S-adic Subshifts and Finite Topological Rank Minimal Systems

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**Theorem (Herman-Putnam-Skau (92))**

Any minimal Cantor system $(X, T)$ is conjugate to a properly ordered Bratteli-Vershik system.
Bratteli-Vershik system

- Local order of edges at each vertex ($\theta_n$).

- Proper order: all the min/max edges of level $n$ have the same extremity at level $n - 1$.
Bratteli-Vershik system

element = infinite path
Bratteli-Vershik system

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$T$ maps path to the next one
(for the order)
Bratteli-Vershik system

- element = infinite path
- $T$ maps path to the next one (for the order)
- adic representation
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A-dic representation
Finite Topological Rank Minimal Systems

**Definition**

A minimal Cantor system \((X, T)\) conjugate to a Bratteli-Vershik system with a uniformly bounded number of vertices per level is said of **finite (topological) rank**.

The **rank** of \((X, T)\) is the smallest bound on the number of vertices among all the BV-representations.
Definition

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Examples:

- odometer
- Sturmian subshift
- coding of minimal Interval Exchange Transformation
- substitutive subshift, linearly recurrent subshift,...
Rigidity properties

Minimal finite rank system:
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- are either equicontinuous either expansive (hence subshifts)

Downarowicz, Maass (08)
Rigidity properties

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- have zero entropy (folklore)
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The rank of the system bounds:
- the number of ergodic invariant probability measure
  see Bezugly, Kwiatkowski, Medynets, Solomyak (13)
- the rational rank of the dimension group
  Giordano, Putnam, Skau, Handelman, Hosseini
- the rational rank of the continuous spectrum of the system
  Bressaud, Durand, Maass
Q. : Provide a practical characterization of finite rank minimal systems.
Main results

A combinatorial characterization of expansive finite rank minimal systems
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**Theorem (DDMP (20))**

A minimal subshift \((X, T)\) has a finite rank if and only if the following limit is finite

\[
\lim_{n \to +\infty} \inf_{W \subseteq \bigcup_{k \geq n} \mathcal{L}_k(X)} |W|.
\]

\(\mathcal{L}_k(X)\): set of words of length \(k\) in \(X\)

Recognizability results Mossé (92), Karhumäki (02), Berthé-Steiner-Thuswaldner-Yassawi (19)
Consequences

Corollary

A minimal subshift \((X, T)\) with a non-superlinear complexity, i.e \(\liminf_{n \to \infty} p_X(n)/n < +\infty\), has a finite rank.

where \(p_X(n) = |\mathcal{L}_n(X)|\).
Corollary

A minimal subshift \((X, T)\) with a non-superlinear complexity, i.e \(\liminf_{n \to \infty} \frac{p_X(n)}{n} < +\infty\), has a finite rank.

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Converse is false.
Consequences

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A minimal subshift \((X, T)\) with a non-superlinear complexity, i.e.
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Converse is false.

The proof use deconnectability properties of the Rauzy graphs.
See Ferenczi 96, Monteil.

Return words of special words form a recognizable family.
\(\mathcal{A}, \mathcal{B}\) finite alphabets, \(\tau: \mathcal{A}^* \rightarrow \mathcal{B}^*\) a non-erasing morphism, 
\(\mathcal{Y} \subset \mathcal{A}^\mathbb{Z}\) be a subshift.
\( \mathcal{A}, \mathcal{B} \) finite alphabets, \( \tau : \mathcal{A}^* \rightarrow \mathcal{B}^* \) a non-erasing morphism, \( Y \subset \mathcal{A}^\mathbb{Z} \) be a subshift.

The morphism \( \tau : \mathcal{A}^* \rightarrow \mathcal{B}^* \) is recognizable in \( Y \), if

1. \( X \) denotes the subshift generated by \( \tau(Y) \);
2. for any \( x \in X \), there is a unique \( (k, y) \in \mathbb{N} \times Y \), s.t. \( x = T^k \tau(y) \) and \( 0 \leq k < |\tau(y_0)| \) (centered representation).
Definition of relative recognizability, similar [BSTY19]

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**Ex:** $\tau : a \mapsto 01; b \mapsto 0$, $Y = \text{Fibonacci subshift}$
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\[ x = \cdots 0100.1010010100100100 \cdots \]
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\[x = \cdots 01|0|0.1|01|0|01|01|0|01|0|0\cdots\]
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x = \cdots 01 | 0 | 0.1 | 01 | 0 | 01 | 01 | 0 | 01 | 0 | 0 \cdots
\]

- $\tau(a)$
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Definition of relative recognizability, similar [BSTY19]
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\[ \tau(a) \mid \tau(b) \mid \tau(a) \mid \tau(b) \mid \tau(a) \mid \tau(b) \mid \tau(a) \mid \tau(b) \mid \tau(a) \mid \tau(b) \]

\[ x = T\tau(\cdots ab.aabaababa \cdots) \]
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  \[x = T^k \tau(y)\] and \(0 \leq k < |\tau(y_0)|\) \textit{ (centered representation)}.

[BSTY19]: When moreover \(X\) is an aperiodic subshift, there is a \(R > 0\) s.t. if \(y, y' \in Y\), \(0 \leq k < |\tau(y_0)|, 0 \leq k' < |\tau(y'_0)|\)

\[
T^k \tau(y)|_{(-R, R)} = T^{k'} \tau(y')|_{(-R, R)}
\]

then \(y_0 = y'_0\) and \(k = k'\).
A, B finite alphabets, \( \tau : A^* \to B^* \) a non-erasing morphism, \( Y \subset A^\mathbb{Z} \) be a subshift.

The morphism \( \tau : A^* \to B^* \) is recognizable in \( Y \), if
- \( X \) denotes the subshift generated by \( \tau(Y) \);
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A finite set \( \mathcal{W} \subset B^* \) is recognizable in a subshift \( X \) if there are a morphism \( \tau : A^* \to B^* \) and a subshift \( Y \subset A^\mathbb{Z} \), s.t.
- \( \tau \) is recognizable in \( Y \);
- \( X \) is the subshift generated by \( \tau(Y) \);
- \( \tau(A) = \mathcal{W} \).
Main results

Theorem (DDMP (20))

A minimal subshift \((X, T)\) has a finite rank if and only if the following limit is finite

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\(\mathcal{L}_k(X)\): set of words of length \(k\) in \(X\)

\(W\) is recognizable in \(X\).
Relations with S-adic subshifts

\( \mathcal{A}, \mathcal{B} \) finite alphabets.
A positive morphism \( \tau: \mathcal{A}^* \rightarrow \mathcal{B}^* \) is a morphism such that any letters \( a \in \mathcal{A}, b \in \mathcal{B}, b \) appears in \( \tau(a) \).
$\mathcal{A}$, $\mathcal{B}$ finite alphabets.

A positive morphism $\tau: \mathcal{A}^* \rightarrow \mathcal{B}^*$ is a morphism such that any letters $a \in \mathcal{A}$, $b \in \mathcal{B}$, $b$ appears in $\tau(a)$.

A primitive S-adic subshift is the orbit closure for the shift action of points of the form

$$\lim_{n \rightarrow +\infty} \tau_0 \circ \cdots \circ \tau_n(a_n^\infty),$$

for a fixed sequence of morphisms $(\tau_n: \mathcal{A}_{n+1}^* \rightarrow \mathcal{A}_n^*)_n$, s.t.

$$\forall n \in \mathbb{N}, \exists N > n, \quad \tau_n \circ \tau_{n+1} \circ \cdots \circ \tau_N \text{ is positive}.$$
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**Theorem (Espinoza, Golestani-Hosseini 20)**

Let \((X, T)\) be a minimal aperiodic subshift. The system \((X, T)\) is of finite rank \(\iff\) it is conjugate to a primitive S-adic subshift with \(\lim \inf_n |A_n| < +\infty\).
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Durand-Leroy 12
- Local order of edges at each vertex \((\theta_n)\).

\[\tau_1 : a \mapsto cba \quad \tau_2 : a \mapsto abc\]
\[b \mapsto cbca \quad b \mapsto abc\]
\[c \mapsto cba \quad c \mapsto abbc\]

BV conjugate to the S-adic system given by \((\tau_n)_{n \geq 1}\)
Theorem (Espinoza, Golestani-Hosseini 20)

Let \((X, T)\) be a minimal aperiodic subshift. The system \((X, T)\) is of finite rank \(\iff\) it is conjugate to a primitive \(S\)-adic subshift with \(\lim \inf_n |A_n| < +\infty\).

Durand-Leroy 12 \(\Rightarrow\)

[BSTY19] similar result in the measurable context + recognizability conditions.
Theorem (Espinoza, Golestani-Hosseini 20)

Let $(X, T)$ be a minimal aperiodic subshift. The system $(X, T)$ is of finite rank $\iff$ it is conjugate to a primitive $S$-adic subshift with $\lim \inf_n |A_n| < +\infty$.

Durand-Leroy 12 $\implies$

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[DDMP20] $\iff$ $+$ recognizability conditions.
Relations with S-adic subshifts

Theorem (Espinoza, Golestani-Hosseini 20)

Let \((X, T)\) be a minimal aperiodic subshift. The system \((X, T)\) is of finite rank \(\Leftrightarrow\) it is conjugate to a primitive S-adic subshift with \(\lim \inf_n |A_n| < +\infty\).

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[BSTY19] similar result in the measurable context + recognizability conditions.

[DDMP20] \(\Leftarrow\) + recognizability conditions.

Theorem (Espinoza, Golestani-Hosseini 20)

Let \((X, T)\) be a minimal Cantor system of finite rank. Then any minimal Cantor system \((Y, S)\) factor of \((X, T)\) is of finite rank.
More rigidity results

Proposition (DDMP 20)

A minimal Cantor system of rank 2 has only one asymptotic component.

An asymptotic component is a set of all the orbits containing asymptotics points (i.e. points $x \neq y$ s.t. $x(-\infty,0) = y(-\infty,0)$).

Ex: the Prouhet-Thue-Morse subshift is of rank at least 3.
More rigidity results

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Corollary (DDMP 20)

A minimal Cantor system \((X, T)\) of rank 2 has a trivial automorphism group: \(\text{Aut}(X, T) = \langle T \rangle\).
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Espinoza Maass 20: For a minimal Cantor subshift of finite rank $\text{Aut}(X, T)/\langle T \rangle$ is finite
Classical examples of finite rank system (I.E.T, Substitutive,...) have sublinear complexity : $p_x(n) \in O(n)$. 
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Any finite rank system has zero entropy.
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Any minimal Cantor system $(X, T)$ of finite rank is strongly orbit equivalent to a subshift of sublinear complexity.
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**Proposition (DDMP 20)**

*Any minimal Cantor system $(X, T)$ of finite rank is strongly orbit equivalent to a subshift of sublinear complexity*

S-adic system has sublinear complexity with morphisms of the form $\tau: A \rightarrow \{b_1, \cdots, b_p\}^*$

$$\forall a \in A, \quad \tau(a) = b_1^{\ell_1(a)} \cdots b_p^{\ell_p(a)}$$

for some $\ell_1(a), \cdots, \ell_p(a) \in \mathbb{N}$. 
Classical examples of finite rank system (I.E.T, Substitutive,...) have sublinear complexity: \( p_x(n) \in \mathcal{O}(n) \).

Any finite rank system has zero entropy.

**Proposition (DDMP 20)**

Let \( (X, T) \) be a S-adic subshift generated by a positive directed sequence \( (\tau_n : A_{n+1}^* \rightarrow A_n^*)_{n \geq 0} \). If \( \lim \inf_n |A_n| \leq 2 \), then the complexity of \( X \) is sub-quadratic along a subsequence, i.e.

\[
\liminf_{n \rightarrow +\infty} \frac{p_X(n)}{n^2} = 0.
\]
Relation with the complexity

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$$\liminf_{n \to +\infty} \frac{p_X(n)}{n^2} = 0.$$ 

For any subexponential function $\varphi : \mathbb{N} \to \mathbb{R}$ (i.e $\limsup_n \varphi(n)/\alpha^n = 0, \forall \alpha > 1$), there exists S-adic subshift $(X, T)$ on 2-letters alphabet s.t.

$$\limsup_n \frac{p_X(n)}{\varphi(n)} > 0.$$
Open questions

Is the topological rank computable (for effective $S$-adic)?
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Is the topological rank computable (for effective S-adic)?

Question

For a finite rank S-adic, does there exists \( d = d(\text{rank}) \) s.t.

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Open questions

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For a finite rank S-adic, does there exist $d = d(\text{rank})$ s.t.

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**Question**

Let $(X, T)$ be a Toeplitz subshift. Is it true that $(X, S)$ has a finite topological rank $\iff$ the complexity of $X$ is non-superlinear?
Open questions

Is the topological rank computable (for effective S-adic)?

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For a finite rank S-adic, does there exists \( d = d(\text{rank}) \) s.t.

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**Question**

Let \((X, T)\) be a Toeplitz subshift. Is it true that \((X, S)\) has a finite topological rank \(\iff\) the complexity of \(X\) is non-superlinear?

**Question**

Let \((X, T)\) be a finite rank subshift. Can \((X, T)\) be mixing for an invariant measure \(\mu\)?