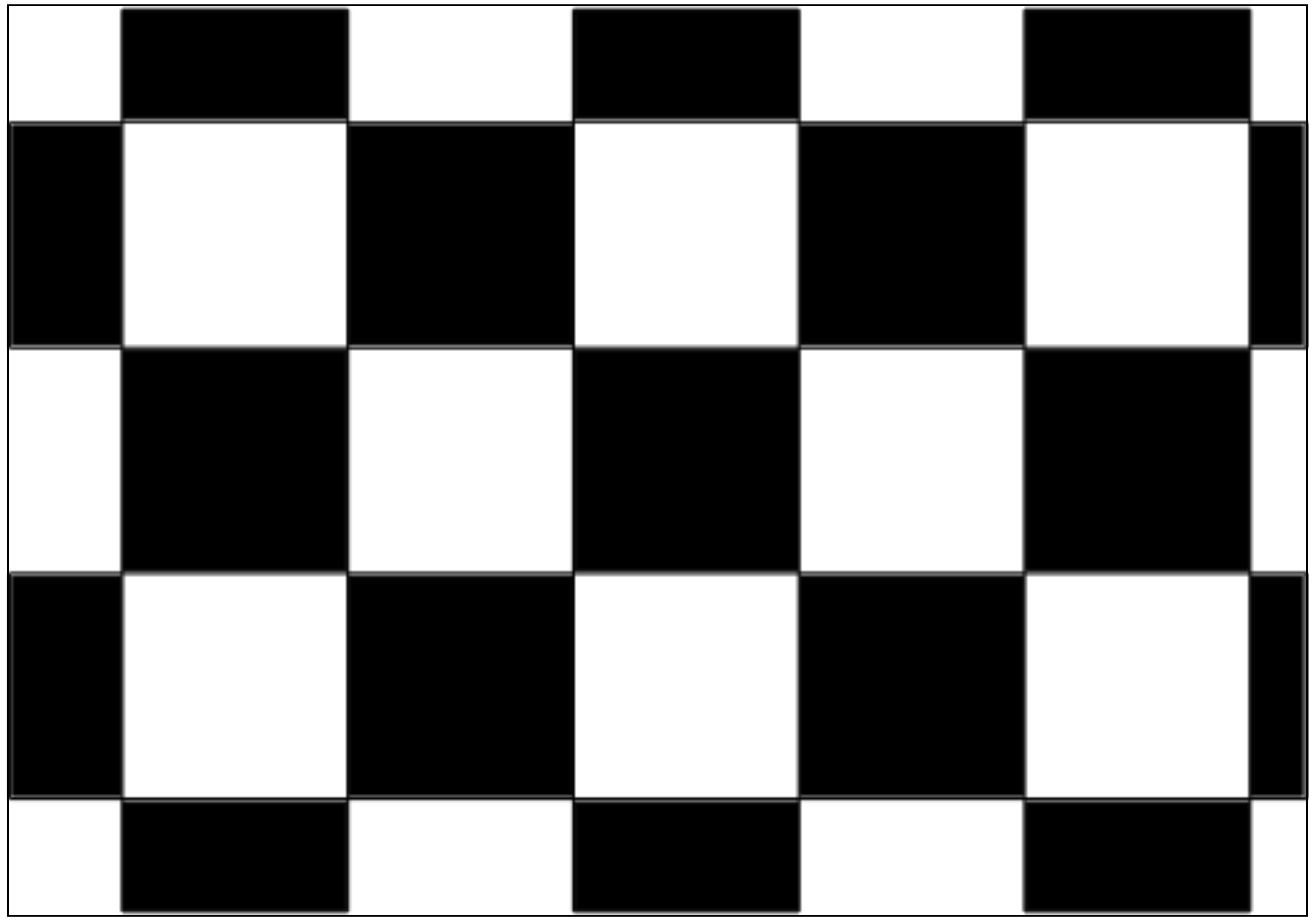
Example

$T_W = ([0,1]^2, W) =$ 

$T_B = ([0,1]^2, B) =$ 

# The definition of tilings

a tiling=a patch $\mathcal{T}$ such that the union of supports of the tiles in $\mathcal{T}$ is $\mathbb{R}^d$



$$\mathcal{T} = \{T_W + (n,m) \mid n+m \text{ odd}\} \cup \{T_B + (n,m) \mid n+m \text{ even}\}$$

crystallographic, i.e. $\mathcal{T} + (n,m) = \mathcal{T}$ for $n+m$ even

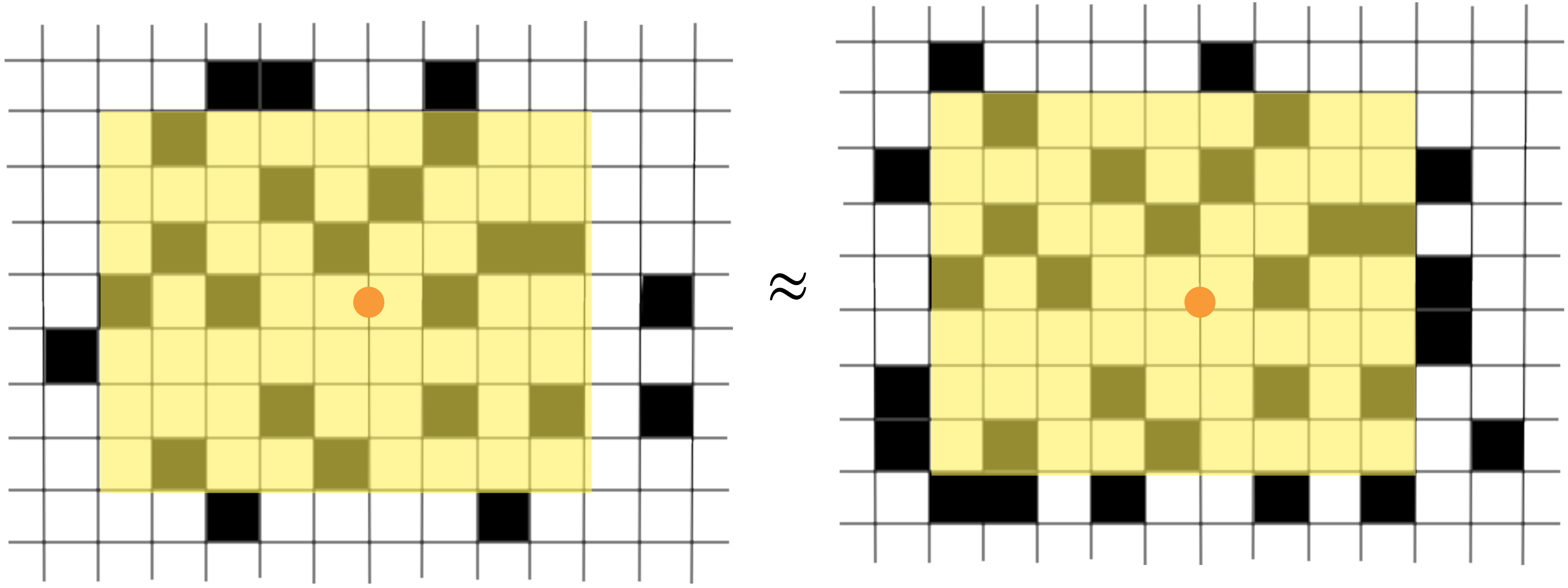
# A construction of non-periodic tilings

interest: non-periodic but "ordered" tiling

$\mathcal{T} + x = \mathcal{T}$ only for $x = 0$

construction: via a substitution rule

# The tiling metric



# Analogies

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# Tiling dynamical systems

Continuous hull

$$X_{\mathcal{T}} = \overline{\{\mathcal{T} + x \mid x \in \mathbb{R}^d\}}$$

$\mathbb{R}^d$ acts on $X_{\mathcal{T}}$ via translation:

$$X_{\mathcal{T}} \times \mathbb{R}^d \ni (\mathcal{S}, x) \mapsto \mathcal{S} + x \in X_{\mathcal{T}}$$

Often there is one and only one invariant Borel probability measure $\mu$

# Tiling dynamical systems

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Often there is one and only one invariant Borel probability measure $\mu$

We say $\mathcal{T}$ has pure discrete dynamical spectrum if there exists a complete orthonormal basis for $L^2(\mu)$ consisting of eigenfunctions for Koopman operators $U_x : f \mapsto f(\cdot - x)$

# Main question

Decide which non-periodic tiling has pure discrete dynamical spectrum.

# A construction of non-periodic tilings

a substitution rule =a recipe for "expanding and subdividing"

= • $\mathcal{A}$: a finite set of tiles (the alphabet)

• $\rho$: the rule of expanding $P \in \mathcal{A}$ and then subdivide it

• $\phi: \mathbb{R}^d \to \mathbb{R}^d$ (the expansion map)

a linear map s.t. $\lambda$: eigenvalue$\Rightarrow |\lambda| > 1$

Example

$\tau = \dfrac{1+\sqrt{5}}{2}$ :expansion factor $(\phi = \tau \times \text{identity})$ $\mathcal{A} = \{\overset{T_1}{[0,\tau]}, \overset{T_2}{[0,1]}\}$

$$\rho_F(T_1) = \{T_1, T_2 + \tau\} \qquad \rho_F(T_2) = \{T_1\}$$

# A construction of non-periodic tilings

Example $\tau = \frac{1+\sqrt{5}}{2}$ :expansion factor $\mathcal{A} = \{\overset{T_1}{[0,\tau]}, \overset{T_2}{[0,1]}\}$

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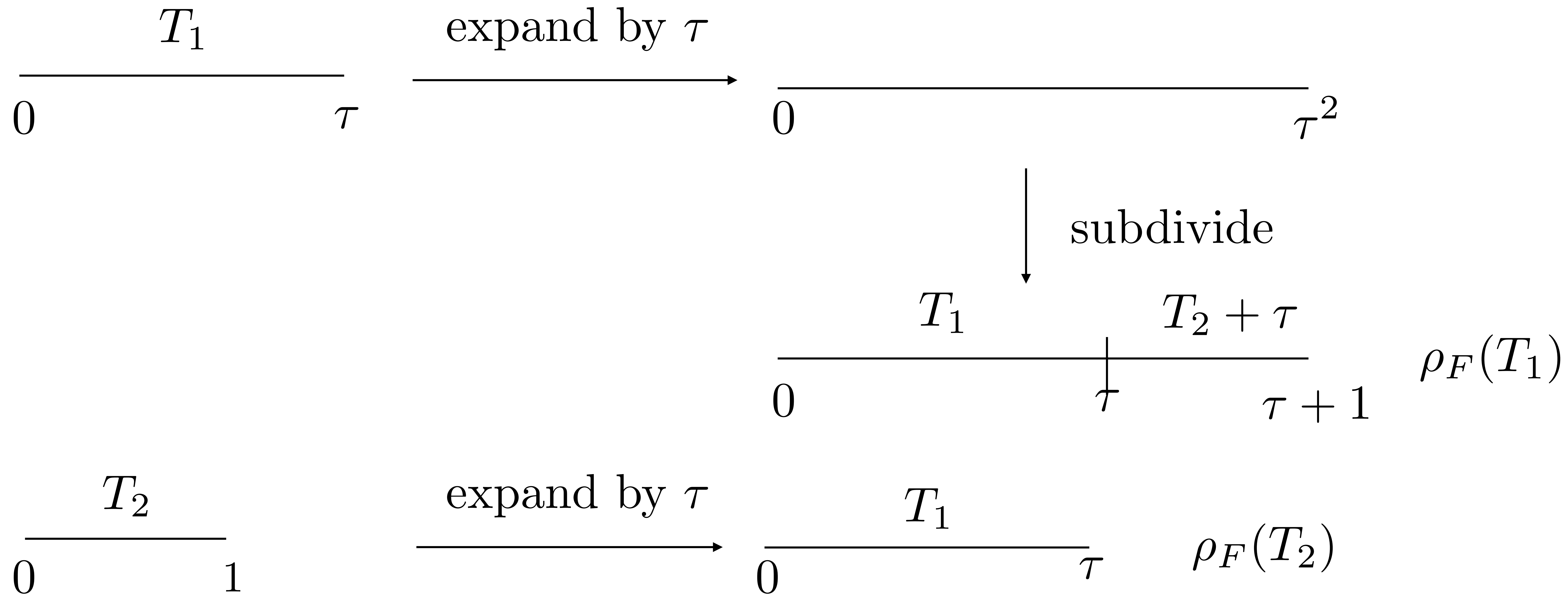
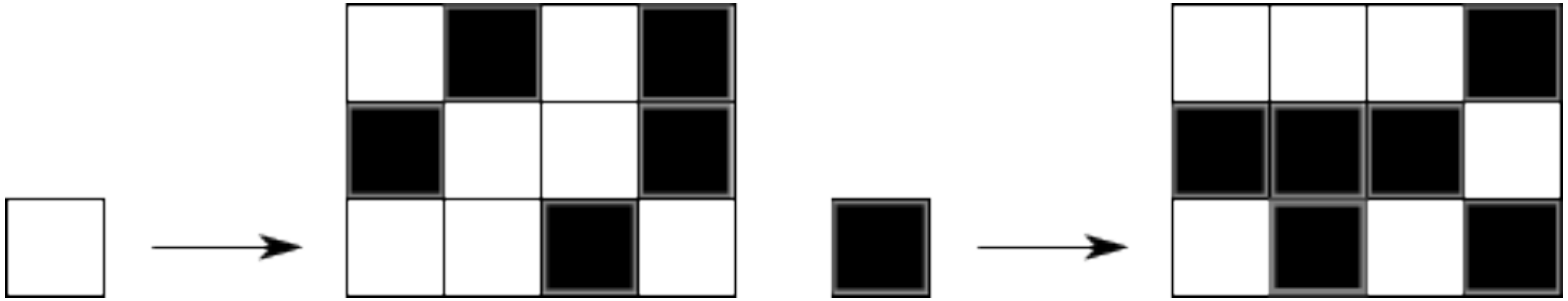


# A construction of non-periodic tilings

Example 2

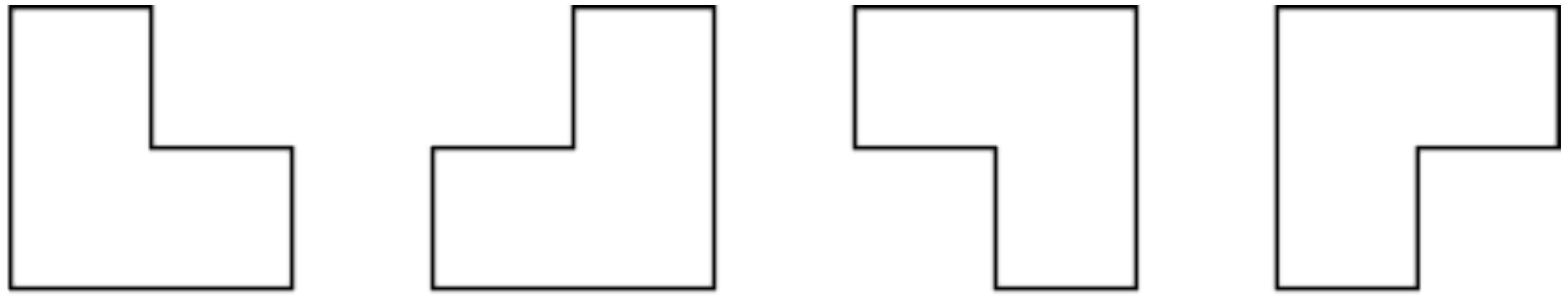
$$\mathcal{A} = \{([0,1]^2, B), ([0,1]^2, W)\}$$

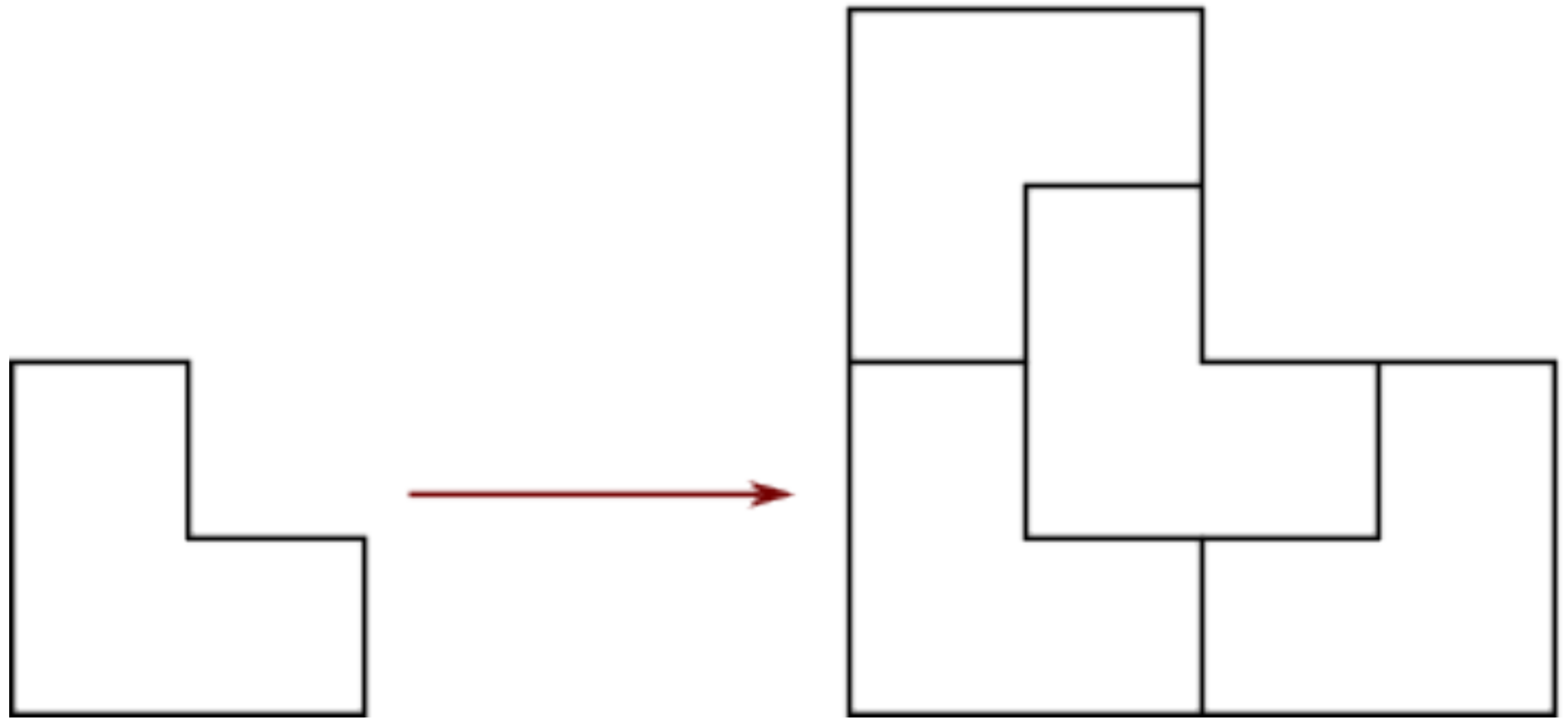
$$\phi = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$$



# A construction of non-periodic tilings

Example 3

$\mathscr{A} = \{$  $\}$

$\rho :$ 

# A construction of non-periodic tilings

In general, $\rho$ is a map that sends a proto-tile $P \in \mathscr{A}$ to a patch $\rho(P)$

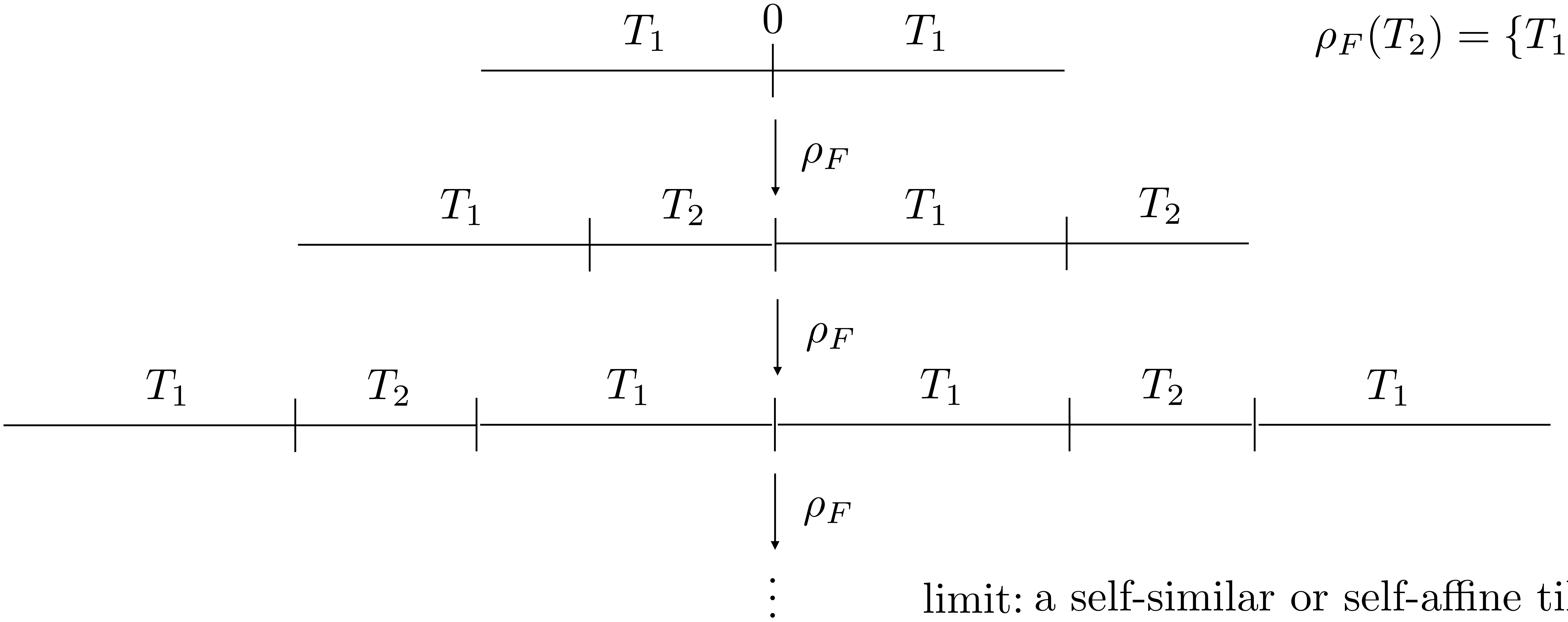
We can iterate $\rho$ to obtain $\rho^n(P), n = 1,2,\ldots$

# A construction of non-periodic tilings

given a substitution rule $(\mathcal{A}, \phi, \rho)$, we can "iterate" $\rho$

$$\rho_F(T_1) = \{T_1, T_2 + \tau\}$$

$$\rho_F(T_2) = \{T_1\}$$




limit: a self-similar or self-affine tiling

# A construction of non-periodic tilings

$\mathcal{T} = \lim_{n\to\infty} \rho^{kn}(\mathcal{P})$ a self-similar tiling

apply $\rho$ "infinitely many times" $\rightsquigarrow$ self-similar or self-affine tiling

Q what if we pick two $\rho_1, \rho_2$, toss a coin each time and decide which of $\rho_1$ and $\rho_2$ we apply by head/tail?

to make sense of $\rho_1 \circ \rho_2$, $\rho_1$ and $\rho_2$ must share a common alphabet

if so, for arbitrary $i_1, i_2, \ldots \in \{1, 2\}$, the limit

$$\lim_{n\to\infty} \rho_{i_1}\rho_{i_2}\cdots\rho_{i_n}(\mathscr{P})$$

convergent?

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$i_{k_n}$

yes, under FLC

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convergent? yes, under FLC

Is $\mathcal{T}$ non-periodic? $\rightsquigarrow$ case-by-case

The spectral properties of $\mathcal{T}$? $\rightsquigarrow$ discuss this later

# A construction of non-periodic tilings

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$$\lim_{n\to\infty} \rho_{i_1}\rho_{i_2}\cdots\rho_{i_{k_n-1}}\rho_{i_{k_n}}(\mathscr{P})$$

convergent? yes, under FLC

S-adic tilings belonging to $(i_n)_{n=1,2,\ldots}$: tilings of the form

$$\mathcal{T} = \lim_{n\to\infty} \rho_{i_1} \circ \rho_{i_2} \circ \cdots \circ \rho_{i_{k_n}}(\mathcal{P}_n)$$

$\{\rho_1, \rho_2, \ldots, \rho_{m_a}\}$: a finite family of substitutions

with a common alphabet $\mathcal{A}$

$i_1, i_2, \ldots \in \{1, 2, \ldots, m_a\}$: a directive sequence

# A construction of non-periodic tilings

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$i_1, i_2, \ldots \in \{1, 2, \ldots, m_a\}$: a directive sequence

in other words: a tiling $\mathcal{T} = \mathcal{T}^{(1)}$ that admits "de-substituted tilings"

$$\mathcal{T}^{(2)}, \mathcal{T}^{(3)}, \mathcal{T}^{(4)}, \ldots$$

such that

$$\rho_{i_n}(\mathcal{T}^{(n+1)}) = \mathcal{T}^{(n)}, n = 1, 2, \ldots$$

# A construction of non-periodic tilings

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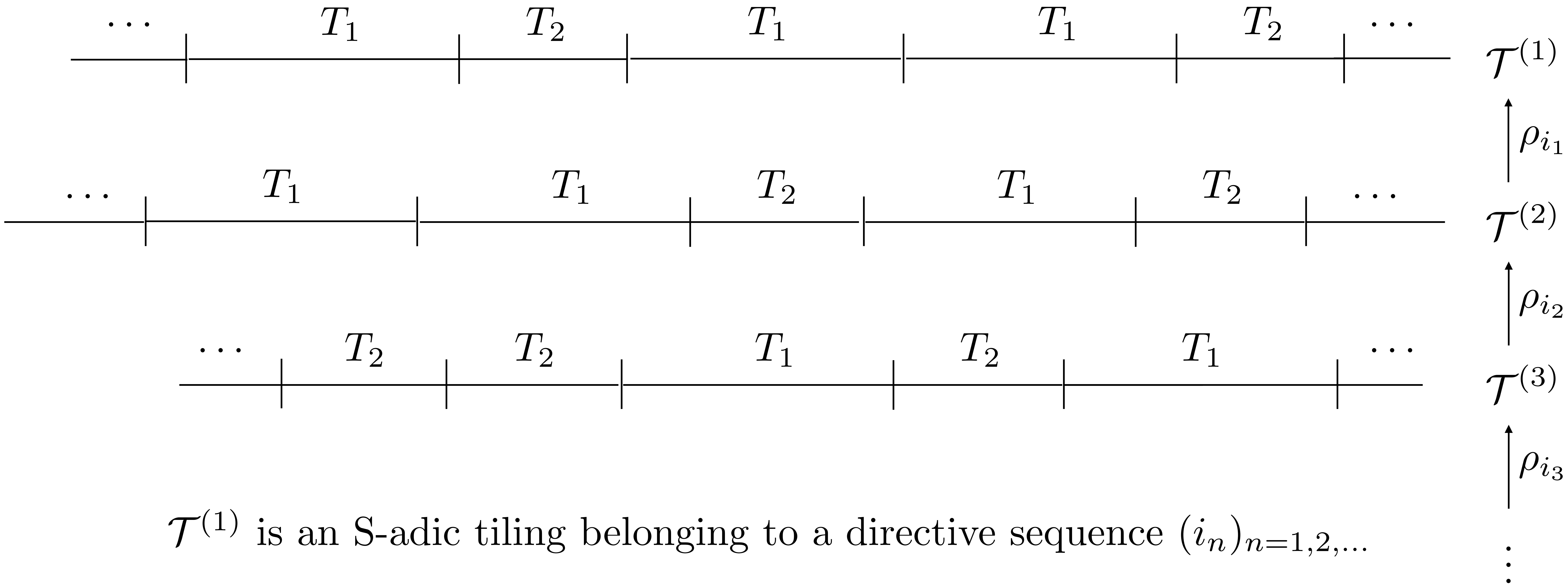


$\mathcal{T}^{(1)}$ is an S-adic tiling belonging to a directive sequence $(i_n)_{n=1,2,\ldots}$

# Main question

Decide which S-adic tiling has pure discrete dynamical spectrum.

Pisot Conjecture:
Self-affine tilings by substitution rules with the Pisot condition have pure discrete spectrum

Today's result
(1)Give a sufficient condition for a given S-adic tiling to be pure discrete
(2)this condition is satisfied for almost all block S-adic tilings

# Contents

(1)Definitions (done)
(2)The first main result (Overlap algorithm)
(3)The Second main result (block substitutions)

# The main idea

(1)Generalize Solomyak's overlap algorithm [Solomyak 1997] to the S-adic setting

(2)apply the overlap algorithm to a class of S-adic tiligs of interest

Goes back to the coincidence condition for constant-length symbolic substitution

# Overlap algorithm

$$\mathcal{T}_1 \xleftarrow{\rho_1} \mathcal{T}_2 \xleftarrow{\rho_2} \mathcal{T}_3 \xleftarrow{\rho_3} \cdots,$$

where $\rho_n$: a substitution rule with a fixed alphabet $\mathscr{A}$ and non-fixed expansion map $\phi_n : \mathbb{R}^d \to \mathbb{R}^d$

Pick a relatively dense subset $\Lambda_n \subset \mathbb{R}^d$ for each $n$ such that $\phi_n(\Lambda_{n+1}) \subset \Lambda_n$

An overlap @$n$ = a triple $(S, x, T)$ such that $S, T \in \mathcal{T}_n$ and $x \in \Lambda_n$ with $\mathrm{int}(S + x) \cap \mathrm{int} T \neq \emptyset$

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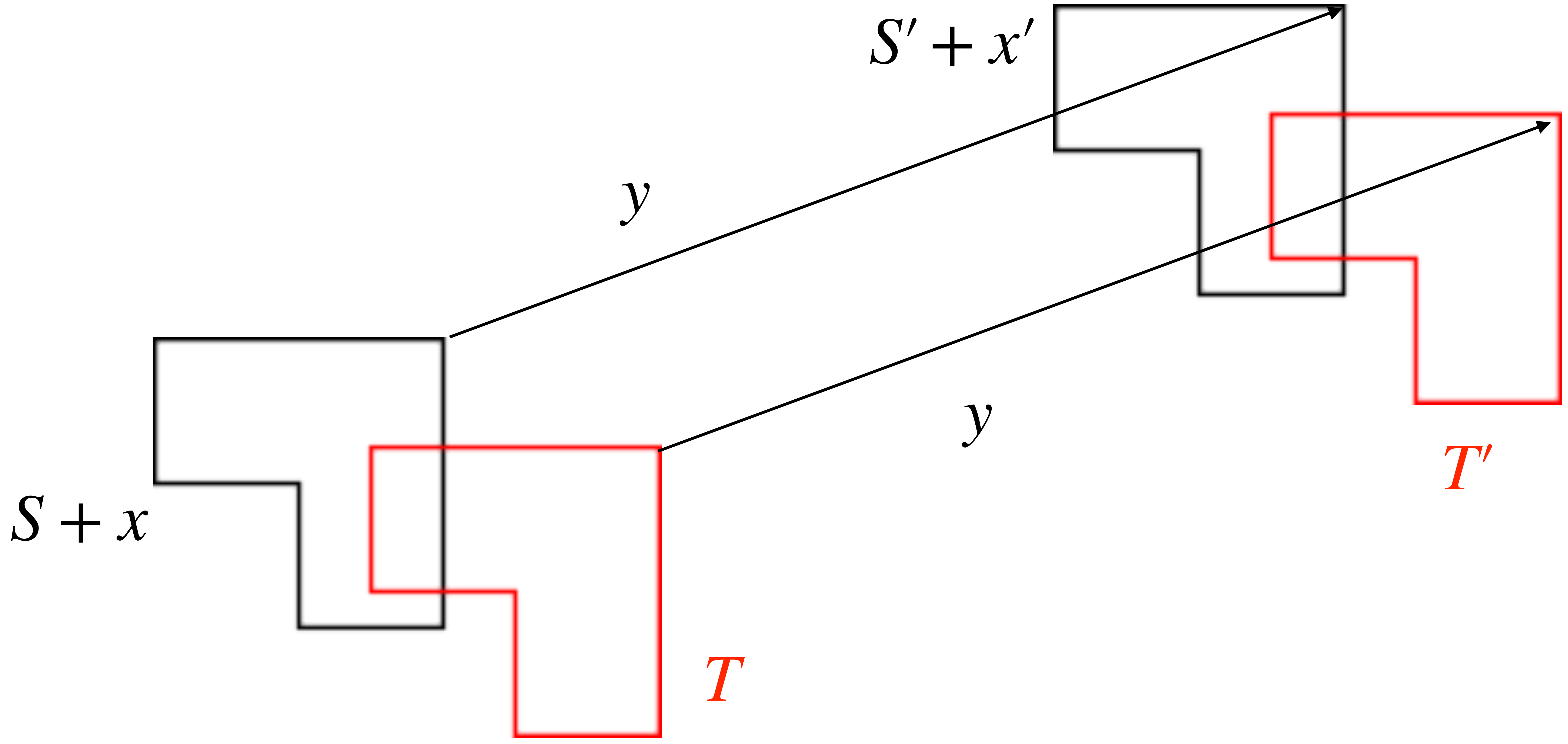


$(S, x, T) \sim (S', x', T')$

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$[S, x, T]$: the equivalence class

$V_n = \{[S, x, T] \mid (S, x, T)$ : an overlap @$n\}$

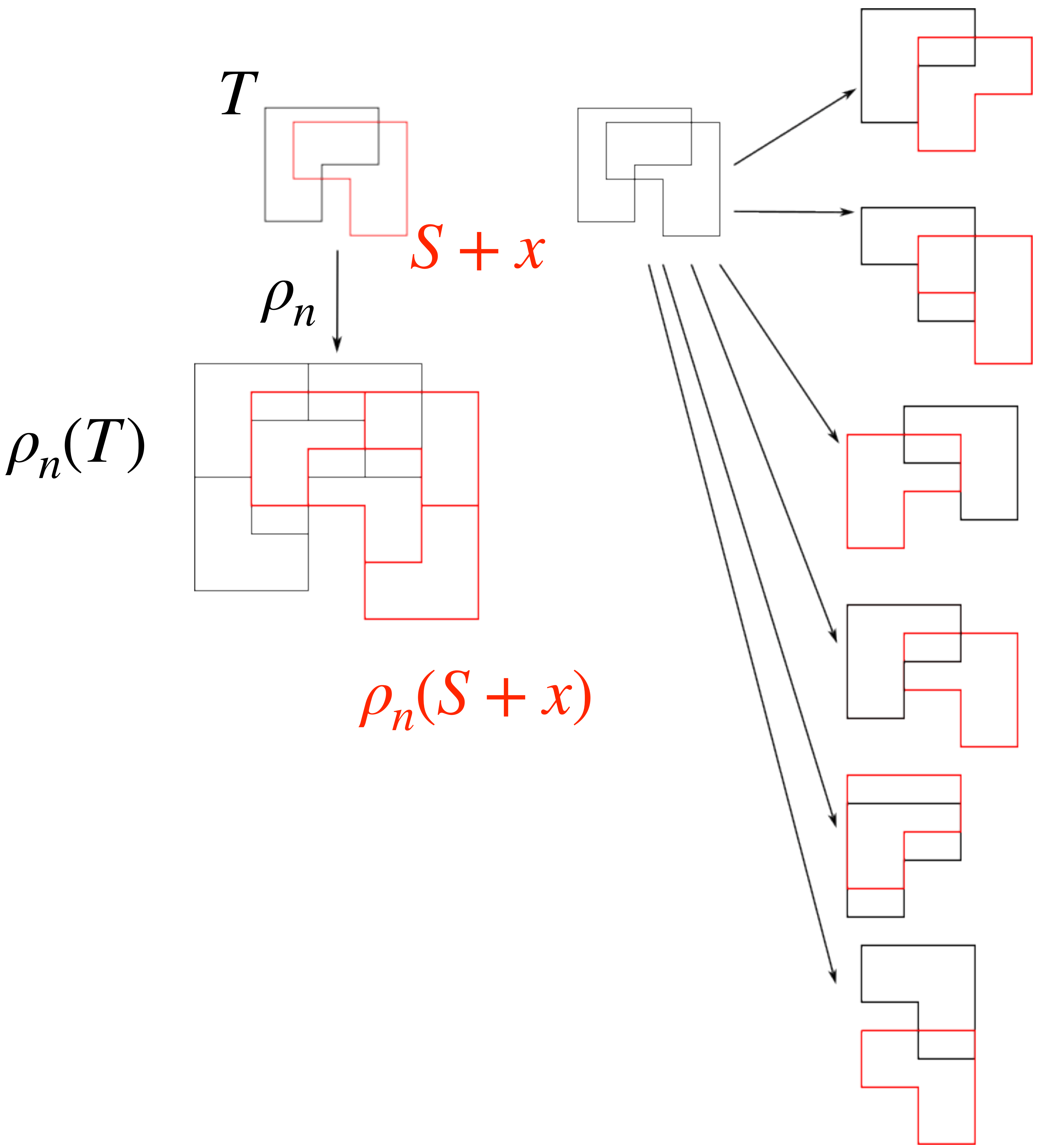
# Overlap algorithm

$[S, x, T]$: the equivalence class

$V_n = \{[S, x, T] \mid (S, x, T)$ : an overlap $@n\}$

$(S, x, T) @ n + 1 \to (S', x', T') @ n$

if $S' \in \rho_n(S), T' \in \rho_n(T)$, and $x' = \phi_n(x)$



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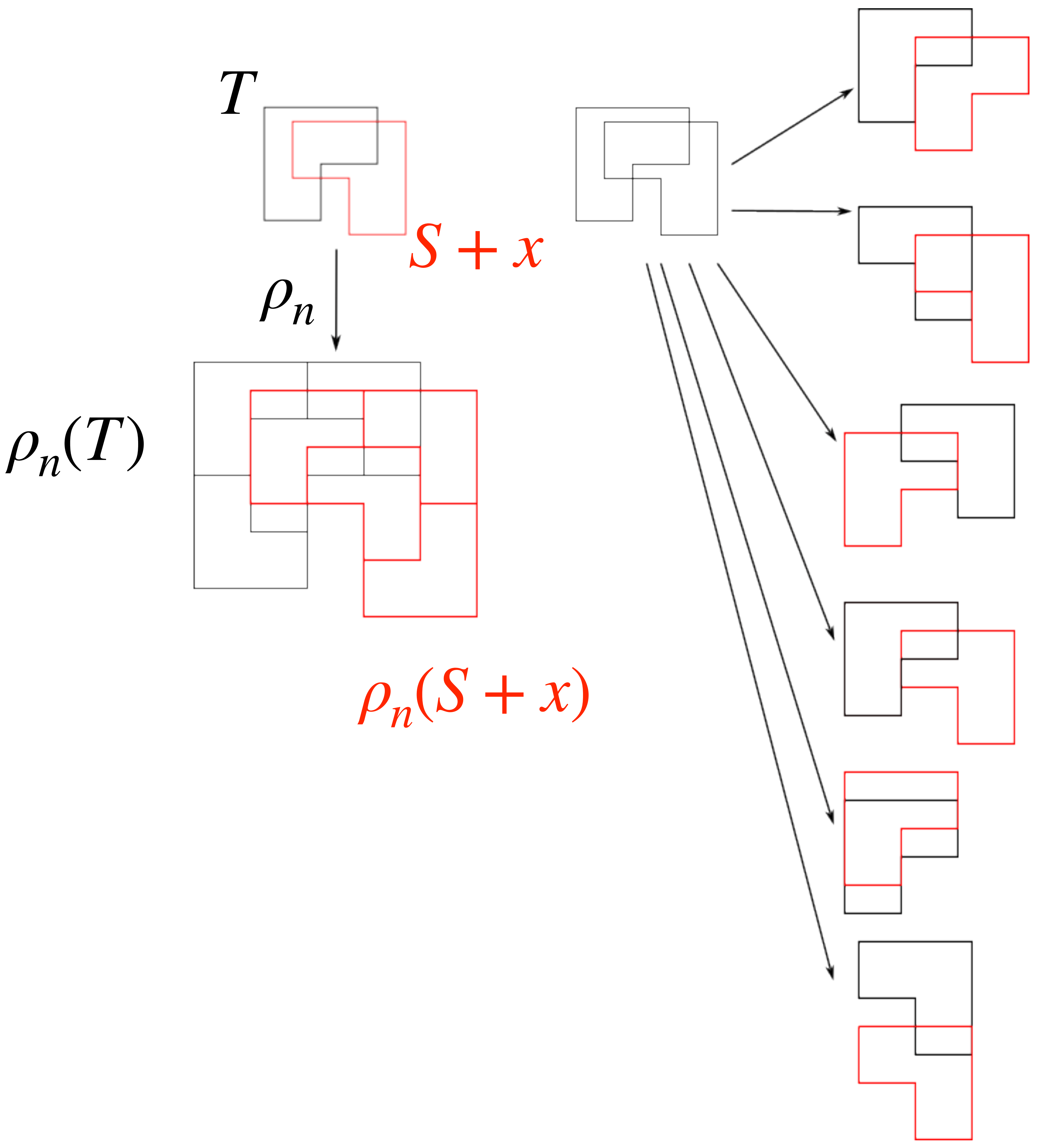
$V_{n+1} \ni v \to w \in V_n$ if there are

$(S, x, T) \in v, (S', x', T') \in w$

such that $(S, x, T) \to (S', x', T')$



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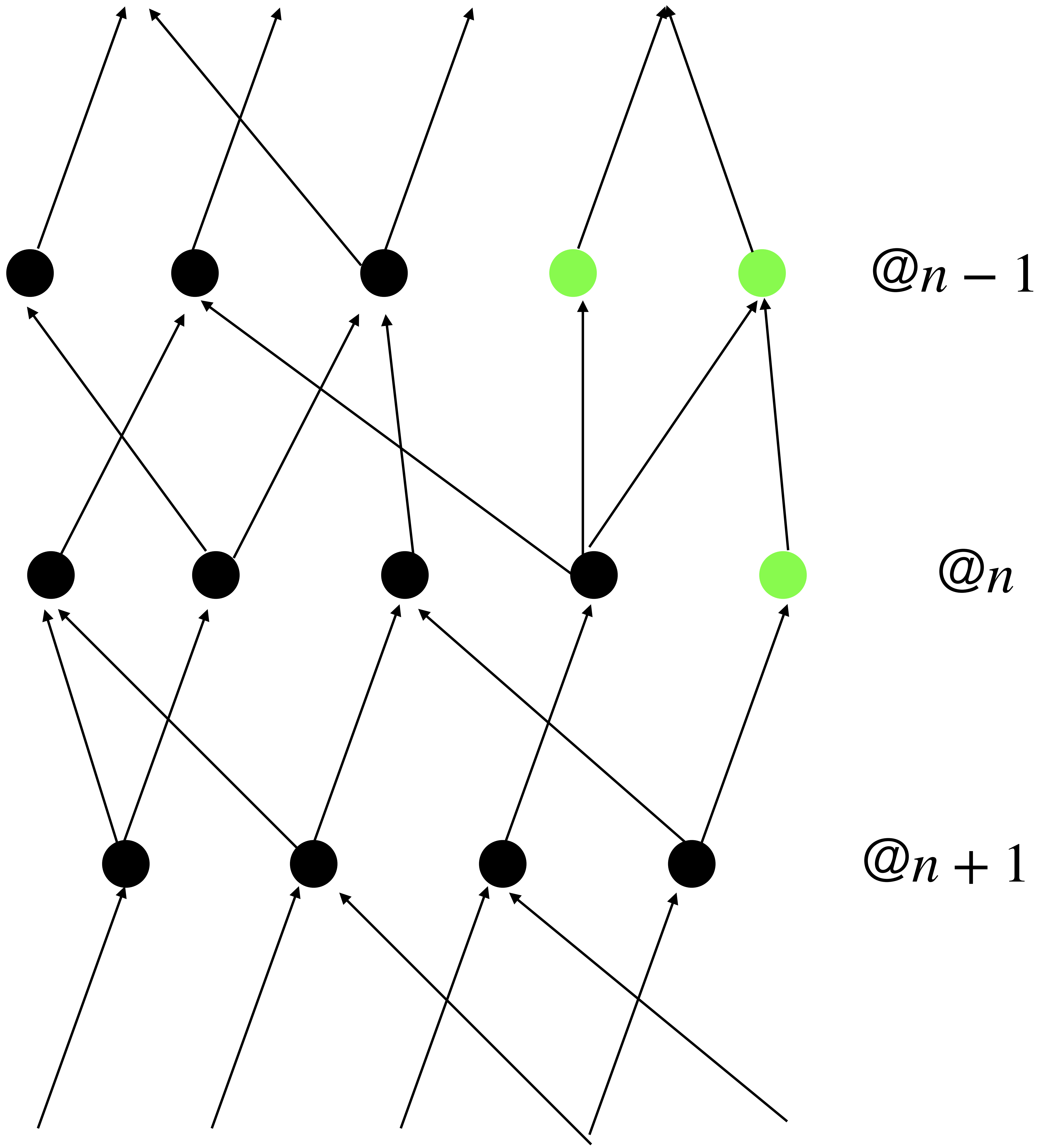


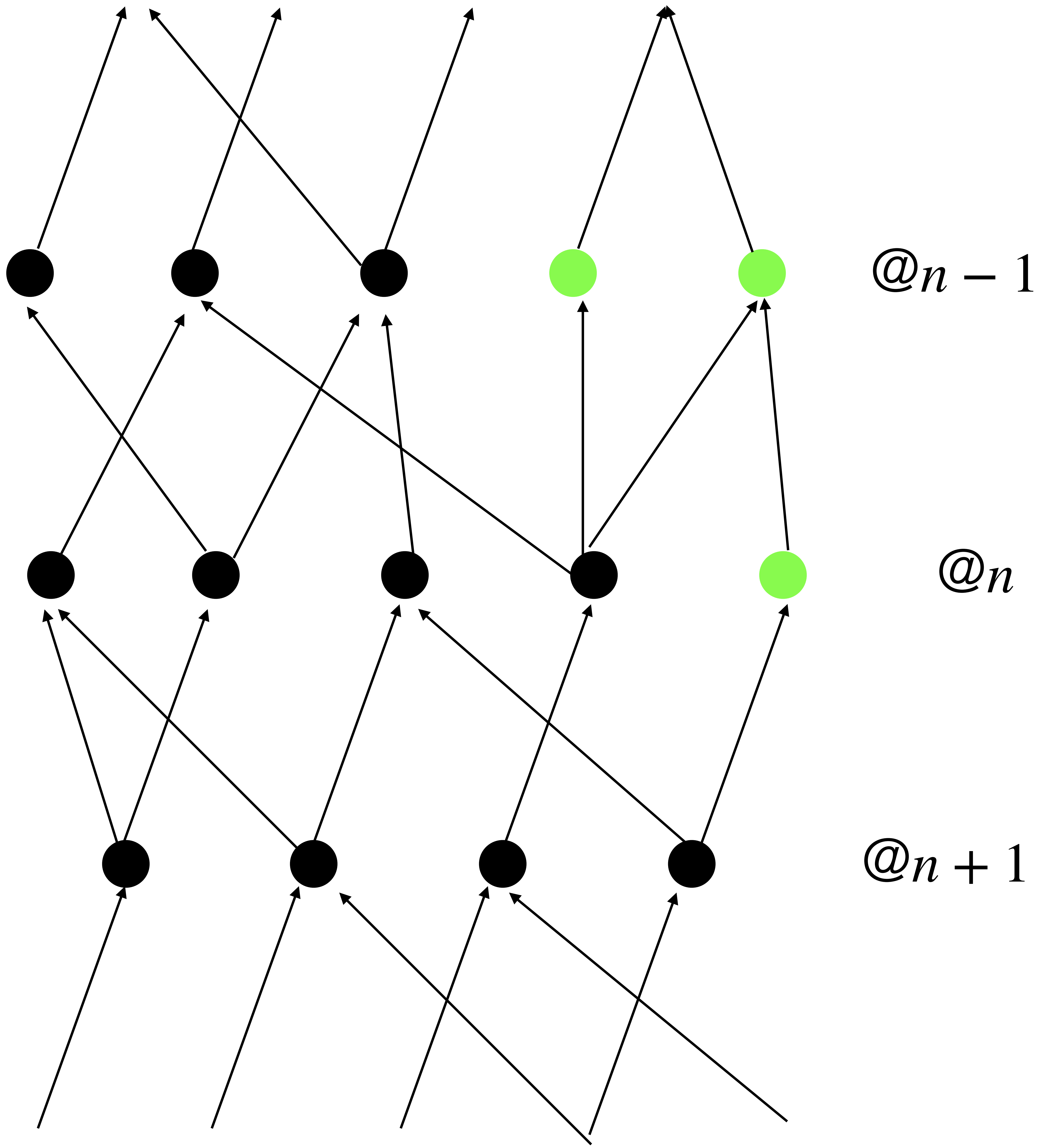
# Overlap algorithm

$[S, x, T]$: the equivalence class

$V_n = \{[S, x, T] \mid (S, x, T)$ : an overlap $@n\}$

An overlap $(S, x, T)$ is a coincidence if

$S + x = T$



# The first main theorem

Theorem (N-Thuswaldner)

If there are $n_1 < m_1 < n_2 < m_2 < \cdots$ such that,

for any $j$ and $v \in V_{m_j}$, there is a path from $v$ to a coincidence $w \in V_{n_j}$

+ a technical condition,

Then $\mathcal{T}_1$ has pure discrete dynamical spectrum

A combinatorial condition⇒an analytic condition

# Contents

(1)Definitions (done)
(2)The first main result (Overlap algorithm) (done)
(3)The Second main result (block substitutions)

# The block cases

$\mathscr{A} = \{T_i = ([0,1]^d, i) \mid i = 1,2,\ldots,n_0\}$: fix

A substitution rule with alphabet $\mathscr{A}$ and a diagonal expansion map

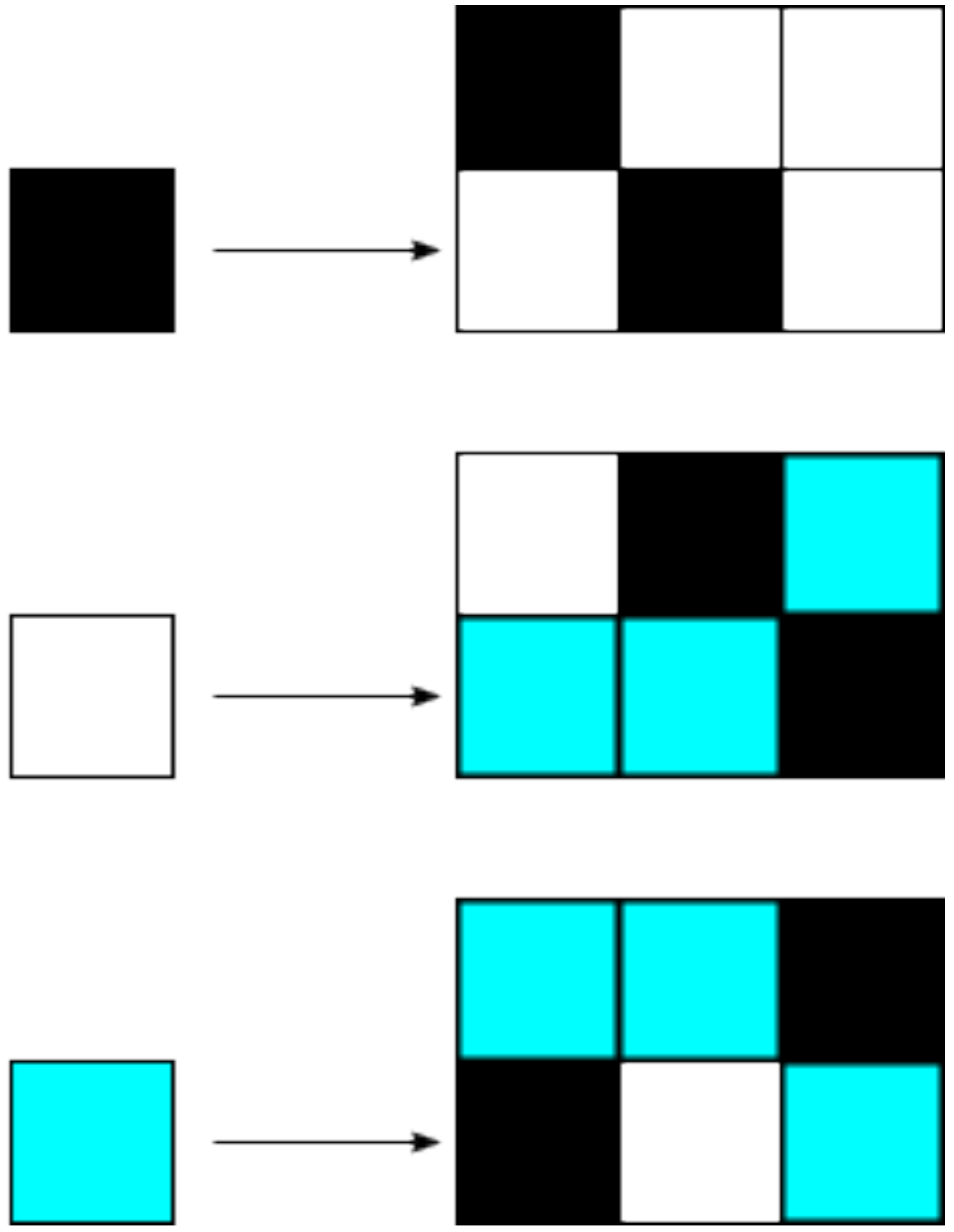
$$\phi = \begin{pmatrix} k_1 & & & \\ & k_2 & & \\ & & \ddots & \\ & & & k_d \end{pmatrix}$$

($k_i \in \mathbb{Z}_{>1}$) is called a block substitution.

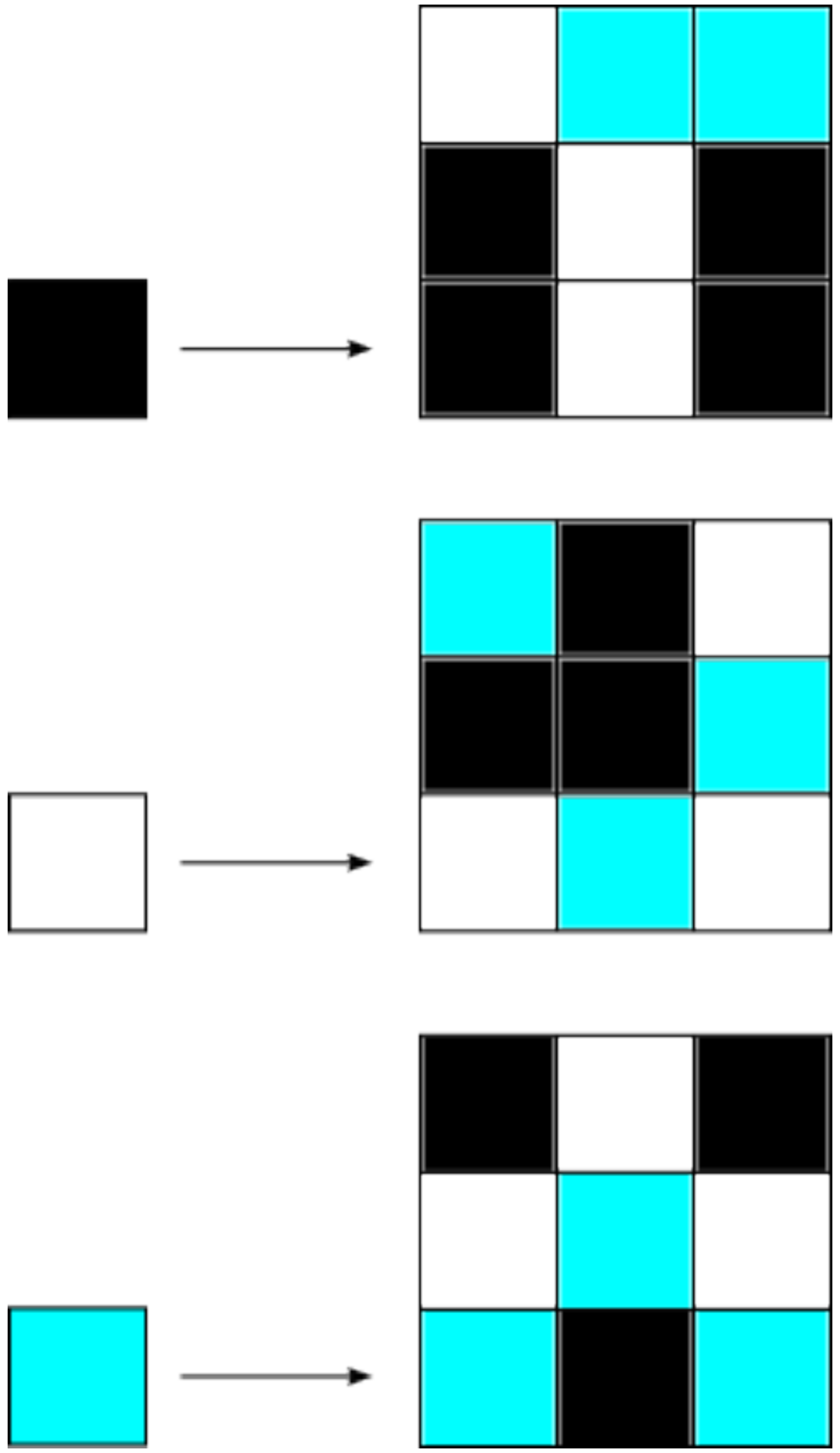
# The block cases

Example

$d = 2, n_0 = 3$



$\xi_1$



$\xi_2$

# The block cases

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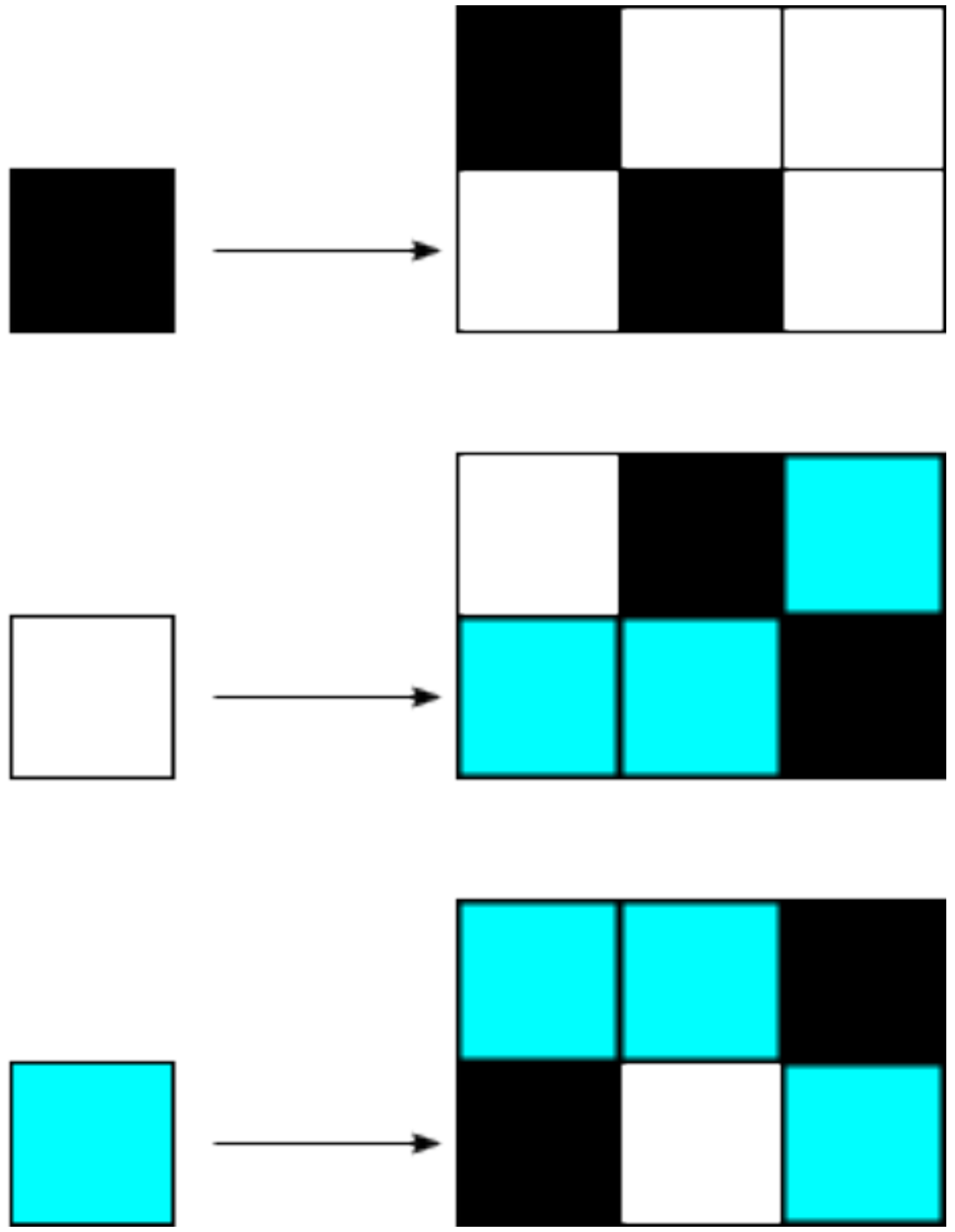

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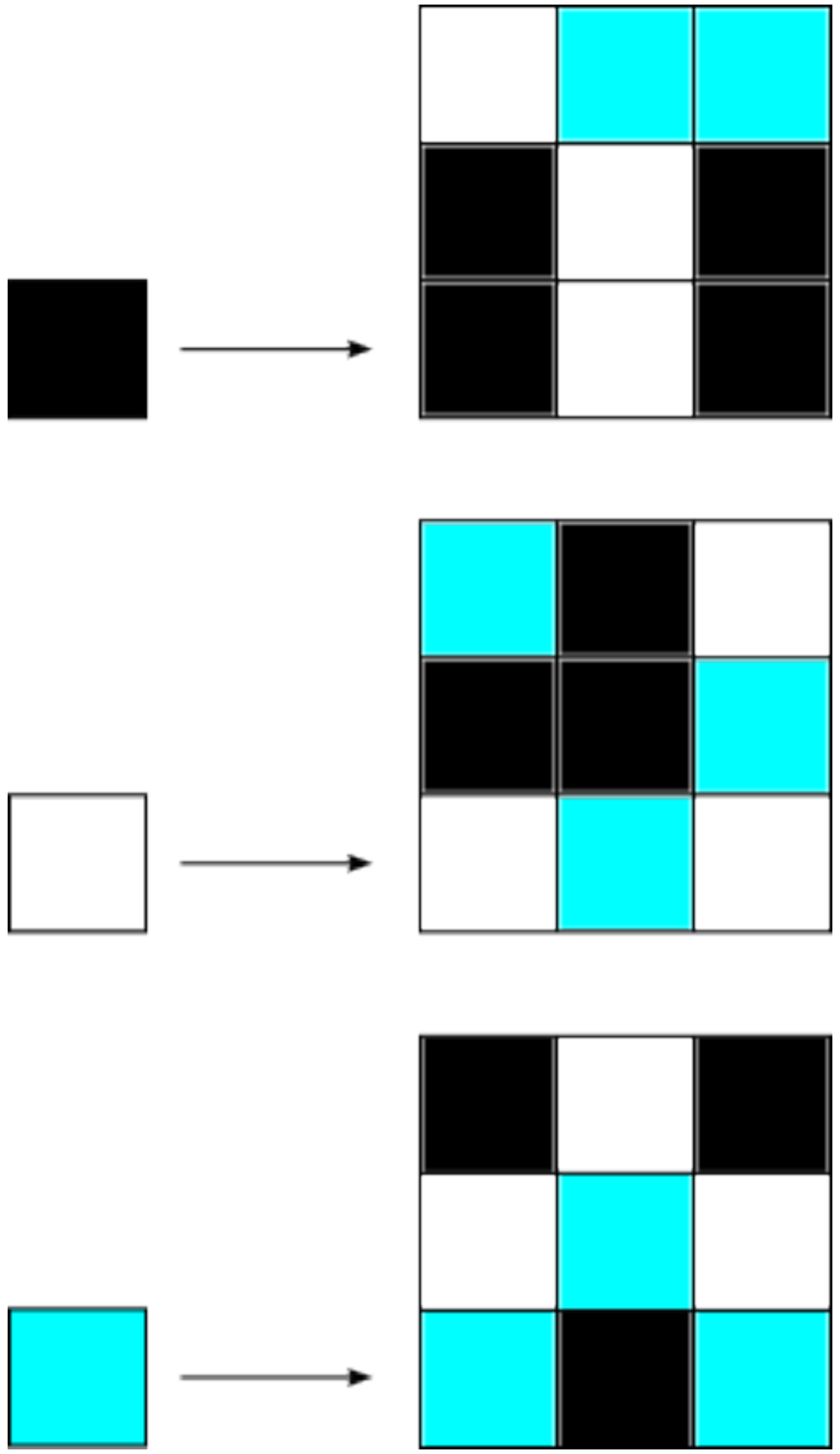
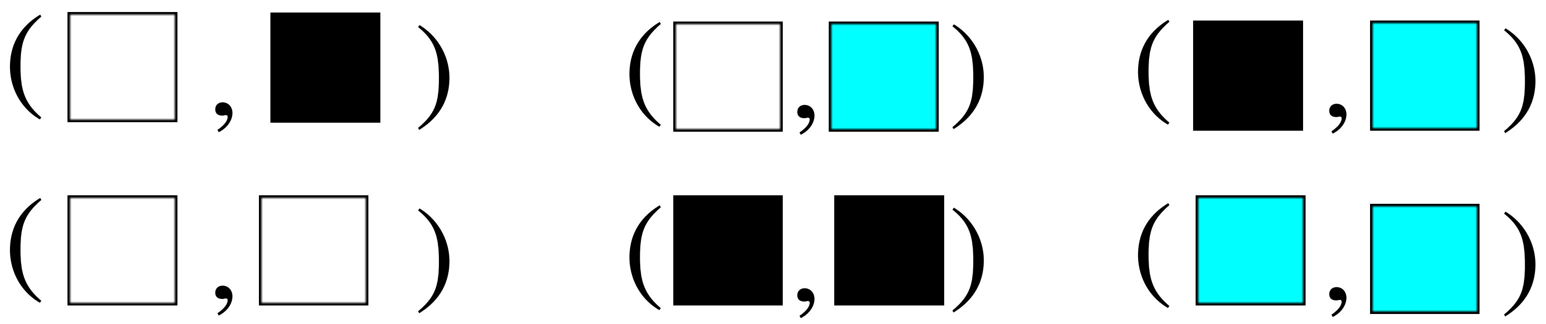
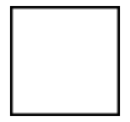


$\xi_2$

Study the overlap graph for $\Lambda_n = \mathbb{Z}^2$

⇝overlaps are





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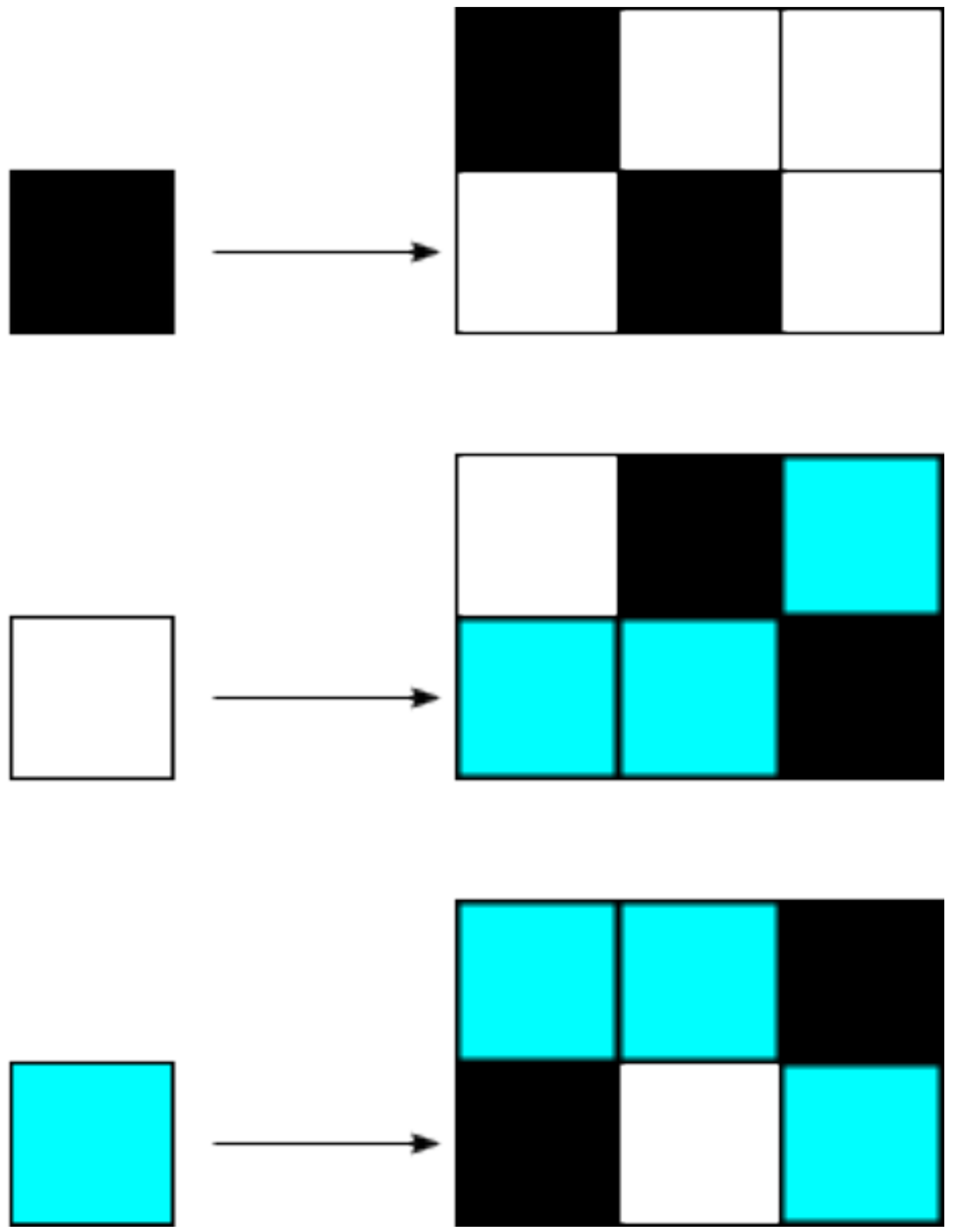
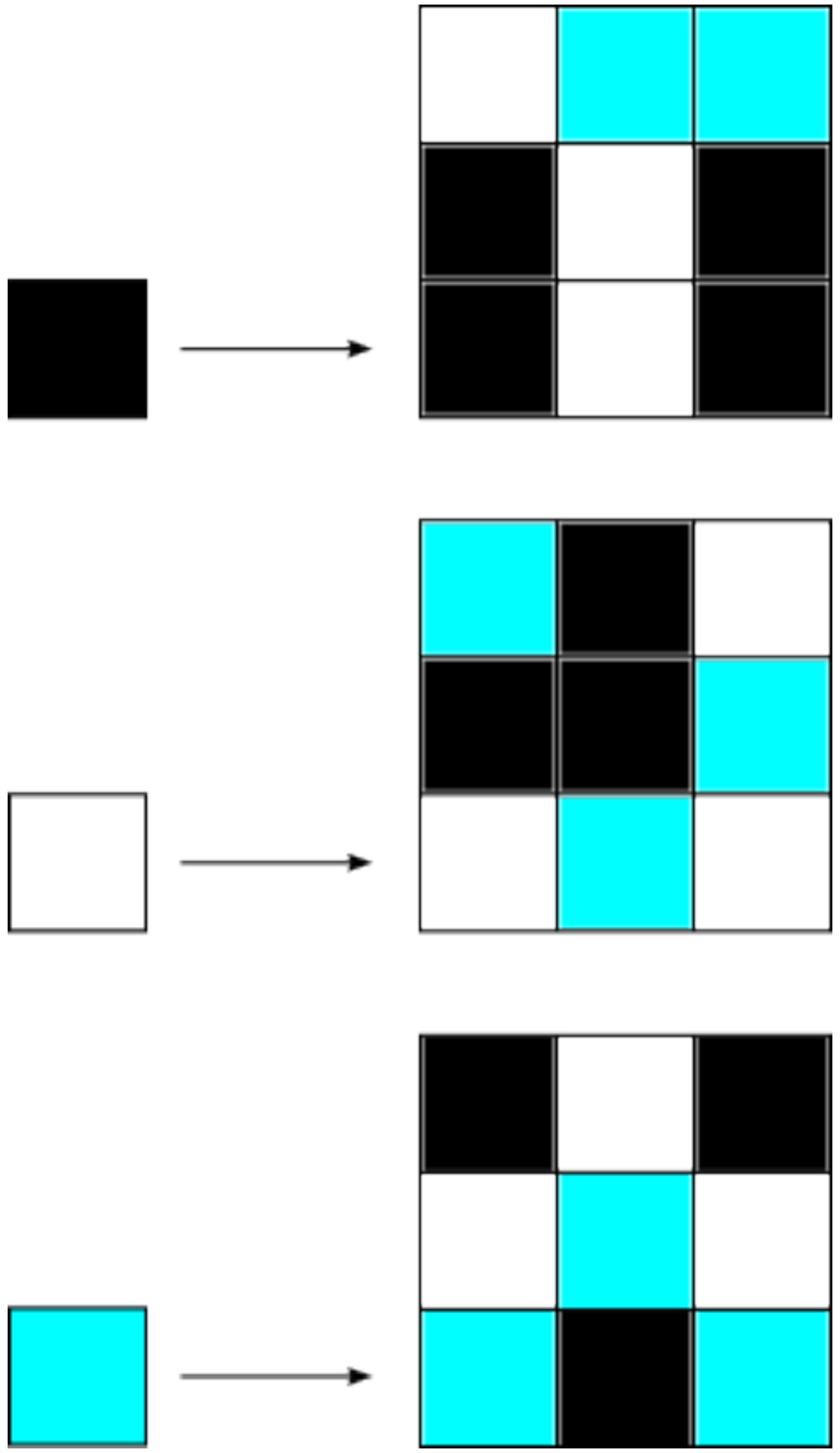
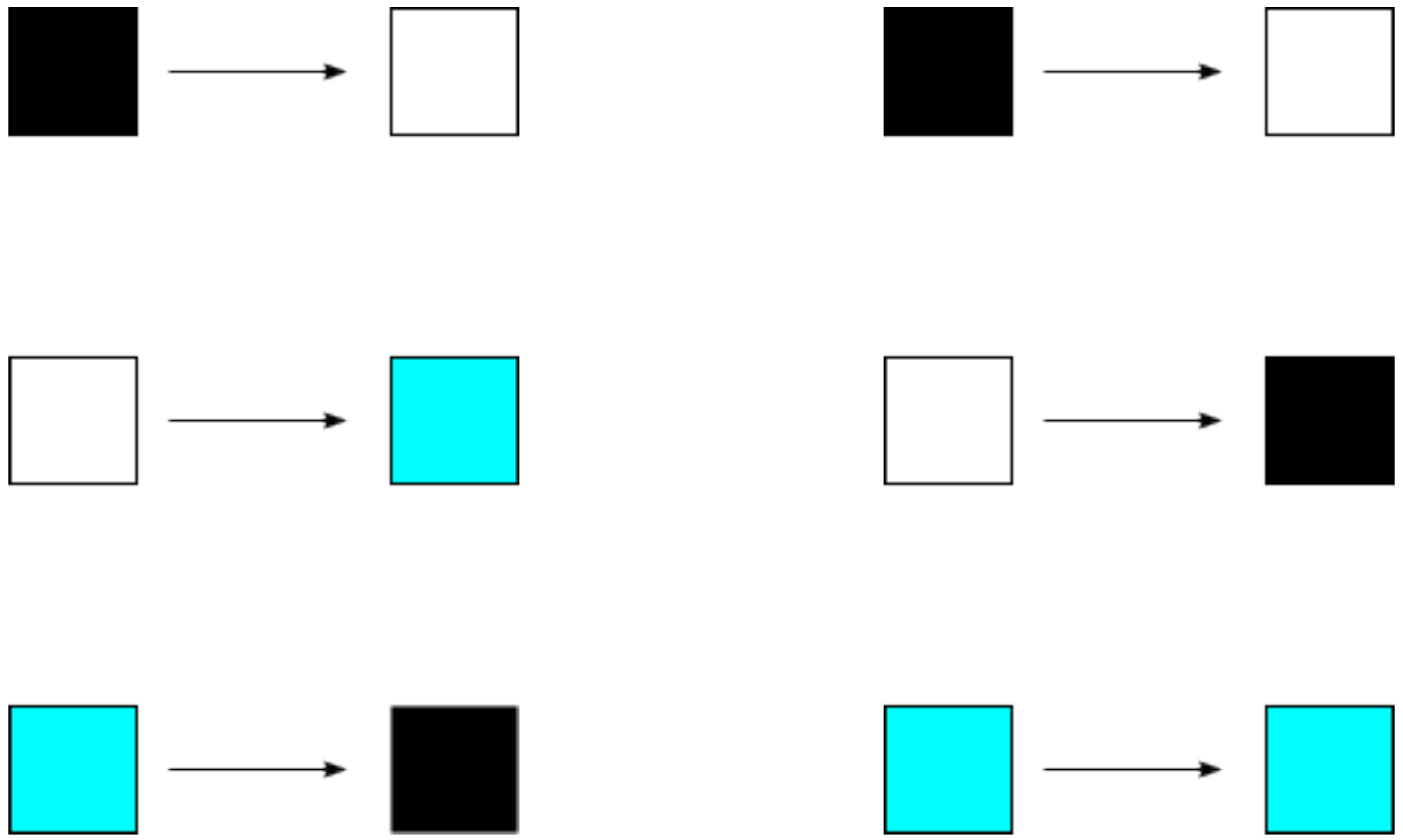


$\xi_1$(bijective)



$\xi_2$(non-bijective)

$\xi_1$ is bijective in the sense that, for any integer-coordinate point inside the patch, $\xi_1$defines a bijection on $\mathscr{A}$

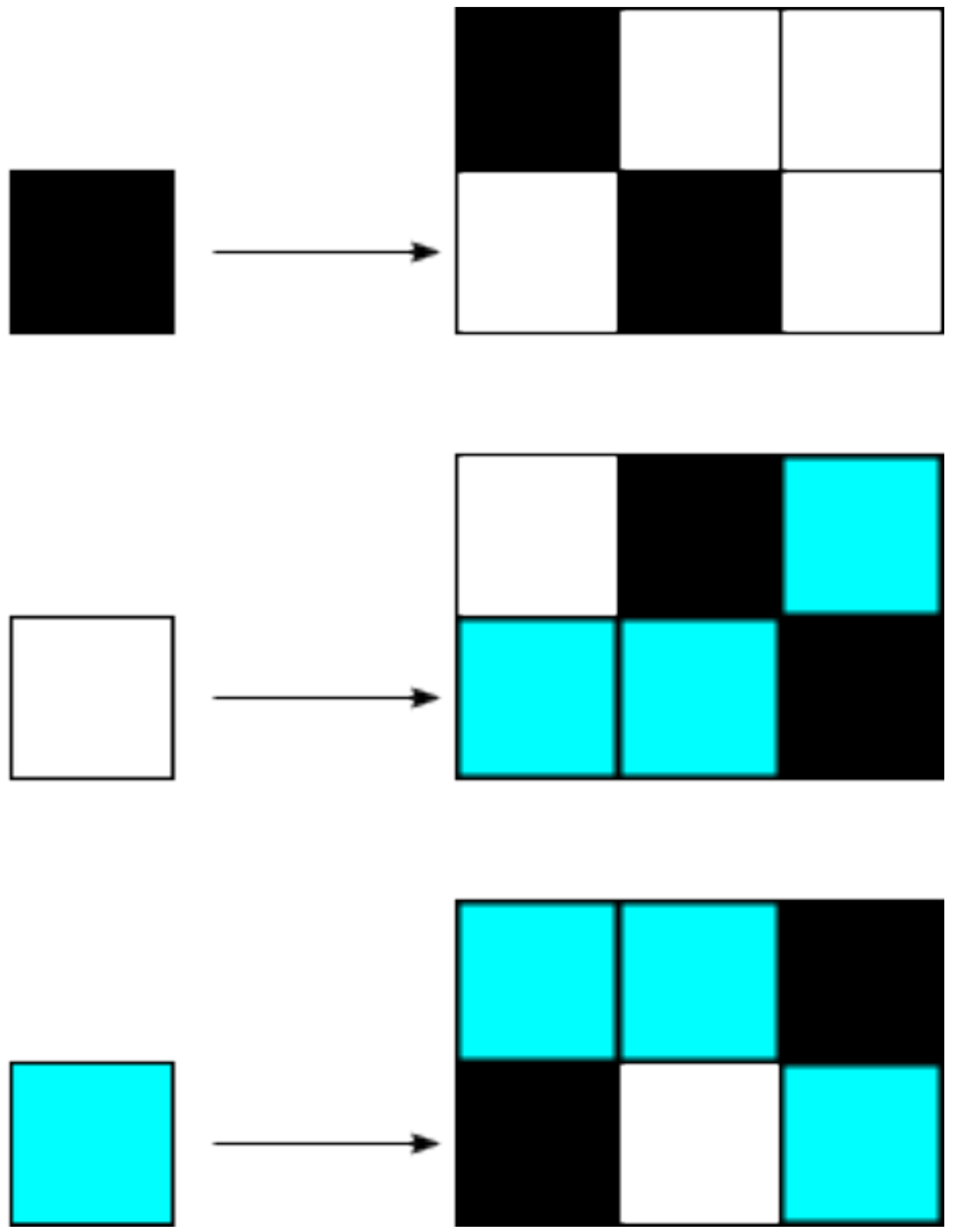


Lower-left　　Top-middle

# The block cases

Example

$d = 2, n_0 = 3$



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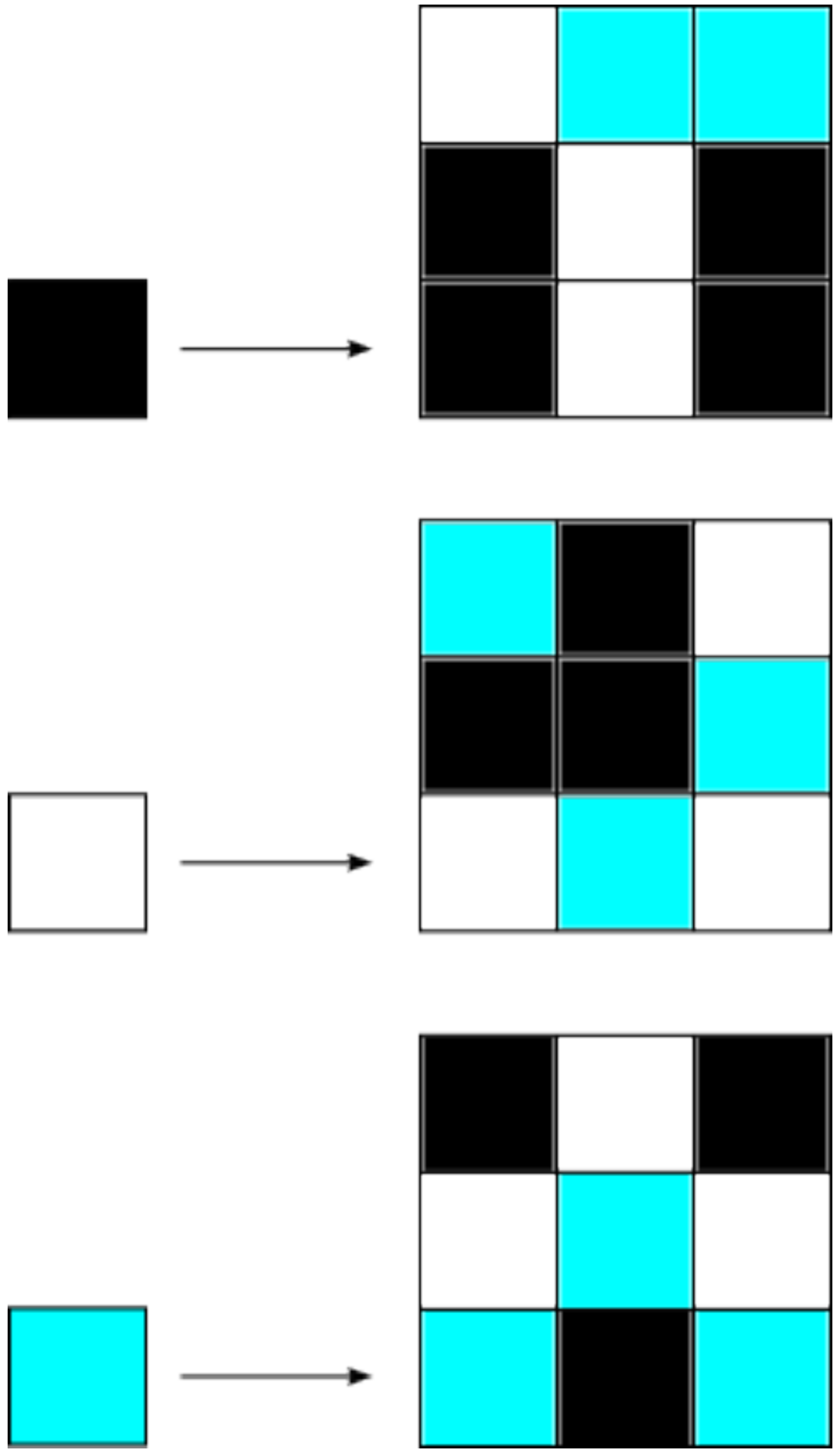
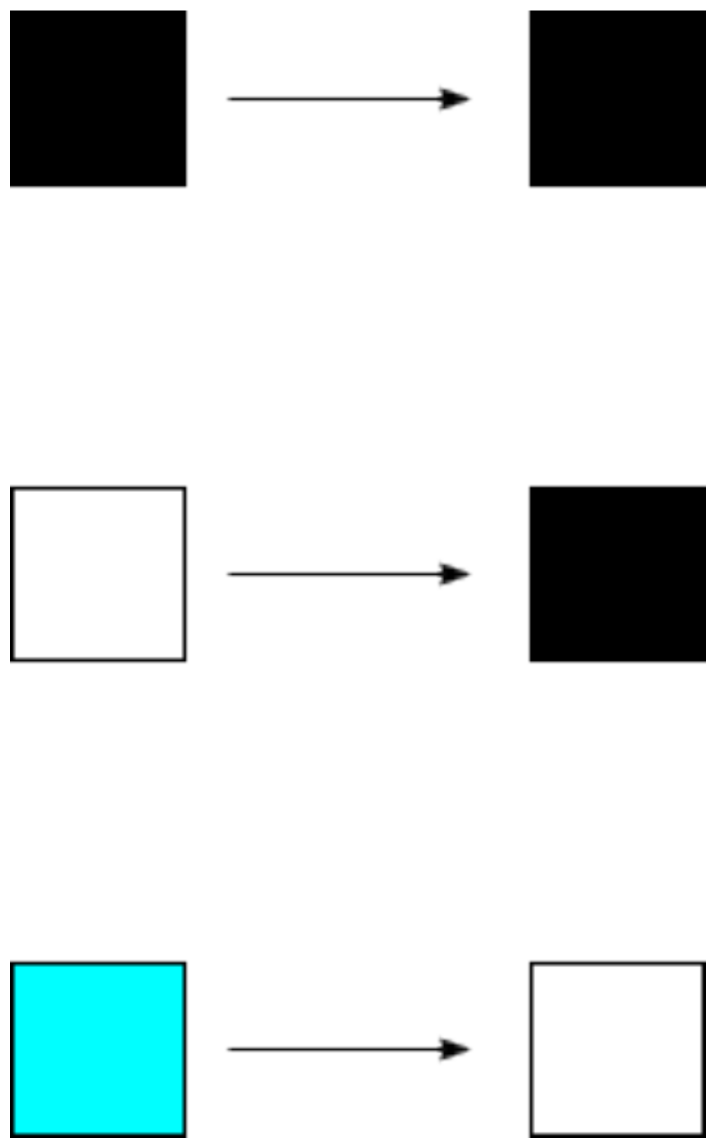


$\xi_2$(non-bijective)

$\xi_2$ is not bijective



Middle-left

# The block cases

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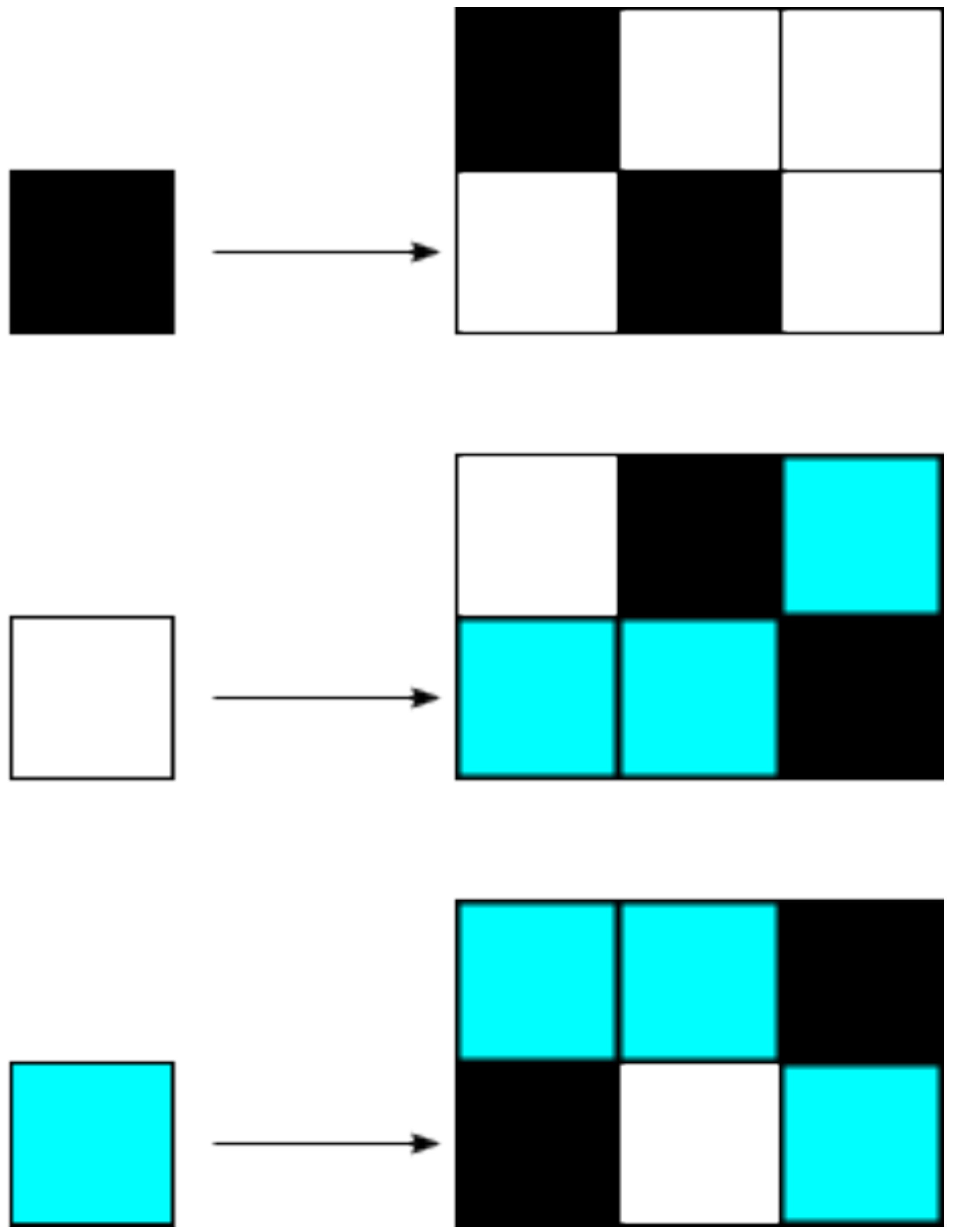

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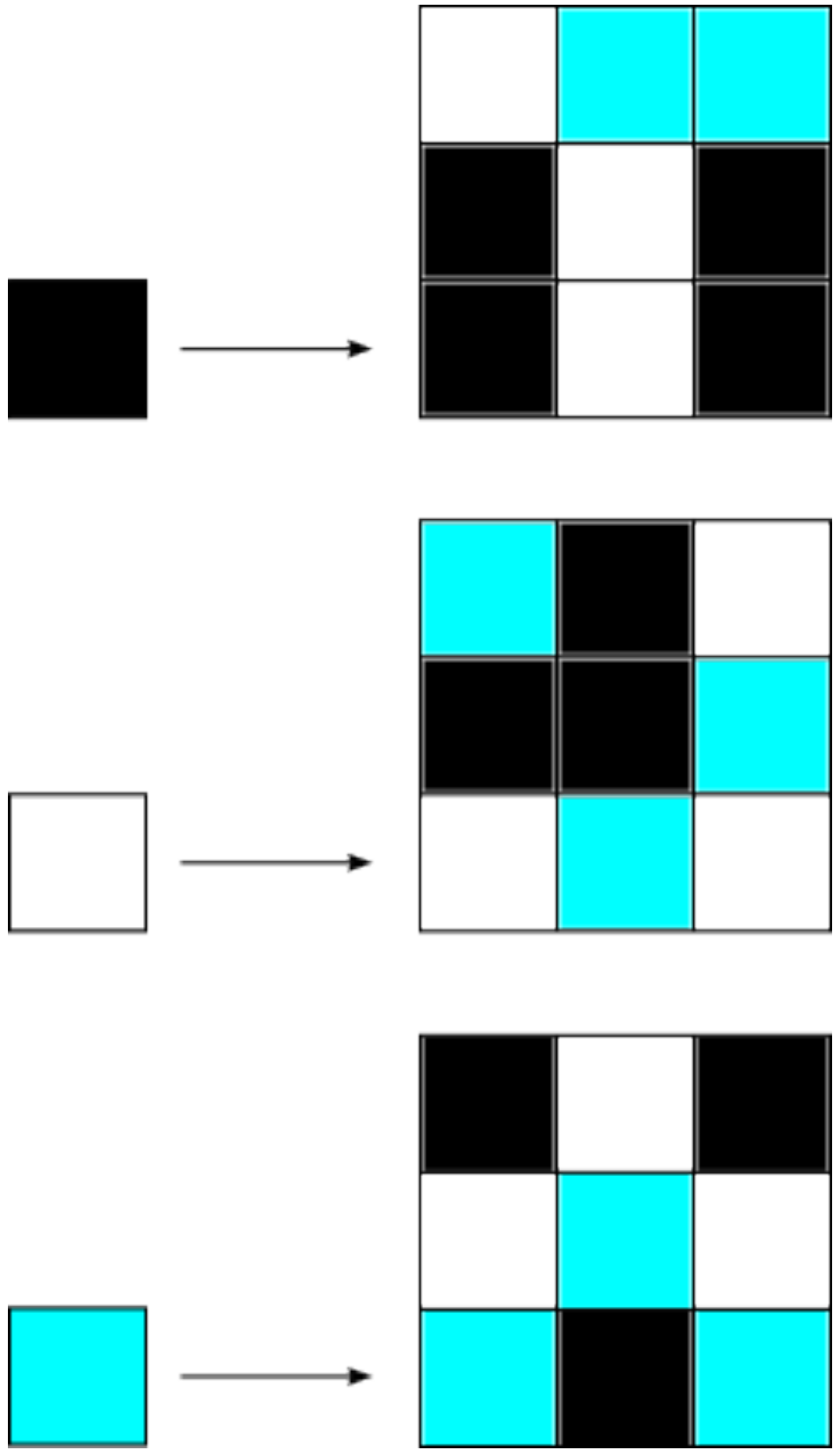
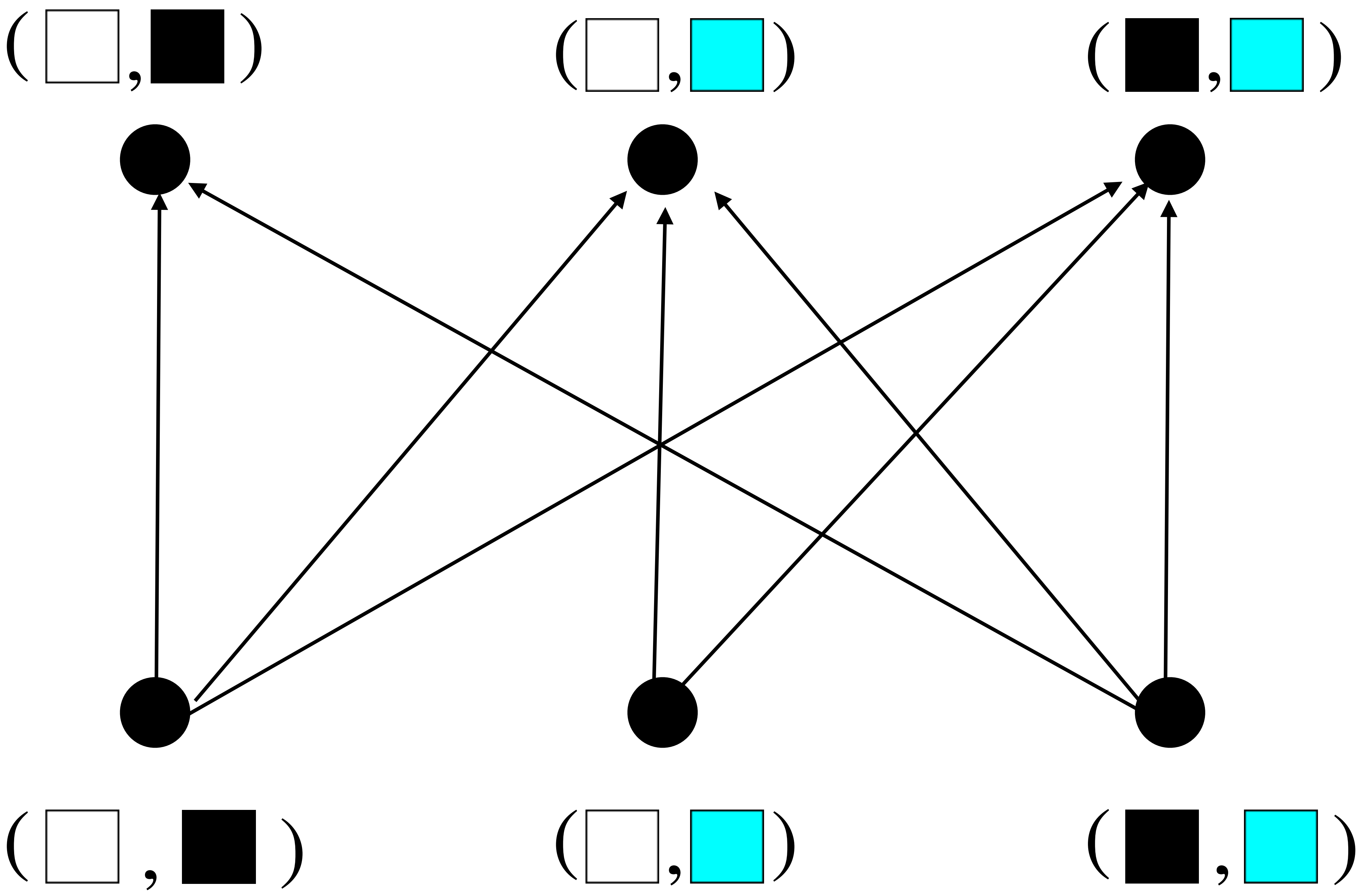


$\xi_2$(non-bijective)

Moreover, the graph for $\xi_1$ is as follows. ($\Lambda_n = \mathbb{Z}^2$)



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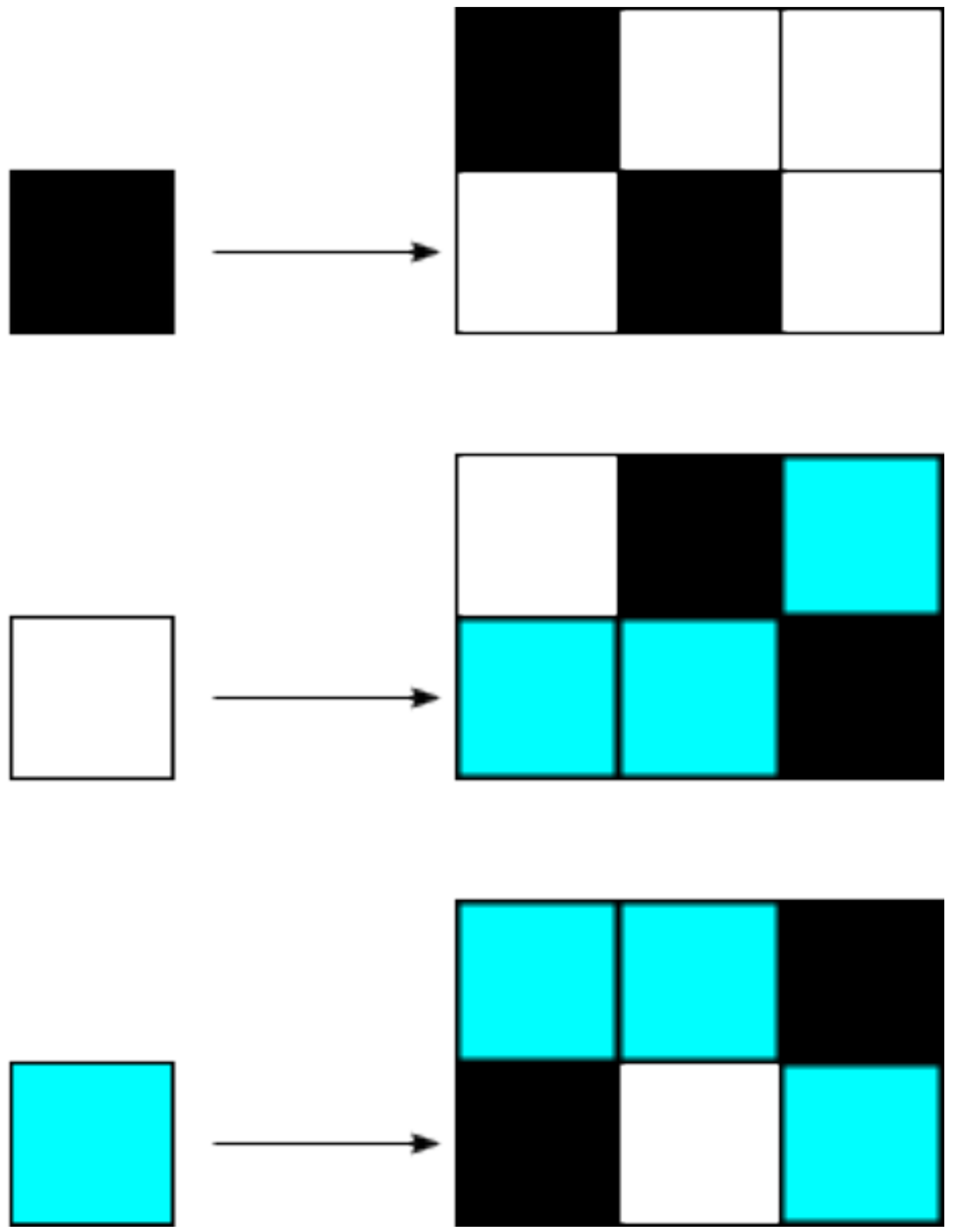
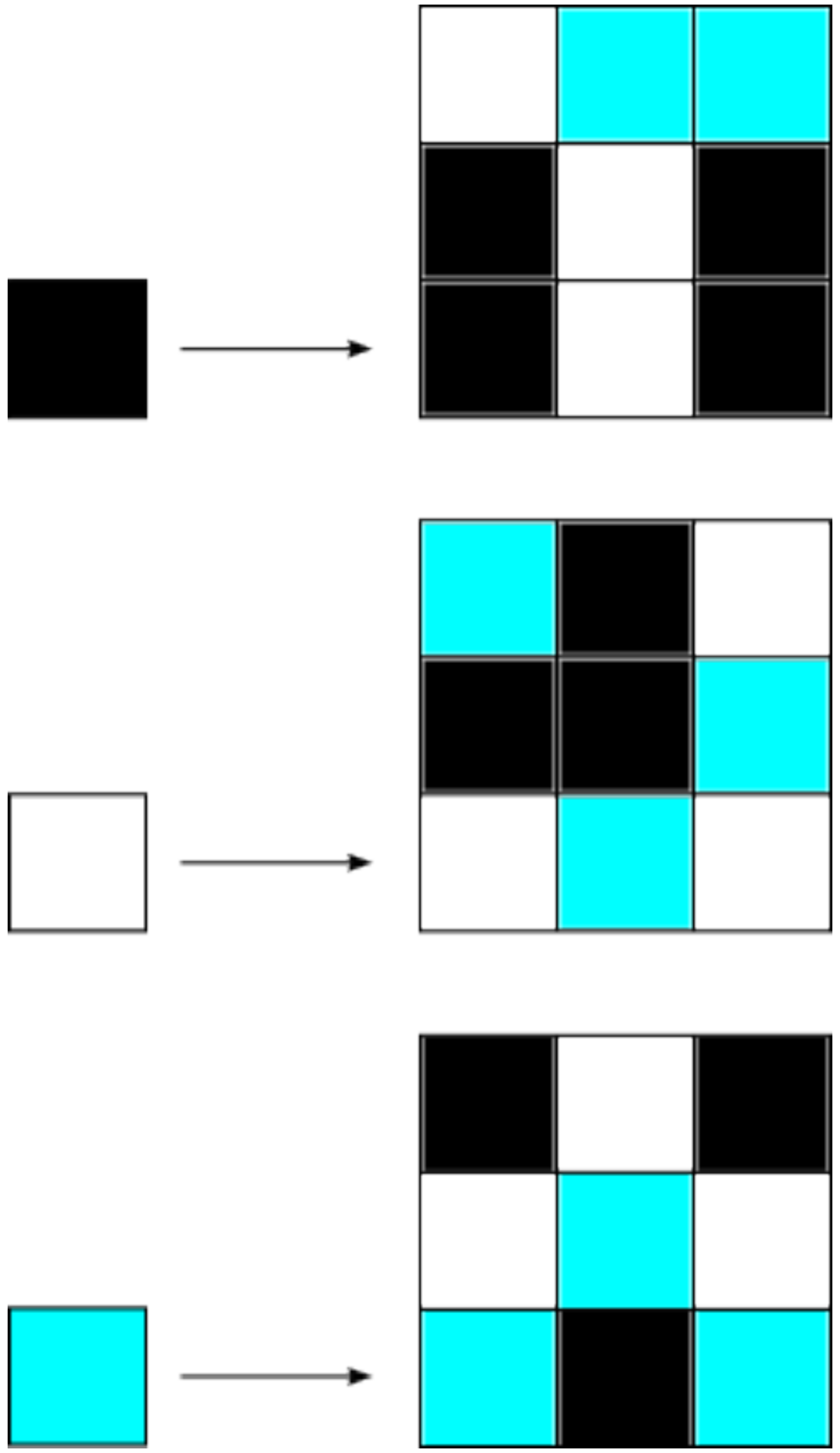
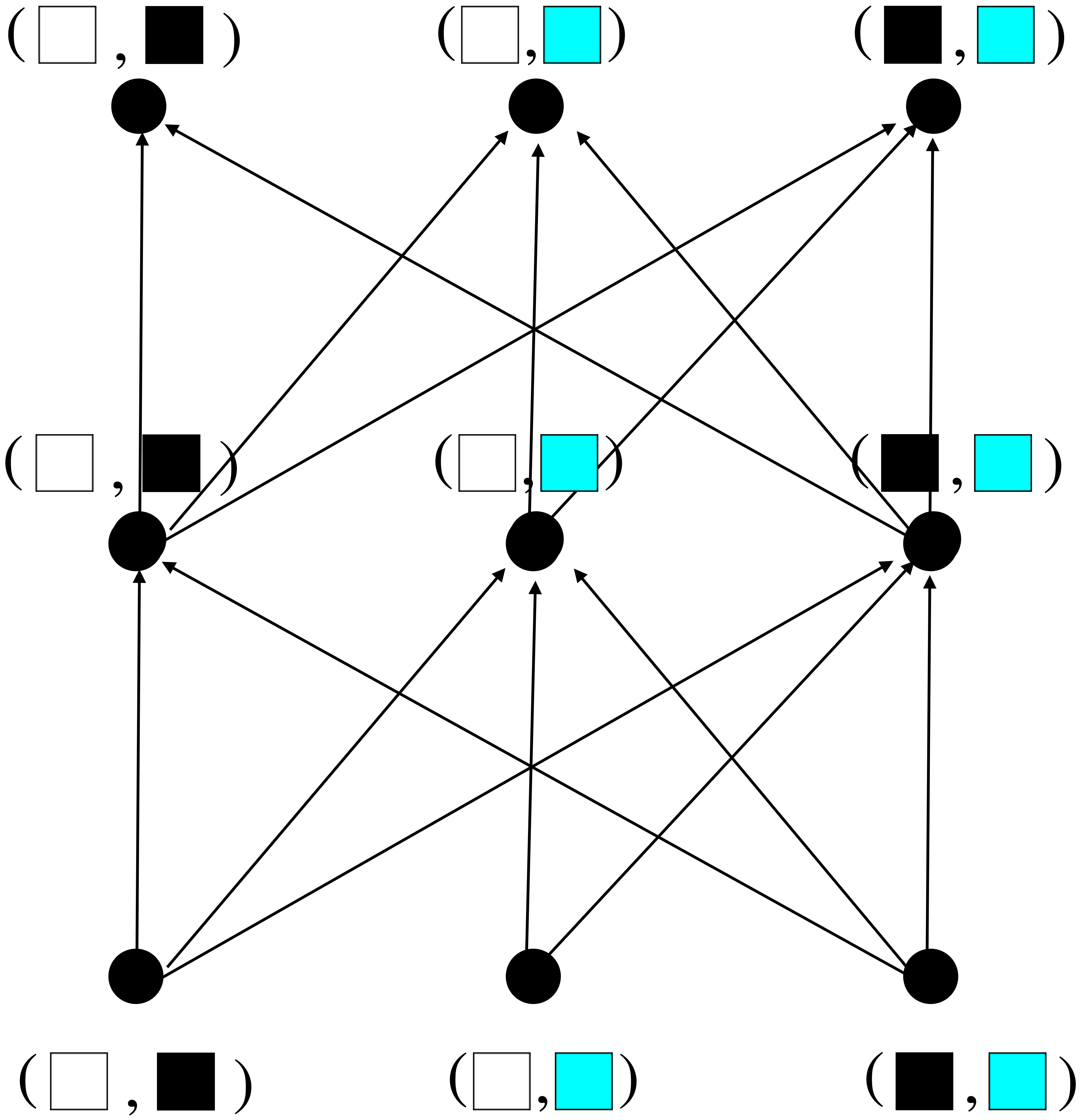


$\xi_1$(bijective)

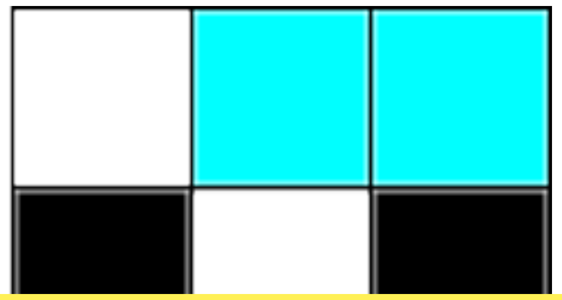


$\xi_2$(non-bijective)

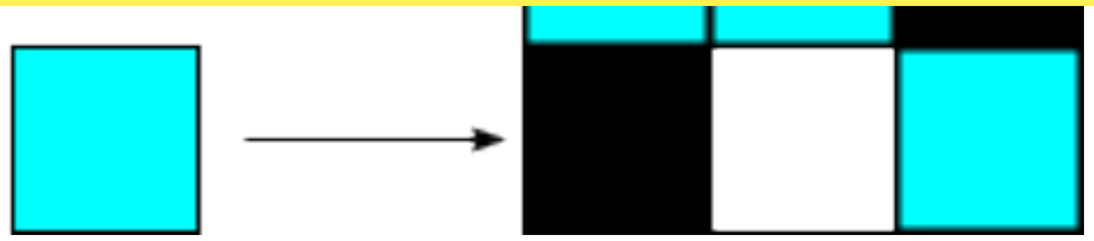


# The block cases

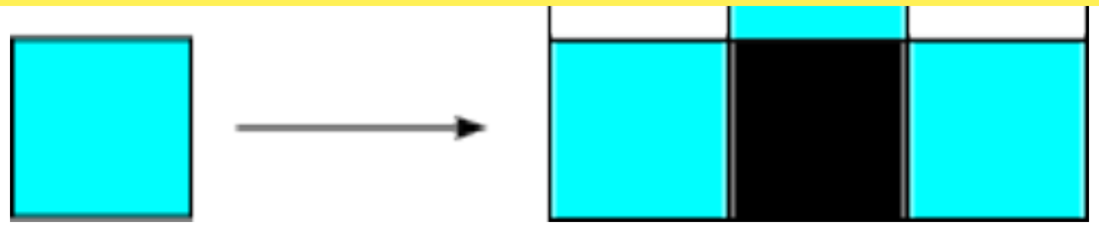
Example



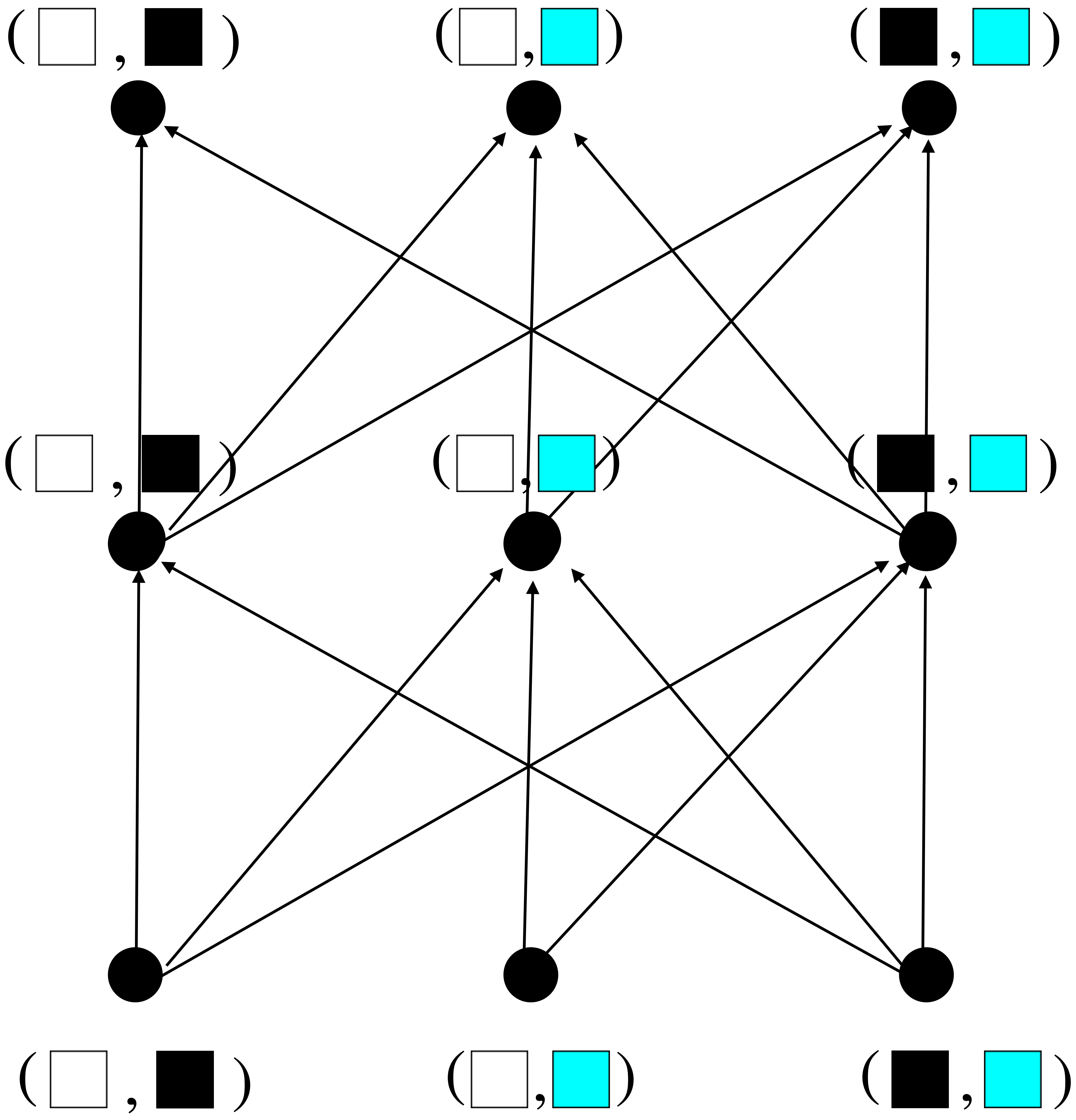
$\xi_1^2$ connects every non-coincidence overlap to every non-coincidence overlap



$\xi_1$(bijective)



$\xi_2$(non-bijective)



# A special case for the second main theorem

Assume in $(i_j)_{j=1,2,\ldots} \in \{1,2\}^{\mathbb{N}}$, both 11 and 2 appear infinitely often.

Then the S-adic tilings belonging to the directive sequence $\xi_{i_1}, \xi_{i_2}, \ldots$

have pure discrete spectrum.

# The second main theorem(N-Thuswaldner)

Let $\xi_1, \xi_2, \ldots$ be a family of all block substitutions in dimension $d$ and the number of colors $n_0$.

Pick

(1) $j_1, j_2, \ldots, j_k$ such that $\xi_{j_1} \circ \xi_{j_2} \circ \cdots \circ \xi_{j_k}$ connects every non-coincidence overlap to every non-coincidence overlap,

(2) $j_c$ such that $\xi_{j_c}$ is not bijective

Assume in $(i_j)_{j=1,2,\ldots} \in \{1,2,\ldots\}^{\mathbb{N}}$, both $j_1 j_2 \cdots j_k$ and $j_c$ appear infinitely often.

Then the S-adic tilings belonging to the directive sequence $\xi_{i_1}, \xi_{i_2}, \ldots$ have pure discrete spectra.

# The second main theorem

Corollary

Let $\mu$ be a product measure on $\{1,2,\ldots\}^{\mathbb{N}}$ such that $\mu([j_b]) > 0$ and $\mu([j_c]) > 0$ where $\xi_{j_b}$ is bijective and the columns generate the whole permutation group, and $\xi_{j_c}$ is non-bijective. Then for $\mu$-a.a.$(i_j) \in \{1,2,\ldots\}^{\mathbb{N}}$, the S-adic tilings belonging to it have pure discrete spectra.

Example

| $\xi_1$ | $\xi_2$ |
|---|---|
| $1 \to 122$ | $1 \to 122$ |
| $2 \to 231$ | $2 \to 232$ |
| $3 \to 313$ | $3 \to 323$ |

$\mu$-a.a. $(i_j) \in \{1,2\ldots\}^{\mathbb{N}}$

# The second main theorem

Corollary

Let $\mu$ be the product measure on $\{1,2,\ldots\}^{\mathbb{N}}$ such that $\mu([j_b]) > 0$ and $\mu([j_c]) > 0$ where $\xi_{j_b}$ is bijective and the columns generate the whole permutation group, and $\xi_{j_c}$ is non-bijective. Then for $\mu$-a.a.$(i_j) \in \{1,2,\ldots\}^{\mathbb{N}}$, the S-adic tilings belonging to it have pure discrete spectra.

Example

| $\xi_1$ | $\xi_2$ |
|---|---|
| $0 \to 03$ | $0 \to 02$ |
| $1 \to 12$ | $1 \to 32$ |
| $2 \to 13$ | $2 \to 01$ |
| $3 \to 02$ | $3 \to 31$ |

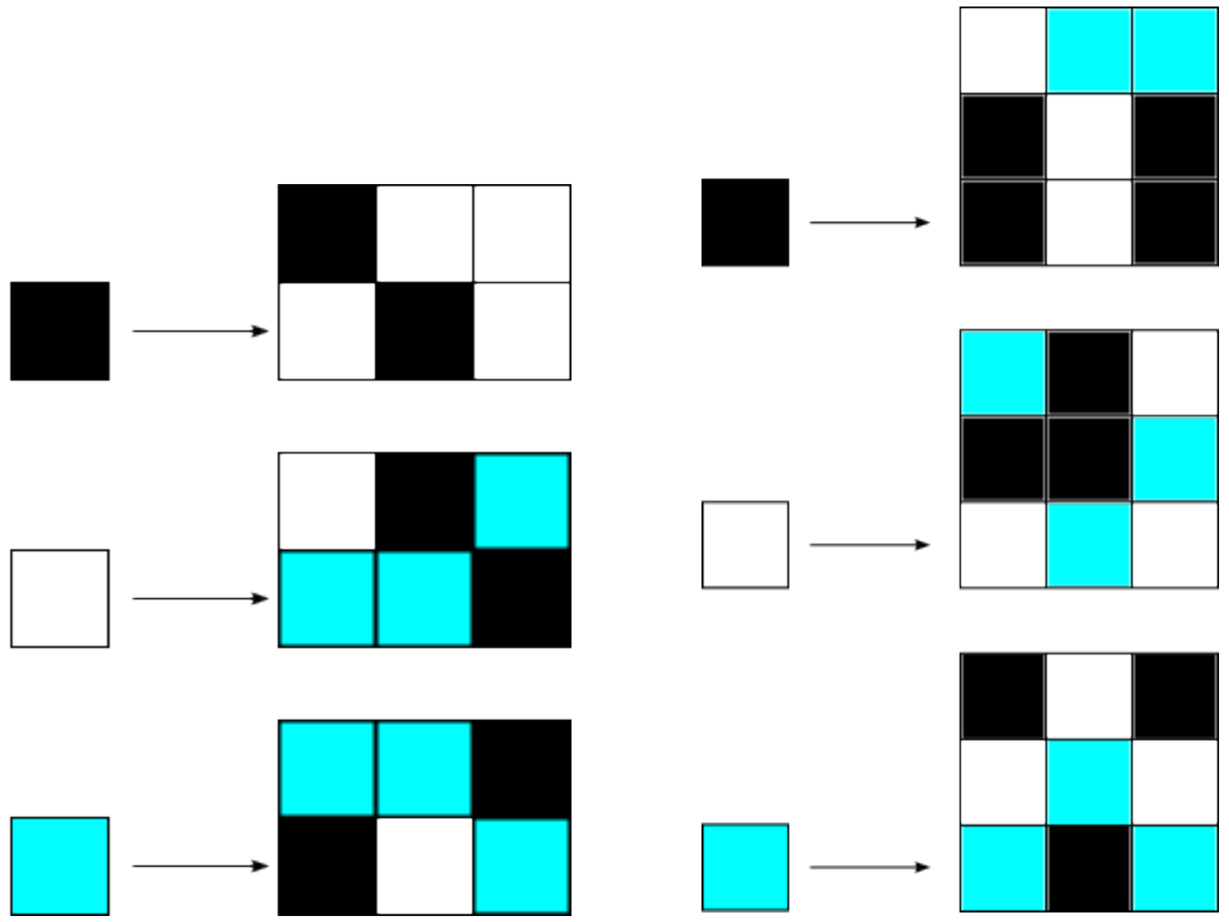
Separately yield tilings with non-zero ac part

If 12 appears infinitely often in the directive sequence→pure discrete spectrum

# A remark

- [Bustos-Mañibo-Yassawi 23+]: similar criterion for one-dimensional S-adic words

- For the 2-dimensional example $\xi_1, \xi_2$ all the tilings generated by these are non-periodic



$\xi_1$ $\xi_2$

# Further questions

- Non-block cases?

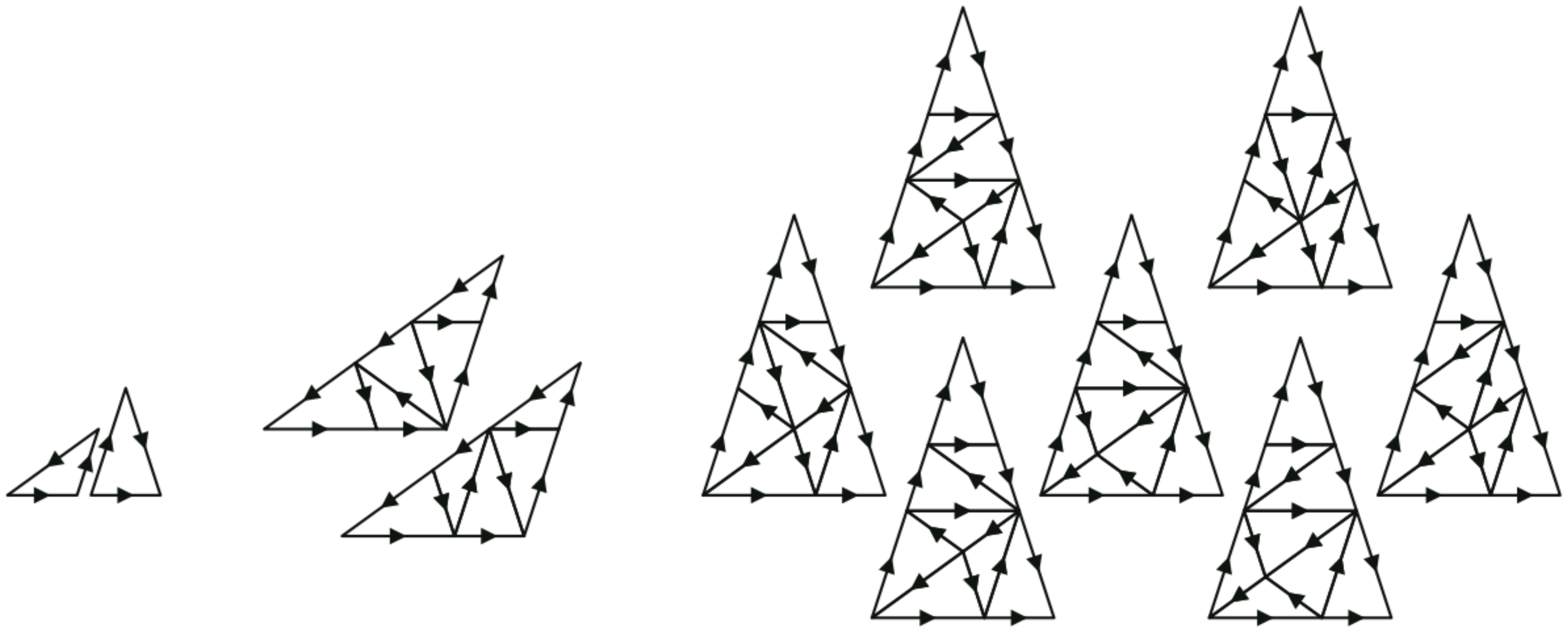


Fig. 8 Results for $S(5)$, $\lambda = 1 + a_2$

Gähler-Kwan-Maloney 2014

- The converse: pure point spectrum⇒overlap coincidence for $\Lambda_n$ = return vectors?

Thank you for your attention.

# The second main theorem

Assume in $(i_j)_{j=1,2,\ldots} \in \{1,2,\ldots\}^{\mathbb{N}}$, both $j_1 j_2 \cdots j_k$ and $j_c$ appear infinitely often.

Then the S-adic tilings belonging to the directive sequence $\xi_{i_1}, \xi_{i_2}, \ldots$

have pure discrete spectrum.

Corollary

Let $(X, \sigma, \mu)$ be an ergodic subshift of $\{1,2,\ldots\}^{\mathbb{N}}$ such that

$\mu([j_1, j_2, \ldots, j_k]) > 0$ and $\mu([j_c]) > 0$. Then for $\mu$-almost all $(i_j) \in X$, the S-adic

tilings belonging to the directive sequence $\xi_{i_1}, \xi_{i_2}, \ldots$ have pure

discrete spectrum.

# The second main theorem

Corollary

Let $(X, \sigma, \mu)$ be an ergodic subshift of $\{1,2,\ldots\}^{\mathbb{N}}$ such that $\mu([j_1, j_2, \ldots, j_k]) > 0$ and $\mu([j_c]) > 0$. Then for $\mu$-almost all $(i_j) \in X$, the S-adic tilings belonging to the directive sequence $\xi_{i_1}, \xi_{i_2}, \ldots$ have pure discrete spectrum.

Corollary

Let $\mu$ be the product measure on $\{1,2,\ldots\}^{\mathbb{N}}$ such that $\mu([j_b]) > 0$ and $\mu([j_c]) > 0$ where $\xi_{j_b}$ is bijective and the columns generate the whole permutation group, and $\xi_{j_c}$ is non-bijective. Then for $\mu$-a.a.$(i_j) \in \{1,2,\ldots\}^{\mathbb{N}}$, the same conclusion holds.

# The second main theorem(N-Thuswaldner)

Corollary

Let $(X, \sigma, \mu)$ be an ergodic subshift of $\{1,2,\ldots\}^{\mathbb{N}}$ such that $\mu([j_1, j_2, \ldots, j_k]) > 0$ and $\mu([j_c]) > 0$. Then for $\mu$-almost all $(i_j) \in X$, the S-adic tilings belonging to the directive sequence $\xi_{i_1}, \xi_{i_2}, \ldots$ have pure discrete spectrum.