S-adic Subshifts and Finite Topological Rank Minimal Systems

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Finite Topological Rank Minimal Systems

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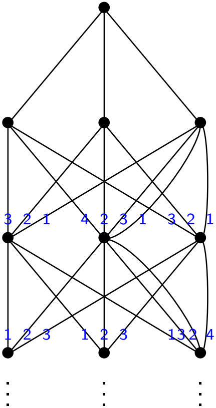
Theorem (Herman-Putnam-Skau (92))

Any minimal Cantor system (X, T) is conjugate to a properly ordered Bratteli-Vershik system.

Bratteli-Vershik system



- Local order of edges at each vertex (θ_n).

- Proper order: all the min/max edges of level n have the same extremity at level $n-1$

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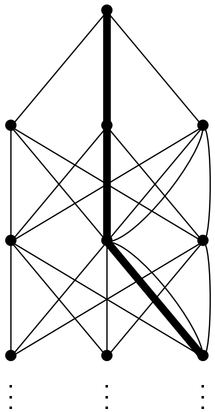
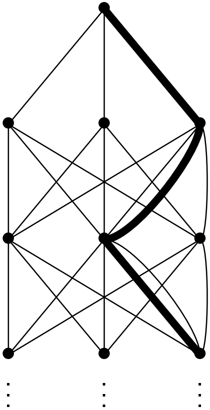


element = infinite path

Bratteli-Vershik system



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T maps path to the next one
(for the order)

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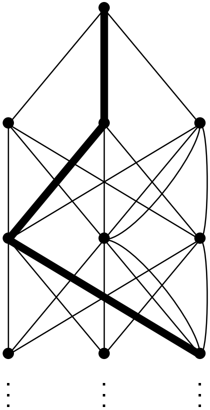

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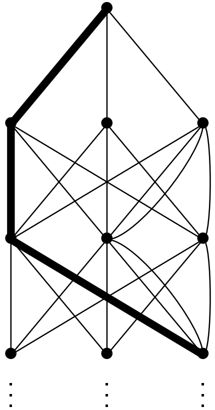

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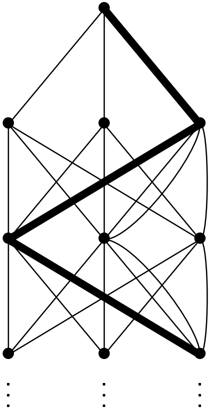

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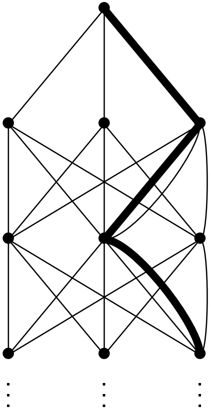

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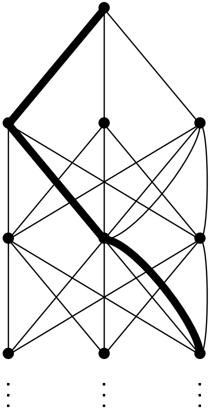

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A-dic representation

Finite Topological Rank Minimal Systems

Definition

A minimal Cantor system (X, T) conjugate to a Bratteli-Vershik system with a uniformly bounded number of vertices per level is said of finite (topological) rank.

The rank of (X, T) is the smallest bound on the number of vertices among all the BV-representations.

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Examples:

- odometer
- Sturmian subshift
- coding of minimal Interval Exchange Transformation
- substitutive subshift, linearly recurrent subshift,...

Rigidity properties

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The rank of the system bounds:

- the number of ergodic invariant probability measure
  see Bezugly, Kwiatkowski, Medynets, Solomyak (13)
- the rational rank of the dimension group
  Giordano,Putnam, Skau, Handelman, Hosseini
- the rational rank of the continuous spectrum of the system
  Bressaud, Durand, Maass

Main results

Q. : Provide a practical characterization of finite rank minimal systems.

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A combinatorial characterization of expansive finite rank minimal systems

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Theorem (DDMP (20))

A minimal subshift (X, T) has a finite rank if and only if the following limit is finite

$$\lim_{n \to +\infty} \inf_{\substack{\mathcal{W} \subset \cup_{k \geq n} \mathcal{L}_k(X) \\ \mathcal{W} \text{ is recognizable in } X}} |\mathcal{W}|.$$

$\mathcal{L}_k(X)$: set of words of length k in X

Recognizability results Mossé (92), Karhumäki (02), Berthé-Steiner-Thuswaldner-Yassawi (19)

Consequences

Corollary

A minimal subshift (X, T) *with a non-superlinear complexity, i.e* $\liminf_{n\to\infty} p_X(n)/n < +\infty$*, has a finite rank.*

where $p_X(n) = |\mathcal{L}_n(X)|$.

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Converse is false.

The proof use deconnectability properties of the Rauzy graphs.
See Ferenczi 96, Monteil.

Return words of special words form a recognizable family.

Definition of relative recognizability, similar [BSTY19]

$\mathcal{A}$, $\mathcal{B}$ finite alphabets, $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ a non-erasing morphism,
$Y \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift.

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- X denotes the subshift generated by $\tau(Y)$;
- for any $x \in X$, there is a unique $(k, y) \in \mathbb{N} \times Y$, s.t.
  $x = T^k\tau(y)$ and $0 \leq k < |\tau(y_0)|$ (centered representation).

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$$x = T\tau(\cdots \mathtt{ab.aabaababa} \cdots)$$

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[BSTY19]: When moreover X is an aperiodic subshift, there is a $R > 0$ s.t. if $y, y' \in Y$, $0 \leq k < |\tau(y_0)|$, $0 \leq k' < |\tau(y'_0)|$

$$T^k\tau(y)|_{(-R,R)} = T^{k'}\tau(y')|_{(-R,R)}$$

then $y_0 = y'_0$ and $k = k'$.

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A finite set $\mathcal{W} \subset \mathcal{B}^*$ is recognizable in a subshift X if there are a morphism $\tau\colon \mathcal{A}^* \to \mathcal{B}^*$ and a subshift $Y \subset \mathcal{A}^{\mathbb{Z}}$, s.t.

- τ is recognizable in Y;
- X is the subshift generated by $\tau(Y)$;
- $\tau(\mathcal{A}) = \mathcal{W}$.

Main results

Theorem (DDMP (20))

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Relations with S-adic subshifts

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A primitive S-adic subshift is the orbit closure for the shift action of points of the form

$$\lim_{n \to +\infty} \tau_0 \circ \cdots \circ \tau_n(a_n^\infty),$$

for a fixed sequence of morphisms $(\tau_n\colon \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*)_n$, s.t.

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Let (X, T) be a minimal aperiodic subshift.
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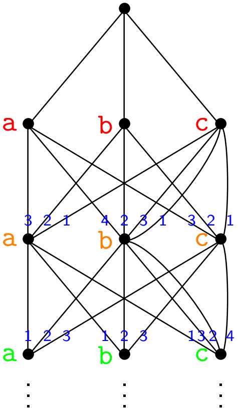
Strategy of proof



- Local order of edges at each vertex (θ_n).

τ_1 : a $\mapsto$ cba
b $\mapsto$ cbca
c $\mapsto$ cba

τ_2 : a $\mapsto$ abc
b $\mapsto$ abc
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BV conjugate to the S-adic system given by $(\tau_n)_{n\geq 1}$

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Theorem (Espinoza, Golestani-Hosseini 20)

Let (X, T) be a minimal Cantor system of finite rank. Then any minimal Cantor system (Y, S) factor of (X, T) is of finite rank.

More rigidity results

Proposition (DDMP 20)

A minimal Cantor system of rank 2 has only one asymptotic component.

An asymptotic component is a set of all the orbits containing asymptotics points (*i.e.* points $x \neq y$ s.t. $x_{(-\infty,0)} = y_{(-\infty,0)}$)

Ex: the Prouhet-Thue-Morse subshift is of rank at least 3.

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Espinoza Maass 20: For a minimal Cantor subshift of finite rank $\mathrm{Aut}(X, T)/\langle T \rangle$ is finite

Relation with the complexity

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S-adic system has sublinear complexity with morphisms of the form $\tau \colon \mathcal{A} \to \{\mathtt{b}_1, \cdots, \mathtt{b}_p\}^*$

$$\forall \mathtt{a} \in \mathcal{A}, \quad \tau(\mathtt{a}) = \mathtt{b}_1^{\ell_1(\mathtt{a})} \cdots \mathtt{b}_p^{\ell_p(\mathtt{a})}$$

for some $\ell_1(\mathtt{a}), \cdots, \ell_p(\mathtt{a}) \in \mathbb{N}$.

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For any subexponential function $\varphi \colon \mathbb{N} \to \mathbb{R}$ (i.e $\limsup_n \varphi(n)/\alpha^n = 0, \forall \alpha > 1$), there exists S-adic subshift (X, T) on 2-letters alphabet s.t.

$$\limsup_n p_X(n)/\varphi(n) > 0.$$

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Let (X, T) be a finite rank subshift. Can (X, T) be mixing for an invariant measure μ ?