Dimension of Bernoulli Convolutions in $\mathbb{R}^d$

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Bernoulli Convolutions

Let $d \geq 1$ be an integer and let $(\lambda_1, ..., \lambda_d) = \lambda \in (0,1)^d$ be with $\lambda_1 > ... > \lambda_d$. Denote by μ_λ the distribution of the random vector

$$\sum_{n \geq 0} \pm (\lambda_1^n, ..., \lambda_d^n),$$

where the $\pm$ signs are chosen independently and with equal weight. The measure μ_λ is called the Bernoulli convolution associated to λ.

Exact dimensional

A Borel probability measure θ on $\mathbb{R}^d$ is said to be exact dimensional if there exists a number $\dim\theta$ such that

$$\lim_{\delta\downarrow 0}\frac{\log\theta(B(x,\delta))}{\log\delta} = \dim\theta \text{ for } \theta\text{-a.e. } x,$$

where $B(x,\delta)$ is the closed ball with centre x and radius δ.

- Feng and Hu (2009): μ_λ is always exact dimensional.

Baisc question

Question 1

what is the value of dim μ_λ *?*

Lyapunov dimension of μ_λ

Setting

$$m := \max \left\{ 0 \le k \le d \ : \ \Pi_{j=1}^{k} \lambda_j \ge 1/2 \right\},$$

the Lyapunov dimension is defined as follows:

$$\dim_L \mu_\lambda := \begin{cases} m + \frac{\log 2 + \sum_{j=1}^{m} \log \lambda_j}{-\log \lambda_{m+1}} & , \text{ if } m < d \\ d \frac{\log 2}{-\sum_{j=1}^{d} \log \lambda_j} & , \text{ if } m = d \end{cases}.$$

- It always holds that

$$\dim \mu_\lambda \le \min \left\{ \dim_L \mu_\lambda, d \right\}.$$

- Question 1*: when $\dim \mu_\lambda = \min \left\{ \dim_L \mu_\lambda, d \right\}$?

Background: $d = 1$ and $\lambda \in (1/2, 1)$

Definition

We say that an affine IFS $\Psi := \{\psi_i\}_{i \in \Lambda}$ on $\mathbb{R}$ has no exact overlaps if its elements generate a free semigroup. That is, if $\psi_{u_1} \neq \psi_{u_2}$ for all distinct $u_1, u_2 \in \Lambda^*$, where Λ^* is the set of finite words over Λ.

Conjecture 1

If $1/\lambda$ has no exact overlaps, then $\dim \mu_\lambda = 1$*?*

Background: $d = 1$ and $\lambda \in (1/2, 1)$

- Erdös (1939): μ_λ is singular whenever $1/\lambda$ is a Pisot number.
- Garsia(1963): $\dim \mu_\lambda < 1$ whenever $1/\lambda$ is a Pisot number.
- Hochman (2014): When $1/2 \leq \lambda < 1$ is algebraic but has no exact overlaps, $\dim \mu_\lambda = 1$.
- Varjú (2019): When $1/2 < \lambda < 1$ is transcendental, then $\dim \mu_\lambda = 1$.

Background: $d = 2$

Przytycki and Urbański (1989) considered the situation in which $\lambda_2 = 1/2$:

- $\dim \mu_{(\lambda_1,1/2)} = \dim_L \mu_{(\lambda_1,1/2)}$ whenever μ_{λ_1} is absolutely continuous.
- $\dim \mu_{(\lambda_1,1/2)} < \dim_L \mu_{(\lambda_1,1/2)}$ when $\lambda_2 = 1/2$ and λ_1^{-1} is Pisot.

Background: $d = 2$

Shmerkin (2006): established that the Hausdorff dimension of the support of μ_λ equals the Lyapunov dimension for Lebesgue a.e. $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \lambda_2 < 1/2 < \lambda_2$.

Main Result

Theorem (Ariel Rapaport with H.R. 2024, Arxiv)

Let $d \in \mathbb{Z}_{>0}$ *and* $(\lambda_1, ..., \lambda_d) = \lambda \in (0,1)^d$ *be with* $\lambda_1 > ... > \lambda_d$*, and suppose that* $P(\lambda_j) \neq 0$ *for every* $1 \leq j \leq d$ *and nonzero polynomial* P *with coefficients* $\pm 1, 0$*. Then* $\dim \mu_\lambda = \min \{\dim_L \mu_\lambda, d\}$*.*

- Rapaport (2023, *Arxiv*) proved this result for cases where $\lambda_1, \ldots, \lambda_d$ are all algebraic.

The proof of Main Result

- let $\mathcal{P}^{(n)} \subset \mathbb{Z}[X]$ be the set of polynomials of degree strictly less than n with integer coefficients bounded in absolute value by 2.
- For $1 \leq j \leq d$ write $\chi_j := -\log \lambda_j$, and set

$$\kappa := \chi_d \dim \mu_\lambda - \sum_{j=1}^{d-1} (\chi_d - \chi_j).$$

- $h_{RW}(\lambda) := \lim_{n\to\infty} \frac{1}{n} H\left(\sum_{0\leq k<n} \pm (\lambda_1^n, ..., \lambda_d^n)\right)$, where $H(\cdot)$ denotes the Shannon entropy. .

The proof of Main Result

- $\Omega := \{(\eta_1, ..., \eta_d) \in (0,1)^d \,:\, \eta_1 > ... > \eta_d\}$.
- Let $\{e_1, ..., e_d\}$ be the standard basis of $\mathbb{R}^d$. Given $J \subset [d]$ denote by π_J the orthogonal projection onto $\mathrm{span}\{e_j \,:\, j \in J\}$. Thus,

$$\pi_J(x) = \sum_{j \in J} \langle e_j, x \rangle \, e_j \text{ for } x \in \mathbb{R}^d,$$

and $\pi_{[0]}$ is identically 0.

The proof of Main Result

To prove the Main Result, the most challenging part is establishing the following Diophantine result. Once this is achieved, we can follow Varjú's strategy to prove the case $d = 1$.

Theorem

Suppose that $\dim \mu_\lambda < \min\{d, \dim_L \mu\}$, $\dim \pi_J \mu_\lambda = |J|$ *for each proper subset* J *of* $[d]$, *and* λ_{j_0} *is transcendental for some* $1 \le j_0 \le d$. *Then for every* $\epsilon > 0$ *and* $N \ge 1$ *there exist* $n \ge N$ *and* $(\eta_1, \dots, \eta_d) = \eta \in \Omega$ *such that,*

1. *for each* $1 \le j \le d$ *there exists* $0 \ne P_j \in \mathcal{P}^{(n)}$ *with* $P_j(\eta_j) = 0$;
2. $h_{RW}(\lambda) < \kappa + \epsilon$;
3. $|\lambda - \eta| \le \exp\left(-n^{1/\epsilon}\right)$.

The proof of Main Result

Given a discrete random vector Y, its Shannon entropy is denoted by $H(Y)$. For a bounded random vector $X = (X_1, ..., X_d)$ in $\mathbb{R}^d$ and $(r_1, ..., r_d) = r \in \mathbb{R}^d_{>0}$, we set

$$H(X; r) := \int_{[0,1)^d} H\left(\lfloor X_1/r_1 + x_1 \rfloor, ..., \lfloor X_d/r_d + x_d \rfloor\right) dx_1 ... dx_d.$$

We refer to $H(X; r)$ as the average entropy of X at scale r.

- The concept of average entropy, originally introduced by Zhiren Wang for $\mathbb{R}$.

Main Finding

- Given $r' \in \mathbb{R}^d_{>0}$, we also set

$$H\left(X; r \mid r'\right) := H\left(X; r\right) - H\left(X; r'\right).$$

- If μ is the distribution of X, we write $H(\mu; r)$ and $H(\mu; r \mid r')$ in place of $H(X; r)$ and $H(X; r \mid r')$.

Main Finding

Definition

Given $\epsilon > 0$ and $m \geq 1$, we say that $\mu \in \mathcal{M}(\mathbb{R}^d)$ is (ϵ, m)-non-saturated across the principal directions at all scales, or simply (ϵ, m)-non-saturated, if for all $1 \leq j \leq d$ and $n \geq 0$,

$$\frac{1}{m} H\left(\mu, \mathcal{E}_{n+m} \mid \mathcal{E}_n \vee \pi_{[d]\setminus\{j\}}^{-1} \mathcal{E}_{n+m}\right) < \chi_j - \epsilon.$$

- Write $[d] = \{1, ..., d\}$, let $\pi_{[d]\setminus\{j\}}$ be the orthogonal projection onto $\mathrm{span}\{e_j\}_{j \in [d]\setminus\{j\}}$.
- $\mathcal{E}_n$ is a partition of $\mathbb{R}^d$ into rectangles with side lengths roughly $\lambda_1^n, ..., \lambda_d^n$.
- For $t \in \mathbb{R}$, we write $\lambda^t := (\lambda_1^t, ..., \lambda_d^t)$.

Main Finding

Theorem

For each $\epsilon > 0$ and $M \geq 1$, there exists $C = C(\lambda, \epsilon, M) > 1$ such that the following holds. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be (ϵ, m)-non-saturated for all $m \geq M$, and let $\nu \in \mathcal{M}(\mathbb{R}^d)$, $0 < \beta < 1/2$, and $t_2 > t_1 > 0$ be with $\frac{1}{t_2 - t_1} H\left(\nu; \lambda^{t_2} \mid \lambda^{t_1}\right) > \beta$. Then,

$$H\left(\nu * \mu; \lambda^{t_2} \mid \lambda^{t_1}\right) \geq H\left(\mu; \lambda^{t_2} \mid \lambda^{t_1}\right) + C^{-1}\beta\left(\log \beta^{-1}\right)^{-1}(t_2 - t_1) - C.$$

- Hochman (2014) shows $f_{\lambda,\epsilon,M}(\beta)$ for some function $f_{\lambda,\epsilon,M}(\cdot)$ in the case $d = 1$.
- Varju (2019) shows $C^{-1}\beta\left(\log \beta^{-1}\right)^{-1}$ for some function in the case $d = 1$.

The idea of Main Finding

- Our approach extends Hochman's qualitative insights to provide a proof of Varjú's (2019) quantitative entropy increase theorem in $\mathbb{R}$.
- To investigate the quantitative entropy increase theorem in $\mathbb{R}^d$, we draw upon the insights from Ariel Rapaport's qualitative version (2023 Arxiv) in $\mathbb{R}^d$.
- Additive combinatorics plays a crucial role in the proof.

Thank you!