Complex dimensions and fractal strings

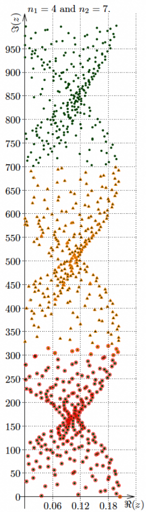
(A confection of Iterated function schemes, zeta functions and nice pictures)

Mark Pollicott

Warwick University

September, 2023

Overview



(I) The setting:

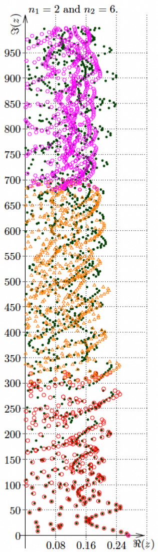
- We associate to a Cantor set $X \subset \mathbb{R}$ a complex function.
- The poles of this function in $\mathbb{C}$ (when they exist) are the complex dimensions.

(II) Iterated function schemes:

- An extension exists when the Cantor set is an attractor for a C^2 IFS (using "thermodynamic formalism" ideas).
- Interpret poles of the complex function as *complex dimensions* of X.

(III) Location of poles

- Plot interesting pictures for specific examples (related to continued fraction): P. Vytnova, J. Slipantschuk, Angxiu Ni. *Figures in these slides are all due to Polina Vytnova.*

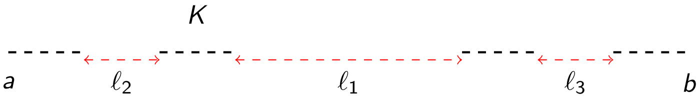


Starting at the beginning (a very good place to start)

(I) The setting

Cantor sets and their gaps

Consider a Cantor set $K \subset \mathbb{R}$ in the real line.



We denote $a = \inf(K)$ and $b = \sup(K)$.

Definition

Let $\mathcal{L} = \{\ell_j\}_{j \geq 1}$ be the lengths of the countable family of maximal bounded intervals in the compliment of the Cantor set: $[a, b] \setminus K$.

We assume an ordering of the lengths of the intervals such that

$$\ell_1 \geq \ell_2 \geq \ell_3 \geq \cdots \geq \ell_n \geq \cdots$$

A complex function

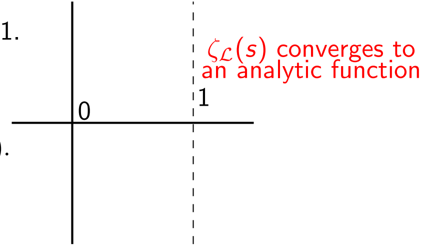
Definition

We formally define a *zeta function* $\zeta_{\mathcal{L}}(s)$ to be the function of a single complex variable formally given by the Dirichlet series

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s \text{ for } s \in \mathbb{C}.$$

- The series $\zeta_{\mathcal{L}}(s)$ converges for $Re(s) > 1$. This is because then

$$|\zeta_{\mathcal{L}}(s)| = \left|\sum_{j=1}^{\infty} \ell_j^s\right| \leq \sum_{j=1}^{\infty} \ell_j \leq (b-a).$$



The simplest simple example

Example (Middle third Cantor set)

An easy example is the middle third Cantor set where

$$\mathcal{L} = \left\{ \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{27}, \frac{1}{81}, \dots \right\}$$

and provided $Re(s) > \frac{\log 2}{\log 3}$:

$$\begin{aligned} \zeta_{\mathcal{L}}(s) &= \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{sn}} \\ &= \frac{1}{3^s} \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{s(n-1)}} = \frac{1}{3^s(1 - \frac{2}{3^s})} \\ &= \frac{1}{3^s - 2}. \end{aligned}$$

Futhermore, $Re(s) > D := \frac{\log 2}{\log 3}$ is the largest half plane on which $\zeta_{\mathcal{L}}(s)$ is analytic.

More generally ...

Given any Cantor set K with interval lengths $\mathcal{L} = \{\ell_j\}$ and a zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s :$$

- The abscissa of convergence of $\zeta_{\mathcal{L}}$ is denoted

$$D = \inf\{\sigma > 0 : \zeta_{\mathcal{L}}(\sigma) < +\infty\}.$$

  Then $\zeta_{\mathcal{L}}$ converges to an analytic function on $Re(s) > D$.

Example (Middle third Cantor set)

When the Cantor set is the middle third Cantor set: $D = \frac{\log 2}{\log 3}$.

- In fact, $D = \overline{\dim}_B(X)$ can be interpreted as the (upper) box dimension of K (by work of Lapidus).

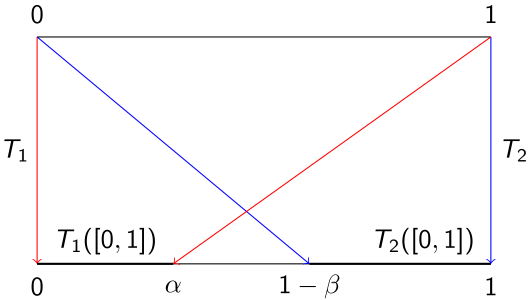
We can generalize this single example in a simple way.

(II) Iterated function schemes

The simplest case: Affine Cantor sets

Let $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$ and consider the affine maps $T_1 : [0,1] \to [0,1]$ and $T_2 : [0,1] \to [0,1]$ defined by

$$T_1(x) = \alpha x \text{ and } T_2(x) = \beta x + (1 - \beta).$$

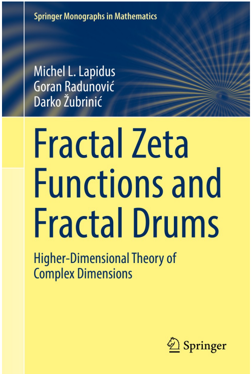


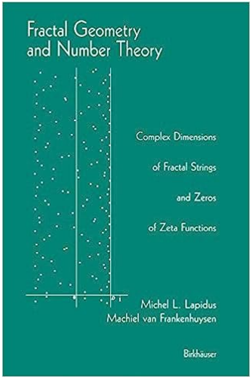
Definition

The associated Cantor set K (i.e., the limit set or attractor, after Hutchinson) is the unique non-empty compact set such that $K = T_1(K) \cup T_2(K)$.

For example, with $\alpha = \beta = \frac{1}{3}$ one recovers the middle third Cantor set.

Lapidus and $\zeta_{\mathcal{L}}(s)$ poles for affine Cantor sets





Lapidus wrote several books on this topic in which he described $\mathcal{L}$ and $\zeta_{\mathcal{L}}$ for affine Cantor sets and studied the meromorphic extension of $\zeta_{\mathcal{L}}(s)$ past the line $Re(s) = D$. The location of the poles in the meromorphic extension he called *complex dimensions*. We can consider a couple of affine examples...

Two affine examples: Poles for $\zeta_{\mathcal{L}}(s)$

Lapidus and Frankenhuysen plotted the following.

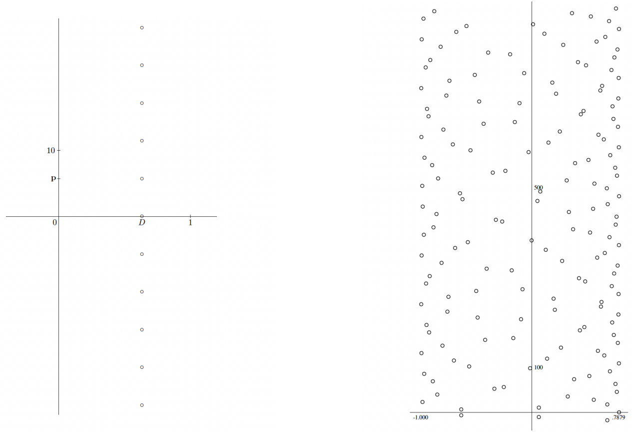


Figure: (a) Poles when $\alpha = \beta = \frac{1}{3}$ (Middle third Cantor set); and (b) Poles when $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{2}$

Location of poles of $\zeta_{\mathcal{L}}(s)$ in the case of affine contractions

Theorem (Lapidus)

(a) For K given by affine contractions

$$T_1(x) = \alpha x \text{ and } T_2(x) = \beta x + (1 - \beta)$$

the function $\zeta_{\mathcal{L}}(s)$ has a meromorphic extension to $\mathbb{C}$.
(b) Moreover, the poles s for $\zeta_{\mathcal{L}}(s)$ are given by the solutions to

$$\alpha^s + \beta^s = 1.$$

Properties of the poles. As a corollary, there are two cases:

- If $\frac{\log \alpha}{\log \beta} \in \mathbb{Q}$ then the poles for $\zeta_{\mathcal{L}}(s)$ are periodic in the imaginary direction.
- If $\frac{\log \alpha}{\log \beta} \in \mathbb{R} \setminus \mathbb{Q}$ then there are poles for $\zeta_{\mathcal{L}}(s)$ in $Re(s) < D$ that accumulate on the line $Re(s) = D$ (i.e., poles $s_n = \sigma_n + it_n$ with $\sigma_n \nearrow D$ as $n \to +\infty$).

These two cases correspond to the two pictures on the previous slide.

Example: $D = \dim_B(K)$ as a complex dimension

The first "complex dimension" is the box counting dimension $D = \dim_B(K)$.

Theorem (Moran's Theorem)

The value $D = \dim_B(K)$ *is a solution to*

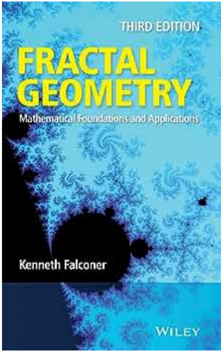
$$\alpha^D + \beta^D = 1.$$

In childhood Pat Moran suffered an attack of appendicitis late one night after going out with his parents for a meal. He was operated on too soon after the meal. He vomited on the operating table and the surgeon had to do a tracheotomy to let him breathe, cutting his vocal cords in the process. This gave him a husky voice for the rest of his life.

During World War II, Moran worked on rockets but also wrote papers on fractal dimensions. After the war his attempt to complete a PhD at Cambridge was unsuccessful.

A nice summary of complex dimensions in the affine case

The book *Fractal Geometry* has a nice account of the results of Lapidus et al for *complex dimensions* for affine Cantor sets. (I used this in my undergraduate course on *Fractal Geometry* this Spring)



How can we try to generalize these results?

More generally: C^2 Iterated function schemes

Let $T_1, T_2 : [0,1] \to [0,1]$ be C^2 maps such that

1. $\sup_x |T_1'(x)| < 1$ and $\sup_x |T_2'(x)| < 1$ (Contractions);
2. $T_1([0,1]) \cap T_2([0,1]) = \emptyset$ (Disjoint images : strong separation condition).

Definition

The associated Cantor set K (i.e., the limit set or attractor, after Hutchinson) is the unique non-empty compact set such that $K = T_1(K) \cup T_2(K)$.

We can recover by taking affine contractions the affine Cantor set.

Example (Recall for affine contractions)

Let $0 < \alpha, \beta < 1$ with $\alpha + \beta < 1$ and

$$T_1(x) = \alpha x \text{ and } T_2(x) = \beta x + (1 - \beta).$$

These are eaily seen to satisfy (1) and (2) above.

A non-affine exmple

Recall the Gauss map $G : [0,1] \to [0,1]$ by

$$G(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \end{cases}$$

The inverse branches (more or less) are the maps $T : [0,1] \to [0,1]$ given by

$$T(x) = \frac{1}{x+n}, \text{ for } n \in \mathbb{N}.$$

Example

Two inverse branches for the Gauss map:
Fix $n, m \in \mathbb{N}$ $(n \neq m)$ then we can define $T_1, T_2 : [0,1] \to [0,1]$ by

$$T_1(x) = \frac{1}{x+n} \text{ and } T_2(x) = \frac{1}{x+m}.$$

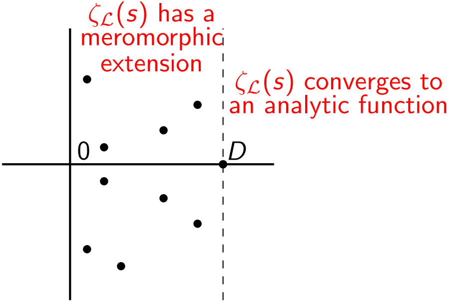
The associated attractor $K(= T_1(K) \cup T_2(K))$ is an affine Cantor set.

Complex dimensions in the general case

What happens if we have a Cantor set associated to general C^2 contractions? Can we still extend $\zeta_{\mathcal{L}}(s)$ (and interpret the poles as complex dimensions)?

Theorem

Let K be the Cantor set associated to a C^2 iterated function scheme. Then the associated zeta function $\zeta_{\mathcal{L}}(s)$ has a meromorphic extension from $Re(s) > D$ to the larger half-plane $Re(s) > 0$.



The dynamical content is in the method: Writing $\zeta_{\mathcal{L}}(s)$ in terms of complex transfer operators (weighted using s) and an old result of T. Morita.

A little "thermodynamical formalism": Transfer Operators

Consider the Banach space of Lipschitz functions

$$C^{Lip}([0,1]) = \{f : [0,1] \to \mathbb{R} : \|f\|_{Lip} < +\infty\}$$

with norm $\|f\| = \|f\|_{Lip} + \|f\|_\infty$ where

$$\|f\|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|} \text{ and } \|f\|_\infty = \sup_x |f(x)|.$$

Definition

We define the family of linear *transfer operators*

$$L_s : C^{Lip}([0,1]) \to C^{Lip}([0,1]) \qquad (Re(s) > 0)$$

by

$$L_s f(x) = |T_1'(x)|^s f(T_1 x) + |T_2'(x)|^s f(T_2 x).$$

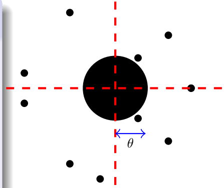
Spectrum of the transfer operators

Lemma (after Morita)

$\exists\ 0 < \theta < 1$, $\forall Re(s) > 0$ spectrum of

$$L_s : C^{Lip}([0,1]) \to C^{Lip}([0,1])$$

has isolated eigenvalues in $|z| > \theta$.



We can then write $\zeta_{\mathcal{L}}(s)$ in terms of the resolvent $(I - L_s)^{-1}$.
In particular,

1. $\zeta_{\mathcal{L}}(s)$ has a meromorphic extension to $Re(s) > 0$; and
2. s in this domain is a pole for $\zeta_{\mathcal{L}}(s)$ if and only if 1 is an eigenvalue for L_s.

So given the meromorphic extension of $\zeta_{\mathcal{L}}(s)$ ... what do the poles look like?

(III) Location of poles

Continued fraction contractions

We want to consider some empirical results so we want to consider the (next) simplest setting.

Given distinct $n, m \in \mathbb{N}$ we can consider Cantor sets K associated to contractions

$$T_1(x) = \frac{1}{n+x} \text{ and } T_2(x) = \frac{1}{m+x},$$

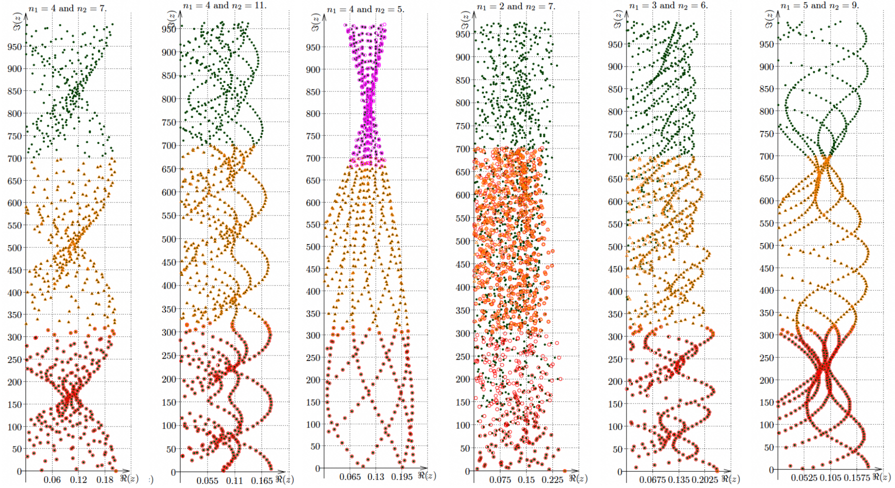
i.e., K is the non-empty compact set such that $T_1(K) \cup T_2(K) = K$.

- In particular, $K \subset [0,1]$ is merely points whose infinite continued fraction expansion contains just m's and n's.
- As before we can associate the lengths $\mathcal{L} = \{\ell_j\}_{j=0}^{\infty}$ of the gaps in the Cantor set and the zeta function

$$\zeta_{\mathcal{L}}(s) = \sum_{j=0}^{\infty} \ell_j^s$$

We can now consider some examples.

Poles of $\zeta_{\mathcal{L}}(s)$



Values of (n, m): (i) $(4, 7)$; (ii) $(4, 11)$; (iii) $(4, 5)$; (iv) $(2, 7)$; (v) $(3, 6)$; (vi) $(5, 9)$.
Plots above are due to Polina Vytnova.

skip to end

Practical plotting: connection to Ruelle-Selberg zeta functions

In the present case the poles for $\zeta_{\mathcal{L}}(s)$ are the same as that of a much better understood complex function: The *Ruelle-Selberg zeta function* $Z(s)$.

- For each word $\underline{i} = (i_1, \cdots, i_k) \in \{1, 2\}^k$ we can consider fixed points $T_{i_1} \circ \cdots \circ T_{i_k}(x_{\underline{i}}) = x_{\underline{i}}$. We denote $|\underline{i}| = k$.
- The Ruelle-Selberg zeta function takes the form

$$Z(s) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \sum_{|\underline{i}|=k} \frac{|x_{\underline{i}}|^{2s}}{1 - x_{\underline{i}}^2}\right).$$

In particular,

- The poles for $Z(s)$ correspond to the poles for $\zeta_{\mathcal{L}}(s)$.
- The poles for $Z(s)$ are easier to estimate than the poles for $\zeta_{\mathcal{L}}(s)$.

Therefore we plot the poles of $Z(s)$ and just claim they are the complex dimensions, i.e., poles of $\zeta_{\mathcal{L}}(s)$

skip to end

A special property of $\zeta_{\mathcal{L}}(s)$ for continued fractions

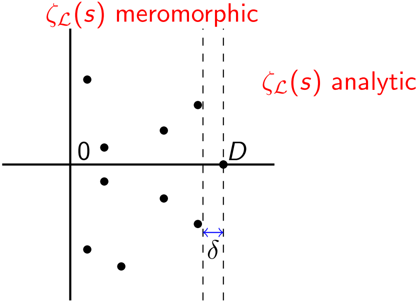
Despite the diversity of plots there is at least one common feature.

In contract to the affine case, for the inverse branches of continued fractions the function $\zeta_{\mathcal{L}}(s)$ cannot have complex poles too close to the line $Re(s) = D$.

Theorem

There exists $\delta > 0$ such that the poles $s_n = \sigma_n + it_n$ satisfy

$$\sup\{\sigma_n : s_n \neq D\} \leq D - \delta.$$



Again the dynamics enters in the proofs: This follows from an application of a result of Naud, which in turn depends on the method of Dolgopyat. This uses that poles correspond to the complex transfer operator L_s having 1 as an eigenvalue.

extra material

References

1. K. Falconer, Fractal Geometry (3rd Edition), Wiley, 2014 pp. 158-166.
2. M. Lapidus and M. van Frankenhuysen, *Fractal Geometry and Number Theory*, Birkhauser, 2012.
3. M. Lapidus , G. Radunovic and D. Zubrinic, *Fractal Zeta Functions and Fractal Drums*, Springer, 2017.
4. T. Morita, Meromorphic extensions of a class of dynamical zeta function and their special values at the origin, Ergod. Th. & Dynam. Sys., 26 (2006) 1127-1158.
5. F. Naud, Expanding maps on Cantor sets and analytic continuation of zeta functions, Ann. Sci. E.N.S., 38 (2005) 116-153.
6. D. Ruelle, Zeta-functions for expanding maps and Anosov flows, Invent. Math., 34 (1976) 231-242.

The end

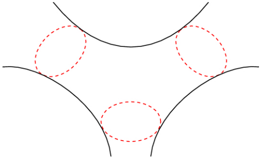
Thanks for your attention.

Retrospective motivation: Borthwick values for closed geodesics

Retrospective motivation Borthwick values

There is another context in which dynamically defined complex function have interesting patterns.

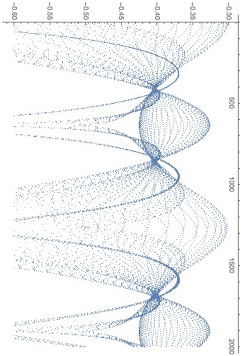
Let M be an infinite area surface of constant curvature -1 (e.g., "pair of pants").



Consider the Ruelle-Selberg zeta function for the periods $\{\ell_n\}$ of orbits for the geodesic flow defined by

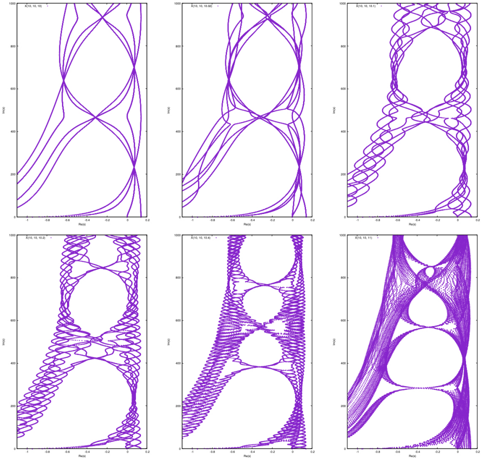
$$Z(s) = \prod_{l=0}^{\infty} \prod_{n} \left(1 - e^{-(s+l)\ell_n}\right)^{-1}, \text{ for } s \in \mathbb{C}$$

In this case, the poles have very interesting empirical patterns:



Poles for slightly different surfaces

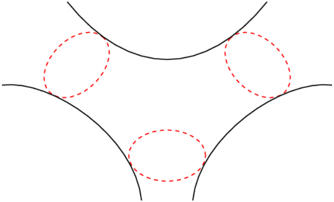
By slightly changing the lengths of the three (dotted red geodesics) which determine the surface, the poles chanage significantly.

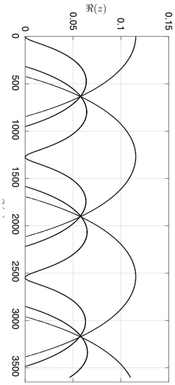


Plots due to Bandtlow, Pohl, Schick and Weisse. skip to end

The only case of Bothwick values we understand ... sort of

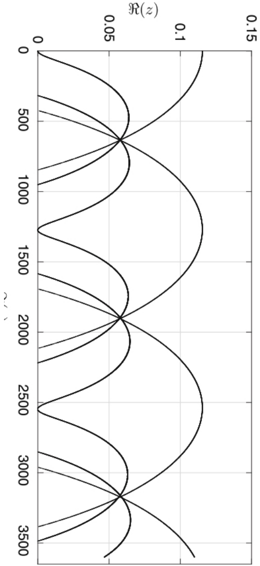
We can consider a surface of negative curvature whereby the three red (defining) geodesics have equal length and are relatively long.





We can consider a surface of negative curvature whereby the three red (defining) geodesics have equal length and are relatively long. The poles of Z(s) seem to be close to well defined curves.

Formulae



As the red curves get longer we can normalize the horizontal and vertical scales and then the poles asymptotically lie on one of four curves

$$\{\sigma(t) + it : t \in \mathbb{R}\}$$

where $\sigma(t)$ is one of the following:

$$\sigma(t) = \frac{1}{2} \log |2 - 2\cos(t)|$$

$$\sigma(t) = \frac{1}{2} \log |2 + 2\cos(t)|$$

$$\sigma(t) = \log \left| 1 - \frac{1}{2} e^{2it} - \frac{1}{2} e^{it} \sqrt{4 - 3e^{2it}} \right|$$

$$\sigma(t) = \log \left| 1 - \frac{1}{2} e^{2it} + \frac{1}{2} e^{it} \sqrt{4 - 3e^{2it}} \right|$$

Back to the complex dimensions via the Modular surface

There is a slightly tenuous construction between geodesic flows and complex dimensions.

- Let $\phi_t : M \to M$ be the geodesic flow on the Modular surface, with $M = PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$.
- One can code the geodesics using continued fractions (after Hedlund).
- We can associate to distinct $n, m \in \mathbb{N}$ the Cantor set $K \subset [0,1]$ whose infinite continued fraction expansions contain only the digits n and m.
- One can then associate a ϕ-invariant measure μ on M corresponding to the Hausdorff measure on K. We can associate to functions $F, G \in C^\infty(M)$ the orrelation function

$$\rho(t) = \int F \circ \phi_t . G d\mu - \int F d\mu \int G d\mu$$

  and ask about the speed at which $\rho(t) \to 0$ as $t \to +\infty$.
- This is controlled by the poles $\{s_n\}$ of the Laplace transform $\widehat{\rho}(s) = \int_0^\infty e^{-st}\rho(t)dt$. These correspond to the complex dimensions $\{z_n\}$ by $z_n = D + s_n$, for $n \geq 1$.