Subshifts of very low complexity

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Introduction

- Today's talk is about subshifts of very low word complexity
- Main question: what can be said about subshifts whose word complexity is near minimum possible?
- Main result: such subshifts have substitutive/S-adic structure, which implies that they are very simple as dynamical systems
    - In particular, they have discrete spectrum

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- Main result: such subshifts have substitutive/S-adic structure, which implies that they are very simple as dynamical systems
  - In particular, they have discrete spectrum

Subshifts and word complexity

- A **subshift** is defined by
  - Finite set A (called the **alphabet**)
  - The **(left) shift action** σ on $A^{\mathbb{Z}}$
  - A set $X \subset A^{\mathbb{Z}}$ which is invariant under σ and closed in product topology
- Example 1: $A = \{0, 1\}$, $X = \{\ldots 000.000\ldots, \ldots 111.111\ldots\}$
- Example 2: $A = \{0, 1\}$, $X = \{0, 1\}^{\mathbb{Z}}$
- Example 3: $A = \{0, 1\}$, X a **Sturmian subshift**
  - Formal definition of Sturmian is a little technical, but an example is given by **Fibonacci sequence**:
    x = 01

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    x = 01001

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    $\mathrm{x} = \ldots .0100101001001 \ldots$
  - Define X to be set of limit points of $\{\sigma^n x \;:\; n \geq 0\}$
  - Sturmian shifts are induced by irrational circle rotations

Subshifts and word complexity

- For any subshift X, define **word complexity function** by:
- for all $n > 0$, $p(n)$ is the number of n-letter words/strings appearing within some $x \in X$
  - Example 1: $A = \{0, 1\}$, $X = \{\ldots 000.000\ldots, \ldots 111.111\ldots\}$: $p(n) = 2$
  - Example 2: $A = \{0, 1\}$, $X = \{0, 1\}^{\mathbb{Z}}$: $p(n) = 2^n$
  - Example 3: $A = \{0, 1\}$, X a **Sturmian subshift**: $p(n) = n + 1$
- The **Morse-Hedlund theorem** states that if $\exists n$ s.t. $p(n) \leq n$, then X is a finite union of periodic points
- Sturmians achieve **minimum possible** $p(n)$ among infinite subshifts
- Slowest possible growth of $p(n)$ implies 'almost' circle rotation
- What about linear growth?

Linear word complexity

- X has **strong linear complexity** if $\exists C$ s.t. $\forall n, p(n) < Cn$
  - Equivalent: $\limsup p(n)/n < \infty$
- X has **weak linear complexity** if $\liminf p(n)/n < \infty$
- Relationship complicated; in fact not only **might** $\liminf p(n)/n$, $\limsup p(n)/n$ be unequal, but they **MUST** be unequal unless an integer! (Heinis, Cassaigne)
- Even weak linear complexity strongly restricts dynamics of X
  - Basic dynamical properties for future results:
  - X is **transitive** if there exists $x \in X$ with $X = \overline{\{\sigma^n x\}}$
  - X is **minimal** if $X = \overline{\{\sigma^n x\}}$ for **every** $x \in X$
  - X is **uniquely ergodic** if only one σ-invariant measure

Some properties of subshifts with linear complexity

- **Theorem:** (Boshernitznan) If X is minimal (transitive) and has weak linear complexity, then X has only finitely many ergodic σ-invariant measures
- **Theorem:** (Dysktra, Ormes, P.) If X is transitive and has weak linear complexity, then X has only finitely many minimal subsystems
- **Theorem:** (Donoso, Durand, Maass, Petite) If X is minimal and has weak linear complexity, then X has finite topological rank
- **Theorem:** (Boshernitzan) If X is minimal (transitive) and $\limsup p(n)/n < 3$, then X is uniquely ergodic
- **Theorem:** (Ormes, P.) If X is transitive and aperiodic and $\limsup p(n)/n < 3/2$, then X is minimal
- Maybe not only finiteness, but **value** of $\limsup p(n)/n$ can be important

Linear complexity and mixing properties

- **Theorem:** (Ferenczi) If X has strong linear complexity, then X cannot support a nontrivial strongly mixing measure.
  - μ strongly mixing: $\forall A, B, \mu(A \cap \sigma^{-n} B) \to \mu(A)\mu(B)$
- **Theorem:** (Chacon) There exists uniquely ergodic X with $\limsup p(n)/n = 2$ with μ weakly mixing.
  - μ weakly mixing: $\forall A, B, \mu(A \cap \sigma^{-n} B) \to \mu(A)\mu(B)$ except for n in set of density 0
  - Equivalent: no nontrivial eigenfunctions, i.e. $f \in L^2$ with $f(\sigma x) = \lambda f(x)$ for $\lambda \neq 1$
- **Theorem:** (Ferenczi) There exists uniquely ergodic X with $\limsup p(n)/n = 5/3$ with μ weakly mixing.
- **Question:** (Ferenczi) Is $5/3$ the minimal possible $\limsup p(n)/n$ for X with weakly mixing μ?

Main results about subshifts with low linear complexity

- **Theorem:** (Creutz, P.) There exists uniquely ergodic X with $\limsup p(n)/n = 3/2$ which has μ weakly mixing.
  - Negatively answers Ferenczi's question
- **Theorem:** (Creutz, P.) If X is transitive with $\limsup p(n)/n < 4/3$ (automatically uniquely ergodic), then μ has discrete spectrum.
  - μ has discrete spectrum if L^2 is spanned by eigenfunctions
  - Equivalent to rotation of compact abelian group
  - Opposite of weak mixing
- Informally: if word complexity close enough to Sturmian, still group rotation, but possibly more complicated than a circle
- Consequence: infimum of $\limsup p(n)/n$ for X with weakly mixing μ is in $[4/3, 3/2]$

Substitutive/S-adic structure

- Key component: $\limsup p(n)/n < 4/3$ implies that X is determined by a sequence of **substitutions**
- A **substitution** from B to A is $\tau : A \to B^*$
- Example: π defined by $\pi(0) = ab, \pi(1) = acd$
- π extendable to B^* by concatenation, e.g. $\pi(001) = ababacd$
- Can compose π from B to A with τ from B to B into $\pi \circ \tau$ from B to A.
- $\tau(0) = 01, \tau(1) = 001$ gives $\pi \circ \tau : 0 \mapsto abacd, 1 \mapsto ababacd$

Substitutive/S-adic structure

- An infinite sequence of substitutions can induce a subshift
- Define $\tau_k : \{0,1\} \to \{0,1\}^*$, assume all $\tau_k(0)$ begin with 0
- Define $\rho_k = \tau_1 \circ \cdots \circ \tau_k$
- Then $\rho_{k+1}(0) = \rho_k(\tau_{k+1}(0)) = \rho_k(0)\ldots$
- Can define $x = \lim \rho_k(0)$, $X = \overline{\{\sigma^n x\}}$
- Example: if all $\tau_k = \tau : 0 \mapsto 01, 1 \mapsto 0$
- $\tau^2 : 0 \mapsto 010, 1 \mapsto 01$
- $\tau^3 : 0 \mapsto 01001, 1 \mapsto 010$
- $\tau^4 : 0 \mapsto 01001010, 1 \mapsto 01001$
- $x = .0100101001001\ldots$; Fibonacci sequence!
- In general, Sturmian comes from τ_k given by continued fraction expansion of rotation number

Substitutive/S-adic structure

- Results of Ferenczi, P.-Schmieding already imply that $\limsup p(n)/n < 2$ means X comes from a sequence $\tau_k : \{0,1\} \to \{0,1\}^*$ and $\pi : \{0,1\} \to A$
  - Need π for silly reason; alphabet of X may not be $\{0,1\}$!
  - π applied after sequence of τ_k
- But in general, only S-adic structure alone doesn't restrict X very much
- When $\limsup p(n)/n < 4/3$, we prove that τ_k are of very specific type

Substitutive/S-adic structure

- **Theorem:** (Creutz-P.) If $\limsup p(n)/n < 4/3$ and X transitive, then X generated by (π and) τ_k where every $\tau_k : 0 \mapsto 0^{m_k-1}1$, $1 \mapsto 0^{n_k-1}1$ and $0 < m_k < n_k \leq 2m_k + 1$. In addition,
  - $n_k \leq 2m_k$ unless $(m_k, n_k) = (1, 3)$
  - $(1, 3)$ can't happen if $\limsup p(n)/n < 5/4$
  - If $(m_k, n_k) = (1, 3)$, then $n_{k+1} = m_{k+1} + 1$
  - $\exists M, \epsilon$ s.t. $m_k > M \implies n_k \leq (2 - \epsilon)m_k$
- $\tau_k(0)$ may not begin with 0, but $\tau_k(1)$ ends with 1, limit x still exists
- X is Sturmian when π identity and all τ_k have $n_k = m_k + 1$
- Heuristic: The closer n_k, m_k are, the 'simpler' X is

Eigenvalues (Host's criterion)

- This substitutive structure implies that X has discrete spectrum (group rotation), but we'll just demonstrate one eigenfunction
- Technique is due to Host
- For all k, define $v_k = \rho_k(0)$ and $u_k = \rho_k(1)$ ($\rho_k = \tau_1 \circ \cdots \tau_k$)
- $x = \lim \rho_n(1) = \lim \rho_k(\tau_{k+1} \ldots)$ is concatenation of u_k and v_k
- **Host's criterion:** Suppose that there exist $\alpha \in (0, 1)$ and summable sequence ϵ_k so that every length L of a finite concatenation of u_k, v_k appearing in u_{k+1} or v_{k+1}, $\langle L\alpha \rangle < \epsilon_k$
  - $\langle x \rangle$ distance from x to nearest integer

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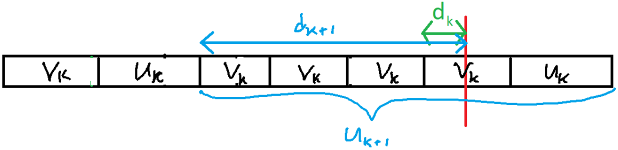
- We'll build eigenfunction with eigenvalue $\lambda := e^{2\pi i \alpha}$
- For $y \in X$, let $d_k(y) = \min\{i \ : \ \sigma^{-i} y \text{ begins with } u_k \text{ or } v_k\}$
- $d_k(\sigma y) = d_k(y) + 1$ unless σy begins with u_k or v_k
  - Exceptions have small measure for large k
- If $f_k = e^{2\pi i \alpha d_k}$, then $f_k(\sigma y) = e^{2\pi i \alpha} f_k(y) = \lambda f_k(y)$ except on set of small measure (approximate eigenfunction)
- Note that for any y, $d_{k+1}(y) - d_k(y)$ is the length of some concatenation of u_k and v_k within u_{k+1} or v_{k+1}



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- Note that for any y, $d_{k+1}(y) - d_k(y)$ is the length L of some concatenation of u_k and v_k within u_{k+1} or v_{k+1}
- $|f_k(y) - f_{k+1}(y)| = |f_k(y)(1 - e^{2\pi i \alpha L})| < \epsilon_k$ by Host's criterion
- f_k uniformly Cauchy, so converge to limit f, which must be an eigenfunction!

Eigenvalues (for low word complexity)

- Recall that in our setting, all $\tau_k : 0 \mapsto 0^{m_k - 1}1, 1 \mapsto 0^{n_k - 1}1$
- $v_{k+1} = \rho_{k+1}(0) = \rho_k(\tau_{k+1}(0)) = \rho_k(0^{m_{k+1}-1}1) = v_k^{m_{k+1}-1} u_k$
- Similarly, $u_{k+1} = v_k^{n_{k+1}-1} u_k$
- We'll outline proof of Host's criterion when $\limsup p(n)/n < 5/4$ (so $m \leq 2n$) and π identity

Eigenvalues (for low word complexity)

- $v_{k+1} = v_k^{m_{k+1}-1} u_k,\ u_{k+1} = v_k^{n_{k+1}-1} u_k$
- $0 < m_k < n_k \leq 2m_k,\ \exists M, \epsilon$ s.t. $m_k > M \implies n_k < (2-\epsilon)m_k$
- Define $\alpha = \cfrac{1}{m_1 + \cfrac{n_1 - m_1}{m_2 + \cfrac{n_2 - m_2}{m_3 + \cfrac{n_3 - m_3}{\ddots}}}}$
- Consider convergents $\frac{c_k}{d_k}$
- $\frac{c_1}{d_1} = \frac{1}{m_1},\ |v_1| = |0^{m_1 - 1}1| = m_1$
- $\frac{c_2}{d_2} = \frac{1}{m_1 + \frac{n_1 - m_1}{m_2}} = \frac{m_2}{m_1 m_2 + n_1 - m_1} = \frac{m_2}{m_1(m_2 - 1) + n_1},$

  $|v_2| = |v_1^{m_2 - 1} u_1| = m_1(m_2 - 1) + n_1$

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- $d_k = |v_k|$
- For all k, $|\frac{c_k}{d_k} - \alpha| < |\frac{c_k}{d_k} - \frac{c_{k+1}}{d_{k+1}}| = \frac{(n_1 - m_1)\cdots(n_k - m_k)}{d_k d_{k+1}}$

Eigenvalues (for low word complexity)

- $0 < m_k < n_k \leq 2m_k$, $\exists M, \epsilon$ s.t. $m_k > M \implies n_k < (2-\epsilon)m_k$
- $d_k = |v_k|$, $|\frac{c_k}{d_k} - \alpha| < \frac{(n_1-m_1)\cdots(n_k-m_k)}{d_k d_{k+1}}$
- $\langle |v_k|\alpha \rangle = |c_k - d_k\alpha| < \frac{(n_1-m_1)\cdots(n_k-m_k)}{d_{k+1}}$
- $d_{k+1} = |v_{k+1}| = |v_k^{m_{k+1}-1}u_k| > m_{k+1}|v_k| = m_{k+1}d_k$
- v_{k+1}, u_{k+1} contain any number p of consecutive v_k up to n_{k+1}
- $\langle |v_k^p|\alpha \rangle < \frac{(n_1-m_1)\cdots(n_k-m_k)n_{k+1}}{d_{k+1}} < \frac{2(n_1-m_1)\cdots(n_{k+1}-m_{k+1})}{m_1\ldots m_{k+1}}$
- Each $\frac{n_i-m_i}{m_i} \leq 1$ since $n \leq 2m$, and $m > M$ gives $\frac{n_i-m_i}{m_i} < 1-\epsilon$
- Yields exponential decay of $\langle |v_k^p|\alpha \rangle$ for $p < n_{k+1}$
- **Host's criterion:** Suppose that there exist α and summable sequence ϵ_k so that every length L of a finite concatenation of u_k, v_k within u_{k+1} or v_{k+1}, $\langle L\alpha \rangle < \epsilon_k$
- Satisfied, so X has nontrivial eigenfunction!

Summary

- We just proved existence of eigenfunction when $\limsup p(n)/n < 5/4$ with no π
- In fact π affects nothing, proof works up to $4/3$, and eigenfunctions span L^2 (discrete spectrum)
- **Theorem:** (Creutz, P.) If X is transitive with $\limsup p(n)/n < 4/3$ (automatically uniquely ergodic), then μ has discrete spectrum.
- If we take $n_k = 2m_k$ for all k and they grow quickly (i.e. the ϵ from previous proof doesn't exist), then $\limsup p(n)/n = 3/2$ and there are no nontrivial eigenvalues
- **Theorem:** (Creutz, P.) There exists uniquely ergodic X with $\limsup p(n)/n = 3/2$ which has μ weakly mixing.

Pisot/S-adic Pisot conjectures

- When π identity and all τ_k same τ, X is a **substitution subshift**
- Has associated matrix M with M_{ij} = number of i in $\tau(j)$
- Example: $\tau(0) = 001, \tau(1) = 0001 \to M = \left(\begin{smallmatrix} 2 & 3 \\ 1 & 1 \end{smallmatrix}\right)$
- **Pisot conjecture:** If largest eigenvalue λ of M is a **Pisot number**, meaning that $\lambda > 1$ and all other eigenvalues have moduli less than 1, then X has discrete spectrum.
- There's an S-adic version as well, too long to get into here. But requires at least all τ_k are Pisot
- Our $\tau : 0 \mapsto 0^{m-1}1, 1 \mapsto 0^{n-1}1$ are Pisot iff $n \leq 2m$ ($(1, 3)$ is not Pisot, but can combine with previous substitution)
- For weak mixing example, each substitution is Pisot, but 2nd eigenvalues approaching 1 quickly, so 'average behavior' (read: Lyapunov exponent) not Pisot.

Thanks for listening!