Low Discrepancy Digital Hybrid Sequences and the t-adic Littlewood Conjecture

Steven Robertson

Manchester University

October 11, 2024

Distribution and Discrepancy

First Definitions

- Let $d \in \mathbb{N}$ and define $\mu_d(\mathcal{B})$ as the d-dimensional Lebesgue measure of the measurable set $\mathcal{B}$.
- Let S be a finite set and let $\#S$ denote the cardinality of S.

Definition:

For a sequence $\mathbf{z} = (\mathbf{z}_n)_{n \geq 1}$ in $\mathbb{R}^d$ and $\mathcal{B} \subset \mathbb{R}^d$,

$$\#(\mathcal{B}, \mathbf{z}, N) = \#\{n \in \mathbb{N} : n < N,\ \mathbf{z}_n \in \mathcal{B}\}$$

Definition:

A d-dimensional sequence $\mathbf{z} = (\mathbf{z}_n)_{n \geq 1}$ is uniformly distributed if for every box $\mathcal{B} \in [0,1]^d$

$$\lim_{N \to \infty} \frac{\#(\mathcal{B}, \mathbf{z}, N)}{N} = \mu_d(\mathcal{B}).$$

Discrepancy

Definition:

The discrepancy of the sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$ is defined as

$$D_N(\mathbf{z}) = \sup_{\mathcal{B}\subset[0,1)^d} \left| \frac{\#(\mathcal{B},\mathbf{z},N)}{N} - \mu_d(\mathcal{B}) \right|,$$

where the supremum is taken over all axis-parallel boxes $\mathcal{B} \subset [0,1)^d$.

Definition:

The star discrepancy of the sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$, denoted $D_N^*(\mathbf{z})$, is defined with the additional condition that $\mathcal{B}$ must have one corner at the origin.

Theorem (Kuipers, Niederreiter, 1974):

For every $N \in \mathbb{N}$ and every sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$, one has

$$D_N^*(\mathbf{z}) \leq D_N(\mathbf{z}) \leq 2^d D_N^*(\mathbf{z}).$$

Bounds on Discrepancy

Theorem (Roth, 1954):

For every $N \in \mathbb{N}$ and every sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$ in the d-dimensional unit cube, one has

$$D_N^*(\mathbf{z}) \gg_d \frac{\log^{\frac{d-1}{2}}(N)}{N}.$$

Conjecture:

For every $N \in \mathbb{N}$ and every sequence $(\mathbf{z}_n)_{n\in\mathbb{N}}$ in the d-dimensional unit cube, one has

$$D_N^*(\mathbf{z}) \gg_d \frac{\log^{d}(N)}{N}.$$

Definition:

Let $(\mathbf{z}_n)_{n\in\mathbb{N}}$ be a sequence in the d-dimensional unit cube. If

$$D_N^*(\mathbf{z}) \ll_d \frac{\log^{d}(N)}{N}$$

for every $N \in \mathbb{N}$, then $(\mathbf{z}_n)_{n\in\mathbb{N}}$ is called low discrepancy.

Low Discrepancy Sequences

Van der Corput Sequences

Definition:

Let $b > 1$ be a natural number and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Example:

Let $b = 5$ and let $n = 1432$. Note that

$$= 2 \cdot 5^0 + 1 \cdot 5^1 + 2 \cdot 5^2 + 1 \cdot 5^3 + 2 \cdot 5^4.$$

Van der Corput Sequences

Definition:

Let $b > 1$ be a natural number and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Example:

Let $b = 5$ and let $n = 1432$. Note that

$$n = 2 \cdot 5^0 + 1 \cdot 5^1 + 2 \cdot 5^2 + 1 \cdot 5^3 + 2 \cdot 5^4.$$

Van der Corput Sequences

Definition:

Let $b > 1$ be a natural number and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Example:

Let $b = 5$ and let $n = 1432$. Note that

$$v_n(5) = \frac{1}{5}(2 \cdot 5^{-0} + 1 \cdot 5^{-1} + 2 \cdot 5^{-2} + 1 \cdot 5^{-3} + 2 \cdot 5^{-4}).$$

Van der Corput Sequences

Definition:

Let $b > 1$ be a natural number and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Example:

Let $b = 5$ and let $n = 1432$. Note that

$$v_n(5) = \frac{1}{5}(2 \cdot 5^{-0} + 1 \cdot 5^{-1} + 2 \cdot 5^{-2} + 1 \cdot 5^{-3} + 2 \cdot 5^{-4}).$$

Theorem (Halton, 1950):

The base-b Van der Corput sequence is Low Discrepancy.

Kronecker Sequences

Definition:

Let $\alpha \in (0,1)$ be a real number. The Kronecker sequence associated to α , denoted $k_\alpha = (k_n(\alpha))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Theorem (Weyl, 1916):

The Kronecker sequence associated to $\alpha \in \mathbb{R}$ is uniformly distributed if and only if $\alpha \notin \mathbb{Q}$.

Diophantine Approximation and Kronecker Sequences

Definition: Bad

The set of **badly approximable** numbers, denoted **Bad**, contains all the $\alpha \in \mathbb{R}$ for which there exists a constant $c_\alpha > 0$ such that for all reduced fractions $\frac{m}{n} \in \mathbb{Q}$

$$\left|\alpha - \frac{m}{n}\right| > \frac{c_\alpha}{n^2}.$$

Theorem: (Niederreiter, 1974)

Let $\alpha \in \mathbb{R}$. Then k_α is low discrepancy if and only if $\alpha \in$ **Bad**.

α is in...	$(k_n(\alpha))_{n\in\mathbb{N}}$ is...

Diophantine Approximation and Kronecker Sequences

Definition: Bad

The set of **badly approximable** numbers, denoted **Bad**, contains all the $\alpha \in \mathbb{R}$ for which there exists a constant $c_\alpha > 0$ such that for all reduced fractions $\frac{m}{n} \in \mathbb{Q}$

$$\left|\alpha - \frac{m}{n}\right| > \frac{c_\alpha}{n^2}.$$

Theorem: (Niederreiter, 1974)

Let $\alpha \in \mathbb{R}$. Then k_α is low discrepancy if and only if $\alpha \in$ **Bad**.

α is in...	$(k_n(\alpha))_{n\in\mathbb{N}}$ is...
$\mathbb{Q}$	Periodic

Diophantine Approximation and Kronecker Sequences

Definition: Bad

The set of **badly approximable** numbers, denoted **Bad**, contains all the $\alpha \in \mathbb{R}$ for which there exists a constant $c_\alpha > 0$ such that for all reduced fractions $\frac{m}{n} \in \mathbb{Q}$

$$\left|\alpha - \frac{m}{n}\right| > \frac{c_\alpha}{n^2}.$$

Theorem: (Niederreiter, 1974)

Let $\alpha \in \mathbb{R}$. Then k_α is low discrepancy if and only if $\alpha \in$ **Bad**.

α is in...	$(k_n(\alpha))_{n\in\mathbb{N}}$ is...
$\mathbb{Q}$	Periodic
$\mathbb{R}\backslash\mathbb{Q}$	Uniformly Distributed

Diophantine Approximation and Kronecker Sequences

Definition: Bad

The set of **badly approximable** numbers, denoted **Bad**, contains all the $\alpha \in \mathbb{R}$ for which there exists a constant $c_\alpha > 0$ such that for all reduced fractions $\frac{m}{n} \in \mathbb{Q}$

$$\left|\alpha - \frac{m}{n}\right| > \frac{c_\alpha}{n^2}.$$

Theorem: (Niederreiter, 1974)

Let $\alpha \in \mathbb{R}$. Then k_α is low discrepancy if and only if $\alpha \in$ **Bad**.

α is in...	$(k_n(\alpha))_{n\in\mathbb{N}}$ is...
$\mathbb{Q}$	Periodic
$\mathbb{R}\backslash\mathbb{Q}$	Uniformly Distributed
Bad	Low Discrepancy

Diophantine Approximation and Kronecker Sequences

Definition: Bad

The set of **badly approximable** numbers, denoted **Bad**, contains all the $\alpha \in \mathbb{R}$ for which there exists a constant $c_\alpha > 0$ such that for all reduced fractions $\frac{m}{n} \in \mathbb{Q}$

$$\left|\alpha - \frac{m}{n}\right| > \frac{c_\alpha}{n^2}.$$

Theorem: (Niederreiter, 1974)

Let $\alpha \in \mathbb{R}$. Then k_α is low discrepancy if and only if $\alpha \in$ **Bad**.

α is in...	$(k_n(\alpha))_{n\in\mathbb{N}}$ is...
$\mathbb{Q}$	Periodic
$\mathbb{R}\backslash\mathbb{Q}$	Uniformly Distributed
Bad	Low Discrepancy
??$\subset$**Bad**	Very Low Discrepancy?

From **Bad** to Worse

Definition: Bad (again)

Let $\|\alpha\|$ denote the distance from $\alpha \in \mathbb{R}$ to the nearest integer. Then

$$\mathbf{Bad} = \left\{ \alpha \in \mathbb{R} : \inf_{n \in \mathbb{N} \setminus \{0\}} n \|n\alpha\| = c_\alpha > 0 \right\}.$$

- Let p be a prime and let $\alpha \in \mathbf{Bad}$. Then $p\alpha \in \mathbf{Bad}$.

Question:

How does $c_{p^n\alpha}$ behave as $n \to \infty$?

Definition:

Define the **p**-adic Badly Approximable Numbers as

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N} \setminus \{0\} \\ k \geq 0}} n \left\| np^k\alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

Key Question:

If $\alpha \in \mathbf{Bad}_p$, what can see say about $D_N\left((k_n(\alpha))_{n \in \mathbb{N}}\right)$?

Two Problems Occur...

Problem 1

The p-adic Littlewood Conjecture, de Mathan and Teulié, 2004:

The set $\mathbf{Bad}_p$ is empty for every prime p.

Problem 2

Theorem (Schmidt, 1972):

Every one-dimensional sequence z satisfies

$$D_N(z) \gg \frac{\log(N)}{N}.$$

Problem 1:

Diophantine Approximation over Function Fields

Positive Characteristic Dictionary

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	Function Field over $\mathbb{F}_p$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	**Function Field over $\mathbb{F}_p$**
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{ \sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p \right\}$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	Function Field over $\mathbb{F}_p$
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{ \sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p \right\}$
$\mathbb{Q}$	$\mathbb{F}_p(t) = \{p(t)/q(t) : p(t), q(t) \in \mathbb{F}_p[t], q(t) \neq 0\}$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	**Function Field over $\mathbb{F}_p$**
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{\sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p\right\}$
$\mathbb{Q}$	$\mathbb{F}_p(t) = \{p(t)/q(t) : p(t), q(t) \in \mathbb{F}_p[t], q(t) \neq 0\}$
$\mathbb{R}$	$\mathbb{F}_p((t^{-1})) = \{\sum_{i=-h}^{\infty} a_i t^{-i} : h \in \mathbb{Z}, a_i \in \mathbb{F}_p\}$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	Function Field over $\mathbb{F}_p$
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{ \sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p \right\}$
$\mathbb{Q}$	$\mathbb{F}_p(t) = \{ p(t)/q(t) : p(t), q(t) \in \mathbb{F}_p[t], q(t) \neq 0 \}$
$\mathbb{R}$	$\mathbb{F}_p((t^{-1})) = \{ \sum_{i=-h}^{\infty} a_i t^{-i} : h \in \mathbb{Z}, a_i \in \mathbb{F}_p \}$
Prime numbers	Irreducible polynomials in $\mathbb{F}_p[t]$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	**Function Field over $\mathbb{F}_p$**
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{ \sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p \right\}$
$\mathbb{Q}$	$\mathbb{F}_p(t) = \{p(t)/q(t) : p(t), q(t) \in \mathbb{F}_p[t], q(t) \neq 0\}$
$\mathbb{R}$	$\mathbb{F}_p((t^{-1})) = \{\sum_{i=-h}^{\infty} a_i t^{-i} : h \in \mathbb{Z}, a_i \in \mathbb{F}_p\}$
Prime numbers	Irreducible polynomials in $\mathbb{F}_p[t]$
$\lvert x \rvert$, $x \in \mathbb{R}$	$\lvert\Theta(t)\rvert = q^{\deg(\Theta(t))}$, $\Theta(t) \in \mathbb{F}_p((t^{-1}))$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	Function Field over $\mathbb{F}_p$
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{ \sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p \right\}$
$\mathbb{Q}$	$\mathbb{F}_p(t) = \{ p(t)/q(t) : p(t), q(t) \in \mathbb{F}_p[t], q(t) \neq 0 \}$
$\mathbb{R}$	$\mathbb{F}_p((t^{-1})) = \{ \sum_{i=-h}^{\infty} a_i t^{-i} : h \in \mathbb{Z}, a_i \in \mathbb{F}_p \}$
Prime numbers	Irreducible polynomials in $\mathbb{F}_p[t]$
$\lvert x \rvert$, $x \in \mathbb{R}$	$\lvert \Theta(t) \rvert = q^{\deg(\Theta(t))}$, $\Theta(t) \in \mathbb{F}_p((t^{-1}))$
$x \mod 1$	$\left\langle \sum_{i=-h}^{\infty} a_i t^{-i} \right\rangle = \sum_{i=1}^{\infty} a_i t^{-i}$

Positive Characteristic Dictionary

- Let $q \in \mathbb{N}$ be a positive power of a prime and let $\mathbb{F}_q$ denote the finite field of cardinality q.
- For this talk, let p be a prime and let $q = p$.

Real numbers	Function Field over $\mathbb{F}_p$
$\mathbb{Z}$	$\mathbb{F}_p[t] = \left\{ \sum_{i=0}^{h} a_i t^i : h \in \mathbb{Z},\ a_i \in \mathbb{F}_p \right\}$
$\mathbb{Q}$	$\mathbb{F}_p(t) = \{ p(t)/q(t) : p(t), q(t) \in \mathbb{F}_p[t], q(t) \neq 0 \}$
$\mathbb{R}$	$\mathbb{F}_p((t^{-1})) = \left\{ \sum_{i=-h}^{\infty} a_i t^{-i} : h \in \mathbb{Z}, a_i \in \mathbb{F}_p \right\}$
Prime numbers	Irreducible polynomials in $\mathbb{F}_p[t]$
$\lvert x \rvert$, $x \in \mathbb{R}$	$\lvert \Theta(t) \rvert = q^{\deg(\Theta(t))}$, $\Theta(t) \in \mathbb{F}_p((t^{-1}))$
$x \mod 1$	$\left\langle \sum_{i=-h}^{\infty} a_i t^{-i} \right\rangle = \sum_{i=1}^{\infty} a_i t^{-i}$
$\lVert x \rVert$, $x \in \mathbb{R}$	$\lvert \langle \Theta(t) \rangle \rvert$, $\Theta(t) \in \mathbb{F}_p((t^{-1}))$.

Digital Van der Corput Sequences

Definition:

Let $b > 1$ be a natural number and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n \geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $b > 1$ be a natural number and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n \geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n \geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $n \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} n_i b^i$. The Base-$b$ Van Der Corput sequence, denoted $(v_n(b))_{n \geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-b Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-b Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(v_n(b))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t) B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(V_n(B(t)))_{n \geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(V_n(B(t)))_{n\geq 1}$, is defined as:

$$v_n(b) = \sum_{i=0}^{\infty} \frac{n_i}{b^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(V_n(B(t)))_{n\geq 1}$, is defined as:

$$V_n(B(t)) = \sum_{i=0}^{\infty} \frac{N_i(p)}{|B(p)|^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(V_n(B(t)))_{n\geq 1}$, is defined as:

$$V_n(B(t)) = \sum_{i=0}^{\infty} \frac{N_i(p)}{|B(p)|^{i+1}}.$$

Digital Van der Corput Sequences

Definition:

Let $B(t) \in \mathbb{F}_p[t]$ be a polynomial and let $N(t) \in \mathbb{F}_p[t]$ such that $N(t) = \sum_{i=0}^{\infty} N_i(t)B(t)^i$ and define $n = N(p)$. The Base-$B(t)$ Digital Van Der Corput sequence, denoted $(V_n(B(t)))_{n \geq 1}$, is defined as:

$$V_n(B(t)) = \sum_{i=0}^{\infty} \frac{N_i(p)}{|B(p)|^{i+1}}.$$

Theorem (Hofer, 2018):

The base-$B(t)$ Digital Van der Corput sequence is Low Discrepancy.

Digital Van der Corput Example

Example:

Let $p = 3$ and let $n = 194$.

Digital Van der Corput Example

Example:

Let $p = 3$ and let $n = 194$. Note that

$$n = 2 \cdot p^0 + 1 \cdot p^1 + 0 \cdot p^2 + 1 \cdot p^3 + 2 \cdot p^4.$$

Digital Van der Corput Example

Example:

Let $p = 3$ and let $n = 194$. Note that

$$n = 2 \cdot p^0 + 1 \cdot p^1 + 0 \cdot p^2 + 1 \cdot p^3 + 2 \cdot p^4.$$

Therefore,

$$N(t) = 2 \cdot t^0 + 1 \cdot t + 0 \cdot t^2 + 1 \cdot t^3 + 2 \cdot t^4.$$

Digital Van der Corput Example

Example:

Let $p = 3$ and let $n = 194$. Note that

$$n = 2 \cdot p^0 + 1 \cdot p^1 + 0 \cdot p^2 + 1 \cdot p^3 + 2 \cdot p^4.$$

Therefore,

$$N(t) = 2 \cdot t^0 + 1 \cdot t + 0 \cdot t^2 + 1 \cdot t^3 + 2 \cdot t^4.$$

In Base $t^2 + 1$, this is

$$N(t) = 1(t^2 + 1)^0 + (t + 2)(t^2 + 1)^1 + 2(1 + t^2)^2.$$

Digital Van der Corput Example

Example:

Let $p = 3$ and let $n = 194$. Note that

$$n = 2 \cdot p^0 + 1 \cdot p^1 + 0 \cdot p^2 + 1 \cdot p^3 + 2 \cdot p^4.$$

Therefore,

$$N(t) = 2 \cdot t^0 + 1 \cdot t + 0 \cdot t^2 + 1 \cdot t^3 + 2 \cdot t^4.$$

In Base $t^2 + 1$, this is

$$N(t) = 1(t^2+1)^0 + (t+2)(t^2+1)^1 + 2(1+t^2)^2.$$

Hence, $V_{194}(1+t) =$

$$\frac{1}{|1+t|}\left(1|t^2+1|^0 + (p+2)|t^2+1|^{-1} + 2|1+t^2|^{-2}.\right)$$

Digital Kronecker Sequences

Definition:

Let $\alpha \in \mathbb{R}$ be a real number. The Kronecker sequence associated to α, denoted $k_\alpha = (k_n(\alpha))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Digital Kronecker Sequences

Definition:

Let $\alpha \in \mathbb{R}$ be a real number. The Kronecker sequence associated to α, denoted $k_\alpha = (k_n(\alpha))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Kronecker sequence associated to α, denoted $k_\alpha = (k_n(\alpha))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Kronecker sequence associated to α, denoted $k_\alpha = (k_n(\alpha))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Digital Kronecker sequence associated to $\Theta(t)$, denoted $K_{\Theta(t)} = (K_n(\Theta(t)))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Digital Kronecker sequence associated to $\Theta(t)$, denoted $K_{\Theta(t)} = (K_n(\Theta(t)))_{n\geq 1}$, is defined as:

$$k_n(\alpha) = n\alpha \mod 1.$$

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Digital Kronecker sequence associated to $\Theta(t)$, denoted $K_{\Theta(t)} = (K_n(\Theta(t)))_{n\geq 1}$, is defined as:

$$K_n(\Theta(t)) = \langle \Theta(t) \cdot N(t) \rangle \Big|_{t=p}.$$

where $N(t) \in \mathbb{F}_p[t]$ is such that $N(p) = n$.

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Digital Kronecker sequence associated to $\Theta(t)$, denoted $K_{\Theta(t)} = (K_n(\Theta(t)))_{n\geq 1}$, is defined as:

$$K_n(\Theta(t)) = \langle \Theta(t) \cdot N(t) \rangle \Big|_{t=p}.$$

where $N(t) \in \mathbb{F}_p[t]$ is such that $N(p) = n$.

Theorem (Niederreiter, Larcher, 1993):

The Digital Kronecker sequence associated to $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ is uniformly distributed if and only if $\Theta(t) \notin \mathbb{F}_p(t)$.

Digital Kronecker Sequences

Definition:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. The Digital Kronecker sequence associated to $\Theta(t)$, denoted $K_{\Theta(t)} = (K_n(\Theta(t)))_{n\geq 1}$, is defined as:

$$K_n(\Theta(t)) = \langle \Theta(t) \cdot N(t) \rangle \Big|_{t=p}.$$

where $N(t) \in \mathbb{F}_p[t]$ is such that $N(p) = n$.

Theorem (Niederreiter, Larcher, 1993):

The Digital Kronecker sequence associated to $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ is uniformly distributed if and only if $\Theta(t) \notin \mathbb{F}_p(t)$.

The analogue of **Bad**

$$\textbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N} \setminus \{0\} \\ k \geq 0}} n \left\| np^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\textbf{Bad}_p$:

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| n p^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N} \setminus \{0\} \\ k \geq 0}} n \left\| n p^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

The analogue of **Bad**

$$\textbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N} \setminus \{0\} \\ k \geq 0}} n \left\| n p^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\textbf{Bad}_p$:

Define $\textbf{Bad}(P(t), p)$

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N} \setminus \{0\} \\ k \geq 0}} n \left\| np^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| n p^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}}$$

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| n p^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}} |N(t)|$$

The analogue of Bad

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| np^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}} |N(t)| \left| \left\langle \qquad \right\rangle \right| \qquad .$$

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| np^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}} |N(t)| \left| \left\langle N(t) \cdot P(t)^k \cdot \Theta(t) \right\rangle \right| \qquad .$$

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| np^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}} |N(t)| \left| \left\langle N(t) \cdot P(t)^k \cdot \Theta(t) \right\rangle \right| > 0.$$

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| np^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}} |N(t)| \left| \left\langle N(t) \cdot P(t)^k \cdot \Theta(t) \right\rangle \right| > 0.$$

Definition:

The set $\mathbf{Bad}(p)$ is defined identically but with $k = 0$.

The analogue of **Bad**

$$\mathbf{Bad}_p = \left\{ \alpha \in \mathbb{R} : \inf_{\substack{n \in \mathbb{N}\setminus\{0\} \\ k \geq 0}} n \left\| n p^k \alpha \right\| = C_{\alpha,p} > 0 \right\}.$$

- Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial.

Definition, The analogue of $\mathbf{Bad}_p$:

Define $\mathbf{Bad}(P(t), p)$ as all the $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ satisfying

$$\inf_{\substack{N(t) \in \mathbb{F}_p[t]\setminus\{0\} \\ k \geq 0}} |N(t)| \left| \left\langle N(t) \cdot P(t)^k \cdot \Theta(t) \right\rangle \right| > 0.$$

Definition:

The set $\mathbf{Bad}(p)$ is defined identically but with $k = 0$.

Theorem, Niederreiter, 1992:

The sequence $K_{\Theta(t)}$ is low discrepancy if and only if $\Theta(t) \in \mathbf{Bad}(p)$.

What is known about $\mathbf{Bad}(P(t), q)$

What is known about $\mathbf{Bad}(P(t), q)$

$P(t)$-adic Littlewood Conjecture, de Mathan, Teulié, 2004:

For any choice of finite field $\mathbb{F}_p$ and any irreducible polynomial $P(t) \in \mathbb{F}_p[t]$,

$$\mathbf{Bad}(P(t), q) = \varnothing$$

Theorem, Adiceam, Nesharim, Lunnon, 2020:

Let $n \in \mathbb{N}$. Then

$$\mathbf{Bad}(t, 3^n) \neq \varnothing$$

Theorem, R., 2022:

For every irreducible polynomial $P(t) \in \mathbb{F}_p[t]$, there is an injection from $\mathbf{Bad}(t, q)$ into $\mathbf{Bad}(P(t), q)$.

Theorem, Garrett, R., 2024

The set $\mathbf{Bad}(P(t), q)$ is non-empty for any choice of irreducible polynomial $P(t) \in \mathbb{F}_p[t]$ when q is a power of 5, 7 or 11.

Problem 2:

Hybrid Sequences

Definition of Hybrid Sequence

Theorem (Schmidt, 1972):

Every one-dimensional sequence z satisfies

$$D_N(z) \gg \frac{\log(N)}{N}.$$

Definition (Spanier, 1995):

Let $d \in \mathbb{N}$. A d-dimensional Hybrid sequence is a concatenation of d different one dimensional low discrepancy sequences.

Theorem (Hofer, 2018):

Let $\Theta(t) \in \mathbf{Bad}(q)$ and let $B(t) \in \mathbb{F}_p[t]$. Then, the 2-dimensional hybrid sequence

$$(\mathbf{H}_n(\Theta(t), B(t))_{n\geq 0} = (K_n(\Theta(t)), V_n(B(t)))_{n\geq 0}$$

satisfies

$$D_{N,\mathbf{H}} \ll \frac{\log^2(N)}{\sqrt{N}}.$$

Main Result

Theorem, R., 2022:

For every irreducible polynomial $P(t) \in \mathbb{F}_p[t]$, there is an injection from $\mathbf{Bad}(t, q)$ into $\mathbf{Bad}(P(t), q)$.

Conjecture: Levin, 2022. Theorem: R. 2024.

Let $\Theta(t) \in \mathbf{Bad}(t, q)$. Additionally, let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial and let $\Phi(t) \in \mathbf{Bad}(P(t), q)$ be induced from $\Theta(t)$. Then, the 2-dimensional hybrid sequence

$$(\mathbf{H}_n(\Phi(t), P(t))_{n\geq 0} = (K_n(\Phi(t)), V_n(P(t)))_{n\geq 0}$$

satisfies

$$D_{N,\mathbf{H}} \ll \frac{\log^2(N)}{N}.$$

Proof of Main Result

Main Idea

- Recall:

$$D_N(\mathbf{z}) = \sup_{\mathcal{B}\subset[0,1)^d} \left| \frac{\#(\mathcal{B}, \mathbf{z}, N)}{N} - \mu_d(\mathcal{B}) \right|.$$

- Let

$$\gamma := \sum_{i=1}^{\infty} \gamma_i p^{-i} \quad \text{and} \quad \lambda := \sum_{i=1}^{\infty} \lambda_i p^{-i}.$$

- Define the box $\mathcal{B} = (0, \gamma] \times (0, \lambda]$.
- The plan is to cover $\mathcal{B}$ in $\ll \log^2(N)$ disjoint boxes $\mathcal{B}_i$, and show that for any $N \in \mathbb{N}$,

$$|\#(\mathcal{B}_i, \mathbf{z}, N) - N \cdot \mu_d(\mathcal{B}_i)| \ll 1,$$

where the implicit constant is independent to γ and λ.

Splitting $\mathcal{B}$ into Boxes

- For $j \in \mathbb{N}$, define
$$\Gamma_j := \sum_{i=1}^{j} \gamma_i p^{-i} \quad \text{and} \quad \Lambda_j := \sum_{i=1}^{j} \lambda_i p^{-i}.$$
- For $j, k \in \mathbb{N}$, define $I_{j,k} := [\Gamma_j, \Gamma_{j+1}) \times [\Lambda_k, \Lambda_{k+1})$.
- Clearly,
$$\mathcal{B} = \bigsqcup_{j,k \in \mathbb{N}} I_{j,k}.$$

Splitting $\mathcal{B}$ into Boxes

- For $j \in \mathbb{N}$, define
$$\Gamma_j := \sum_{i=1}^{j} \gamma_i p^{-i} \quad \text{and} \quad \Lambda_j := \sum_{i=1}^{j} \lambda_i p^{-i}.$$
- For $j, k \in \mathbb{N}$, define $I_{j,k} := [\Gamma_j, \Gamma_{j+1}) \times [\Lambda_k, \Lambda_{k+1})$.
- Clearly,
$$\mathcal{B} = \bigsqcup_{j,k \in \mathbb{N}} I_{j,k}.$$

Splitting $\mathcal{B}$ into Boxes

- For $j \in \mathbb{N}$, define
$$\Gamma_j := \sum_{i=1}^{j} \gamma_i p^{-i} \quad \text{and} \quad \Lambda_j := \sum_{i=1}^{j} \lambda_i p^{-i}.$$
- For $j, k \in \mathbb{N}$, define $I_{j,k} := [\Gamma_j, \Gamma_{j+1}) \times [\Lambda_k, \Lambda_{k+1})$.
- Clearly,
$$\mathcal{B} = \bigsqcup_{j,k \in \mathbb{N}} I_{j,k}.$$

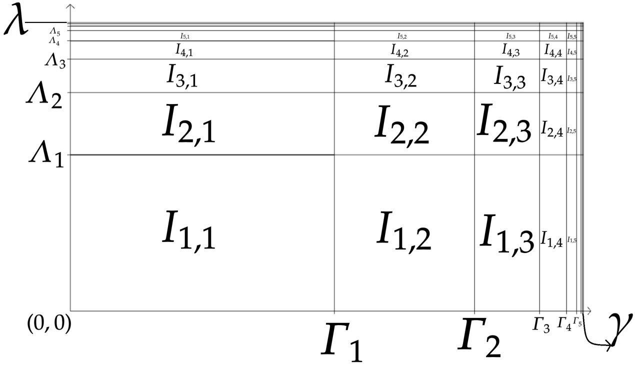
Splitting $\mathcal{B}$ into Boxes

- For $j \in \mathbb{N}$, define

$$\Gamma_j := \sum_{i=1}^{j} \gamma_i p^{-i} \quad \text{and} \quad \Lambda_j := \sum_{i=1}^{j} \lambda_i p^{-i}.$$

- For $j, k \in \mathbb{N}$, define $I_{j,k} := [\Gamma_j, \Gamma_{j+1}) \times [\Lambda_k, \Lambda_{k+1})$.
- Clearly,

$$\mathcal{B} = \bigsqcup_{j,k \in \mathbb{N}} I_{j,k}.$$



Covering $\mathcal{B}$ in Finitely Many Boxes

- Let $M := \log_p(N)$ and define

$$S' := \bigsqcup_{j,k \leq M} I_{j,k} \subset S.$$

- Recall, if $\Theta(t) \in \mathbf{Bad}(t,p)$ then there exists $D(\Theta(t)) \in \mathbb{N}$ such that

$$|N(t)| \cdot |\langle \Theta(t) \cdot t^k \cdot N(t) \rangle| > p^{-D(\Theta(t))}$$

for every $N(t) \in \mathbb{F}_p[t] \backslash \{0\}$ and every $k \in \mathbb{N}$.

- Define the sets

$$S_1 = \bigsqcup_{\substack{j,k \leq M \\ j+k+2 \leq M - D(\Theta)}} I_{j,k} \qquad S_2 = \bigsqcup_{\substack{j,k \leq M \\ j+k+2 > M - D(\Theta)}} I_{j,k}$$

- Clearly, $S' = S_1 \sqcup S_2$.

Covering $\mathcal{B}$ in Finitely Many Boxes

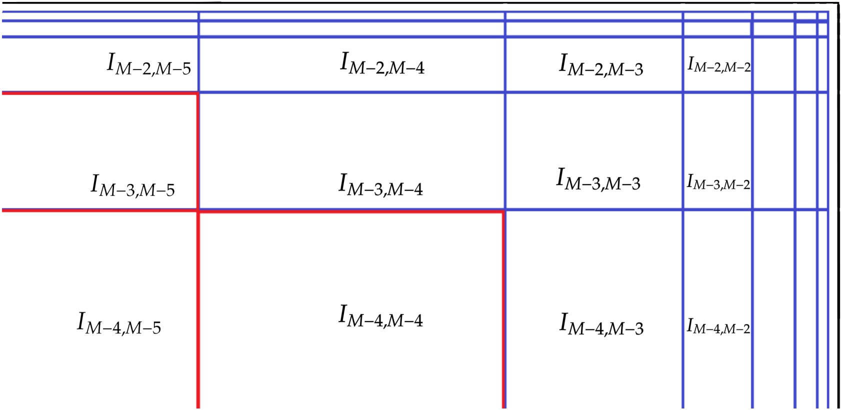
- Recall

$$S_1 = \bigsqcup_{\substack{j,k \leq M \\ j+k+2 \leq M - D(\Theta)}} I_{j,k} \qquad S_2 = \bigsqcup_{\substack{j,k \leq M \\ j+k+2 > M - D(\Theta)}} I_{j,k}$$

Covering $\mathcal{B}$ in Finitely Many Boxes

- Recall

$$S_1 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 \le M - D(\Theta)}} I_{j,k} \qquad S_2 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 > M - D(\Theta)}} I_{j,k}$$



Covering $\mathcal{B}$ in Finitely Many Boxes

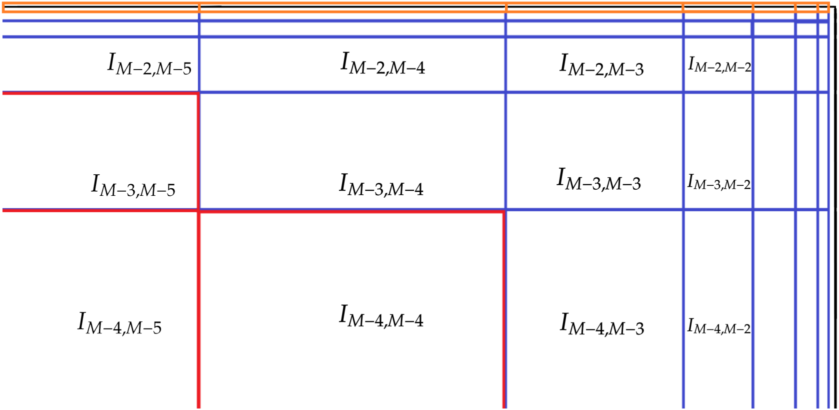
- Recall

$$S_1 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 \le M - D(\Theta)}} I_{j,k} \qquad S_2 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 > M - D(\Theta)}} I_{j,k}$$

$$S_3 = \bigsqcup_{j<M} [\Gamma_j, \Gamma_{j+1}) \times \left[\Lambda_{M+1}, \Lambda_{M+1} + \lambda_{M+1} p^{-(M+1)}\right).$$

Covering $\mathcal{B}$ in Finitely Many Boxes

$$S_3 = \bigsqcup_{j<M} [\Gamma_j, \Gamma_{j+1}) \times \left[\Lambda_{M+1}, \Lambda_{M+1} + \lambda_{M+1} p^{-(M+1)}\right).$$



Covering $\mathcal{B}$ in Finitely Many Boxes

- Recall

$$S_1 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 \le M - D(\Theta)}} I_{j,k} \qquad S_2 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 > M - D(\Theta)}} I_{j,k}$$

$$S_3 = \bigsqcup_{j<M} [\Gamma_j, \Gamma_{j+1}) \times \left[\Lambda_{M+1}, \Lambda_{M+1} + \lambda_{M+1} p^{-(M+1)}\right).$$

$$S_4 = \bigsqcup_{k<M} \left[\Gamma_{M+1}, \Gamma_{M+1} + \gamma_{M+1} p^{-(M+1)}\right) \times [\Lambda_k, \Lambda_{k+1}).$$

Covering $\mathcal{B}$ in Finitely Many Boxes

$$S_4 = \bigsqcup_{k<M} \left[\Gamma_{M+1}, \Gamma_{M+1} + \gamma_{M+1} p^{-(M+1)}\right) \times [\Lambda_k, \Lambda_{k+1}).$$

Covering $\mathcal{B}$ in Finitely Many Boxes

- Recall

$$S_1 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 \le M - D(\Theta)}} I_{j,k} \qquad S_2 = \bigsqcup_{\substack{j,k \le M \\ j+k+2 > M - D(\Theta)}} I_{j,k}$$

$$S_3 = \bigsqcup_{j<M} [\Gamma_j, \Gamma_{j+1}) \times \left[\Lambda_{M+1}, \Lambda_{M+1} + \lambda_{M+1} p^{-(M+1)}\right).$$

$$S_4 = \bigsqcup_{k<M} \left[\Gamma_{M+1}, \Gamma_{M+1} + \gamma_{M+1} p^{-(M+1)}\right) \times [\Lambda_k, \Lambda_{k+1}).$$

$$S_5 = \left[\Gamma_{M+1}, \Gamma_{M+1} + \gamma_{M+1} p^{-(M+1)}\right) \times \left[\Lambda_{M+1}, \Lambda_{M+1} + \lambda_{M+1} p^{-(M+1)}\right).$$

Covering $\mathcal{B}$ in Finitely Many Boxes

$$S_5 = \left[\Gamma_{M+1}, \Gamma_{M+1} + \gamma_{M+1} p^{-(M+1)}\right) \times \left[\Lambda_{M+1}, \Lambda_{M+1} + \lambda_{M+1} p^{-(M+1)}\right).$$

Counting Points in $I_{j,k}$

- The box $\mathcal{B}$ has been covered by $\ll \log^2(N)$ sub-boxes, $I_{j,k}$.

Recall

for $j \in \mathbb{N}$, define

$$\Gamma_j := \sum_{i=1}^{j} \gamma_i p^{-i} \quad \text{and} \quad \Lambda_j := \sum_{i=1}^{j} \lambda_i p^{-i}.$$

For $j, k \in \mathbb{N}$, define $I_{j,k} := [\Gamma_j, \Gamma_{j+1}) \times [\Lambda_k, \Lambda_{k+1})$.

- Sub-box $I_{j,k}$ has width $\gamma_{j+1} p^{-(j+1)}$ and height $\lambda_{k+1} p^{-(k+1)}$.
- Trivially, $\gamma_{j+1}, \lambda_{k+1} < q$.
- Therefore, $I_{j,k}$ is the disjoint union of at most p^2 boxes of the form

$$I_1 \times I_2 := \left[\frac{a}{p^{j+1}}, \frac{a+1}{p^{j+1}}\right) \times \left[\frac{b}{p^{k+1}}, \frac{b+1}{p^{k+1}}\right)$$

for some $a < p^{j+1}$ and $b < p^{k+1}$.

Main Lemma

- Recall that μ_2 is 2-dimensional Lebesgue measure.
- Clearly, $\mu_2(I_1 \times I_2) = p^{-(j+k+2)}$.

Main Lemma

For every choice of box $I_1 \times I_2$and for every $N \in N$

$$|\#(I_1 \times I_2, \mathbf{H}(\Theta(t), t), N) - N\mu_2(I_1 \times I_2)| \leq p^{D(\Theta(t))}.$$

- Goal: Calculate $\#(I_1 \times I_2, \mathbf{H}(\Theta(t), t), N)$.
- This amounts to counting how many $n < N$ satisfy both

$$V_n(t) \in \left[\frac{a}{p^{j+1}}, \frac{a+1}{p^{j+1}}\right) \quad \text{and} \quad K_n(\Theta(t)) \in \left[\frac{b}{p^{k+1}}, \frac{b+1}{p^{k+1}}\right).$$

- Assume $I_1 \times I_2 \subset S_1$.

Proof of Main Lemma: Part 1

Lemma 1

Let $j, a \in \mathbb{N}$ such that $a < p^j$. Then every choice of $n \in \mathbb{N}$ such that

$$V_n(t) \in \left[\frac{a}{p^j}, \frac{a+1}{p^j}\right)$$

has the same first j coefficients in its base p expansion.

Definition:

An $n \times m$ matrix $B = (b_{i,j})_{0 \leq i \leq n, 0 \leq j \leq m}$ is **Hankel** if $b_{i,j} = b_{i+1,j-1}$ for all $0 \leq i \leq n-1, 0 \leq j \leq m-1$.

- **Example:** $\begin{pmatrix} a & b & c & d \\ b & c & d & e \\ c & d & e & f \\ d & e & f & g \end{pmatrix}$

Hankel Matrix from a Sequence

Hankel Matrix from a Sequence

- Let $A = (a_i)_{i \in \mathbb{Z}}$ be an infinite sequence, $k \in \mathbb{Z}$ and $m, n \in \mathbb{N}$.
- Let $\Theta(t) = \sum_{i=0}^{\infty} a_i t^{-i}$.

Definition:

Define the Hankel matrix $H_\Theta(k, m, n) := (a_{j+i+k})_{0 \le i \le m, 0 \le j \le n}$, viz.

$$H_\Theta(n, m) := \begin{pmatrix} a_k & a_{k+1} & a_{k+2} & \cdots & a_{k+n-1} & a_{k+n} \\ a_{k+1} & a_{k+2} & a_{k+3} & \cdots & \cdots & a_{k+n+1} \\ a_{k+2} & a_{k+3} & \iddots & \iddots & \iddots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \iddots & \vdots \\ a_{k+m-1} & \vdots & \iddots & \iddots & \iddots & a_{k+n+m-1} \\ a_{k+m} & a_{k+m+1} & \cdots & \cdots & a_{k+n+m-1} & a_{k+n+m} \end{pmatrix}$$

Proof of Main lemma: Part 2

Lemma 2:

Let $\Theta(t) = \sum_{i=0}^{\infty} a_i t^{-i} \in \mathbb{F}_p((t^{-1}))$ be a Laurent series. Furthermore, let $n \in \mathbb{N}$, define $m = \lfloor \log_q(n) \rfloor$ and expand $n = \sum_{i=0}^{m} n_i p^i$. Then,

$$K_n(\Theta(t)) \in \left[\frac{b}{p^k}, \frac{b+1}{p^k}\right]$$

if and only if there exists some fixed $\mathbf{z} \in \mathbb{F}_p^k$ such that

$$H_\Theta(1, l-1, m) \begin{pmatrix} n_0 \\ \vdots \\ n_m \end{pmatrix} = \mathbf{z}.$$

Above, the precise value of $\mathbf{z}$ depends only on k.

Proof of Main Lemma: Part 3

- Let $\Theta = \sum_{i=1}^{\infty} a_i t^{-i}$ and let $n \in \mathbb{N}$ be such that

$$(K_n(\Theta(t)), V_n(P(t))) \in \left[\frac{a}{p^{j+1}}, \frac{a+1}{p^{j+1}}\right) \times \left[\frac{b}{p^{k+1}}, \frac{b+1}{p^{k+1}}\right).$$

- By Lemma 2, the base-p coefficients of n satisfy

$$\begin{pmatrix} a_1 & \cdots & a_{k+1} & a_{k+2} & \cdots & a_{M+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{j+1} & \cdots & a_{j+k+1} & a_{j+k+2} & \cdots & a_{M+j} \end{pmatrix} \begin{pmatrix} n_0 \\ \vdots \\ n_k \\ n_{k+1} \\ \vdots \\ n_M \end{pmatrix} = \mathbf{z}.$$

- By Lemma 1, the coefficients in red are fixed.

Proof of Main Lemma: Part 3

- Let $\Theta = \sum_{i=1}^{\infty} a_i t^{-i}$ and let $n \in \mathbb{N}$ be such that

$$(K_n(\Theta(t)), V_n(P(t))) \in \left[\frac{a}{p^{j+1}}, \frac{a+1}{p^{j+1}}\right) \times \left[\frac{b}{p^{k+1}}, \frac{b+1}{p^{k+1}}\right).$$

- By Lemma 2, the base-p coefficients of n satisfy

$$\begin{pmatrix} a_1 & \dots & a_{k+1} & a_{k+2} & \dots & a_{M+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{j+1} & \dots & a_{j+k+1} & a_{j+k+2} & \dots & a_{M+j} \end{pmatrix} \begin{pmatrix} n_0 \\ \vdots \\ n_k \\ n_{k+1} \\ \vdots \\ n_M \end{pmatrix} = \mathbf{z}.$$

- By Lemma 1, the coefficients in red are fixed.

Proof of Main Lemma: Part 3

- Let $\Theta = \sum_{i=1}^{\infty} a_i t^{-i}$ and let $n \in \mathbb{N}$ be such that

$$(K_n(\Theta(t)), V_n(P(t))) \in \left[\frac{a}{p^{j+1}}, \frac{a+1}{p^{j+1}}\right) \times \left[\frac{b}{p^{k+1}}, \frac{b+1}{p^{k+1}}\right).$$

- By Lemma 2, the base-p coefficients of n satisfy

$$\begin{pmatrix} a_1 & \dots & a_{k+1} & a_{k+2} & \dots & a_{M+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{j+1} & \dots & a_{j+k+1} & a_{j+k+2} & \dots & a_{M+j} \end{pmatrix} \begin{pmatrix} n_0 \\ \vdots \\ n_k \\ n_{k+1} \\ \vdots \\ n_M \end{pmatrix} = \mathbf{z}.$$

- By Lemma 1, the coefficients in red are fixed.

Proof of Main Lemma: Part 3

- Let $\Theta = \sum_{i=1}^{\infty} a_i t^{-i}$ and let $n \in \mathbb{N}$ be such that

$$(K_n(\Theta(t)), V_n(P(t))) \in \left[\frac{a}{p^{j+1}}, \frac{a+1}{p^{j+1}}\right) \times \left[\frac{b}{p^{k+1}}, \frac{b+1}{p^{k+1}}\right).$$

- By Lemma 2, the base-p coefficients of n satisfy

$$\begin{pmatrix} a_{k+2} & \dots & a_{M+1} \\ \vdots & & \vdots \\ a_{j+k+2} & \dots & a_{M+j} \end{pmatrix} \begin{pmatrix} n_{k+1} \\ \vdots \\ n_M \end{pmatrix} = \mathbf{z} - \begin{pmatrix} a_1 & \dots & a_{k+1} \\ \vdots & & \vdots \\ a_{j+1} & \dots & a_{j+k+1} \end{pmatrix} \begin{pmatrix} n_0 \\ \vdots \\ n_k \end{pmatrix}.$$

- By Lemma 1, the coefficients in red are fixed.

Proof of Main Lemma: Part 4

Lemma 3 (Adiceam, Nesharim, Lunnon, 2020):

Let $\Theta(t) \in \mathbb{F}_q((t^{-1}))$. Then $\Theta(t) \in \mathbf{Bad}(t,p)$ with deficiency $D(\Theta)$ if and only if for any positive $k, l \in \mathbb{N}$, the Hankel matrix $H_\Theta(k, l, l + D(\Theta))$ has full rank over $\mathbb{F}_p$.

- If $I_1 \times I_2 \subset S_1$,

$$\underbrace{\begin{pmatrix} a_{k+2} & \cdots & a_{M+1} \\ \vdots & & \vdots \\ a_{j+k+2} & \cdots & a_{M+j} \end{pmatrix}}_{\text{full rank}} \begin{pmatrix} n_{k+1} \\ \vdots \\ n_M \end{pmatrix} = \mathbf{z} - \begin{pmatrix} a_1 & \cdots & a_{k+1} \\ \vdots & & \vdots \\ a_{j+1} & \cdots & a_{j+k+1} \end{pmatrix} \begin{pmatrix} n_0 \\ \vdots \\ n_k \end{pmatrix}.$$

- Let $N = \sum_{i=0}^{M} N_i p^i$ and recall $n = \sum_{i=0}^{m} n_i p^i < N$.
- Hence, $n_i \leq N_i$.

Open Problems and Conjectures

Open Problems and Conjectures

Conjecture 1:

For any choice of irreducible polynomial $P(t) \in \mathbb{F}_p[t]$, the set $\mathbf{Bad}(P(t), q)$ is non-empty unless $q = 2$.

Conjecture 2:

Let $P(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial. Then the hybrid sequence $(K_n(\Theta), V_n(P(t)))$ generated from some $\Theta(t) \in \mathbf{Bad}(P(t), q)$ is low discrepancy.

Conjecture 3:

Let $\Theta(t) \in \mathbb{F}_p((t^{-1}))$ be a Laurent series, let $k \in \mathbb{N}$ and let $P_1(t), \ldots, P_k(t) \in \mathbb{F}_p[t]$ be coprime irreducible polynomials. Assume that $\Theta(t) \in \mathbf{Bad}(P_i(t), q)$ for all $1 \leq i \leq k$. Then the $(k+1)$-dimensional digital Kronecker-Halton sequence defined by $\Theta(t)$ and the polynomials $P_i(t)$ is low discrepancy.