Rational self-affine tiles associated to standard and nonstandard digit systems

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A nonstandard digit system

Consider a digit system $(\alpha, \mathcal{D})$ with *base* $\alpha = 4$ and *digits* $\mathcal{D} = \{0, 1, 8, 9\}$.

We say this digit system is **nonstandard** because $\mathcal{D}$ is not a residue set of $\mathbb{Z}$ modulo 4.

We can geometrically visualize that this digit system, despite being nonstandard, allows us to represent all the real numbers.

Consider the set of **integer parts** given by

$$\Delta = \{\pm(d_k \dots d_1 d_0)_4 \mid d_j \in \{0, 1, 8, 9\}\}.$$

It is easy to see that

$$\Delta = 4\mathbb{Z} \cup (4\mathbb{Z} + 1).$$

Since $\mathcal{D}$ is nonstandard, Δ is not a group.

We consider the set of **fractional parts**

$$\mathcal{F} := \{(0.d_{-1}d_{-2}\dots)_4 \mid d_j \in \mathcal{D}\}.$$

We see next how to express this set as the solution of a set equation.

Let $x = (0.d_{-1}d_{-2}\dots)_4 \in \mathcal{F}$, $d_j \in \mathcal{D}$.

Multiplying x by the base $\alpha = 4$ implies moving the “decimal point" one place to the right

$$4x = (d_{-1}.d_{-2}d_{-3}\dots)_4 \in \mathcal{F} + d_{-1}$$

Proposition

The set $\mathcal{F} = \mathcal{F}(\alpha, \mathcal{D})$ of fractional parts of the digit system $(\alpha, \mathcal{D})$ is the only non empty compact subset of $\mathbb{R}$ satisfying

$$\alpha\mathcal{F} = \bigcup_{d\in\mathcal{D}} \mathcal{F} + d$$

In our example we obtain

$$\mathcal{F}(\alpha, \mathcal{D}) = [0,1] \cup [2,3].$$

$\mathcal{F}$:

$4\mathcal{F}$:

$\mathcal{F} \cup (\mathcal{F}+1) \cup (\mathcal{F}+8) \cup (\mathcal{F}+9)$:

If we translate $\mathcal{F}$ by the set of integer parts

$$\Delta = 4\mathbb{Z} \cup (4\mathbb{Z} + 1),$$

we obtain a tiling of $\mathbb{R}$.

Since Δ is not a group, it is not a lattice tiling.

The existence of such a tiling implies that almost every $x \in \mathbb{R}$ can be uniquely decomposed as the sum of a point in Δ and a point in $\mathcal{F}$ (integer part + fractional part).

This is always true for standard digit systems, but characterizing nonstandard digit systems that give rise to tilings is a hard open problem in number theory.

The smallest lattice containing the digit set $\mathcal{D} = \{0, 1, 8, 9\}$ is $\mathbb{Z}$.

The collection $\mathbb{Z} + \mathcal{F}$ is a multiple tiling of order 2 of $\mathbb{R}$, meaning almost every point gets covered by exactly 2 translates of $\mathcal{F}$

- Standard digit systems: one lattice tiling.
- Nonstandard digit systems: one tiling, one lattice multiple tiling.

Next, we consider the analogous n-dimensional setting.

Integral self-affine tiles in $\mathbb{R}^n$.

Let $A \in \mathbb{Z}^{n\times n}$ be an expanding matrix, meaning that all its eigenvalues have modulus greater than 1.

Let $\mathcal{D} \subset \mathbb{Z}^n$ be a set of digits such that $|\mathcal{D}| = |\det A|$.

We say that $(A, \mathcal{D})$ is a standard digit system if $\mathcal{D}$ is a complete set of residues of $\mathbb{Z}^n/A\mathbb{Z}^n$. Otherwise we call it nonstandard.

Definition

We define the set $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ as the unique non empty compact set that satisfies the set equation

$$A\mathcal{F} = \bigcup_{d\in\mathcal{D}} \mathcal{F} + d.$$

If $\mathcal{F}$ has positive Lebesgue measure, it is called a *self-affine tile*.

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} \mathcal{F} + d.$$

In order to define a tiling, we want the union on the right to be essentially disjoint.

If λ is the Lebesgue measure, then $\lambda(A\mathcal{F}) = |\det A|\lambda(\mathcal{F})$, and that is why we require $|\mathcal{D}| = |\det A|$.

$$\mathcal{F} = \bigcup_{d \in \mathcal{D}} A^{-1}(\mathcal{F} + d).$$

The existence and uniqueness of $\mathcal{F}$ is guaranteed by Hutchinson's theorem because A^{-1} is a contraction.

$\mathcal{F}$ is called self-affine because it is equal to a union of contracted copies of itself.

Without loss of generality we can assume $\mathbf{0} \in \mathcal{D}$ (we can always "shift" the tiling by shifting the digits).

An expansion of a vector $x \in \mathbb{R}^n$ in base A has the form

$$x = A^k d_k + \cdots + d_0 + A^{-1} d_{-1} + \cdots$$

where $d_j \in \mathcal{D}$, $d_k \neq \mathbf{0}$.

$\mathcal{F}$ corresponds to the set of fractional parts in base A.

Lagarias and Wang considered integral self-affine tiles in $\mathbb{R}^n$ associated to standard and nonstandard digit systems, and proved:

- Integral self-affine tiles are the closure of their interiors and their boundary has measure zero.
- Integral self-affine tiles give a tiling of $\mathbb{R}^n$.
- If $\mathcal{F}$ is an integral self-affine tile, the collection $\mathcal{F} + \mathbb{Z}^n$ gives a multiple tiling of $\mathbb{R}^n$.

Today I will present a generalization of these theorems.

Example of $\mathcal{F}(A, \mathcal{D})$ in $\mathbb{R}^2$

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \mathcal{D} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\}.$$

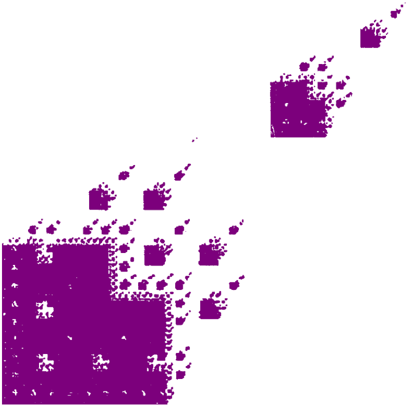


Image Credits: Lagarias and Wang

Rational self-affine tiles

Rational self-affine tiles were introduced by W. Steiner and J. Thuswaldner. They consider a base which is an algebraic number $\alpha \in \mathbb{C}$. They showed that it is equivalent to looking at digit systems where the base is the companion matrix A of the minimal polynomial of α.

They consider a digit set $\mathcal{D}$ that is a residue set for $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$, and a certain representation space $\mathbb{K}_\alpha$ of the form $\mathbb{R}^n \times \prod_\mathfrak{p} K_\mathfrak{p}$, where each $K_\mathfrak{p}$ is a completion of $\mathbb{Q}(\alpha)$ w.r.t. a certain absolute value.

They defined rational-self affine tiles as subsets of $\mathbb{K}_\alpha$, and proved that they always tile the space.

Our setting is a generalization of this, because we don't require A to have irreducible characteristic polynomial. We also define our representation space in a more general and simpler way.

Moreover, we allow digit systems that are nonstandard, and generalize the results obtained by Lagarias and Wang.

From here onwards we set $A \in \mathbb{Q}^{n \times n}$ to be an expanding rational matrix.

Suppose we wanted to consider a digit system $(A, \mathcal{D})$ for some digit set $\mathcal{D} \subset \mathbb{R}^n$.

Analogously as before, we would have a set $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ as the solution of the set equation

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} \mathcal{F} + d.$$

If $\mathcal{F}$ has positive measure and we require the union on the right to be essentially disjoint, this amounts to $|\mathcal{D}| = |\det A|$, which is not doable if the determinant is not an integer.

We will construct a space $\mathbb{K}_A$ where A acts as an integer matrix, meaning that it scales the measure of sets by an integer factor. We will consider rational self-affine sets as subsets of $\mathbb{K}_A$.

Consider the ring

$$\mathbb{Z}^n[A] := \bigcup_{k=1}^{\infty} \left(\mathbb{Z}^n + A\mathbb{Z}^n + \cdots + A^{k-1}\mathbb{Z}^n \right)$$

Definition (Digit system)

Let $\mathcal{D} \subset \mathbb{Z}^n[A]$ be such that

$$|\mathcal{D}| = |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|.$$

Then $(A, \mathcal{D})$ constitutes a *digit system*, where A is the *base* and $\mathcal{D}$ is the *digit set*.
When $\mathcal{D}$ is a complete set of residue class representatives of $\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]$, we say that $(A, \mathcal{D})$ is a *standard digit system*. Otherwise, we say it is *nonstandard*.

From now on, set

$$B := A^{-1}$$

and

$$a := |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]| = |\mathcal{D}|, \qquad b := |\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]|.$$

Proposition

$$|\det A| = \frac{a}{b}.$$

We define on $\mathbb{Z}^n(B)$ the *valuation* $\nu : \mathbb{Z}^n(B) \to \mathbb{Z} \cup \{\infty\}$ as

$$\nu(y) := \inf\{k \in \mathbb{Z} \mid y \in B^k\mathbb{Z}^n[B] \setminus B^{k+1}\mathbb{Z}^n[B]\}.$$

Consider the space $\mathbb{Z}^n((B))$ of Laurent series of powers of B with coefficients in $\mathbb{Z}^n$. That is, an element $y \in \mathbb{Z}^n((B))$ is of the form

$$y = \sum_{j=\nu(y)}^{\infty} B^j y_j, \quad y_j \in \mathbb{Z}^n, \tag{1}$$

where $y_{\nu(y)}$ is nonzero, and $\nu(0) := \infty$.

Then $\nu : \mathbb{Z}^n((B)) \to \mathbb{Z} \cup \{\infty\}$ is an extension of the valuation to all of $\mathbb{Z}^n((B))$. We will see that in the one dimensional case it coincides the the b-adic valuation (better known when b is a prime).

For this reason, we refer to series of the form (1) as *B-adic series*.

On $\mathbb{Z}^n((B))$ the B-*adic metric* is defined by

$$\mathbf{d}_B(y, y') := b^{-\nu(y-y')},$$

where $b = |\mathbb{Z}^n[B]/B\mathbb{Z}^n[B]|$, with the convention that $b^{-\infty} = 0$.

Back to the digit system, suppose we take series of the form

$$\sum_{j=k}^{\infty} A^{-j} d_j, \quad d_j \in \mathcal{D}.$$

In some sense, $\mathcal{D}$ has too many digits for the base A, which leads to different strings of digits converging to the same point.

What we want to do is to consider a pair $(x, y) \in \mathbb{R}^n \times \mathbb{Z}^n((B))$, where

$$x = \sum_{j=k}^{\infty} A^{-j} d_j, \quad d_j \in \mathcal{D}$$

and

$$y = \sum_{j=k}^{\infty} B^{j} d_j, \quad d_j \in \mathcal{D}.$$

That is, they are both the same series but one converges in $\mathbb{R}^n$ and the other one in $\mathbb{Z}^n((B))$.

In this way, we could achieve uniqueness of expansion almost everywhere and hence define a tiling.

A rational example

Suppose our base is $-\frac{3}{2}$ and let $\mathcal{D} = \{0, 1, 2\}$.

Then $\mathcal{D}$ is a complete set of representatives of $\mathbb{Z}[-\frac{3}{2}]/(-\frac{3}{2})\mathbb{Z}[-\frac{3}{2}]$, meaning this is a standard digit system.

The set of integer parts is given by

$$\mathbb{Z}[\tfrac{1}{2}] = \Big\{ \sum_{j=0}^{k} (-\tfrac{3}{2})^j d_j \mid d_j \in \mathcal{D} \Big\}.$$

We want this to be the translation set for our tiling, but this is not possible because it is not discrete.

However, note that the points of $\mathbb{Z}[\frac{1}{2}]$ that are close in the Euclidean distance, are far appart in the 2-adic metric.

For example,

$$|0 - \tfrac{1}{32}| = \tfrac{1}{32} \quad \text{but} \quad |0 - \tfrac{1}{32}|_2 = 32.$$

Consider the space

$$\mathbb{R} \times \mathbb{Q}_2.$$

We can define a metric **d** on $\mathbb{R} \times \mathbb{Q}_2$ given by

$$\mathbf{d}((x_1, x_2), (y_1, y_2)) := \max\{|x_1 - y_1|, |x_2 - y_2|_2\}.$$

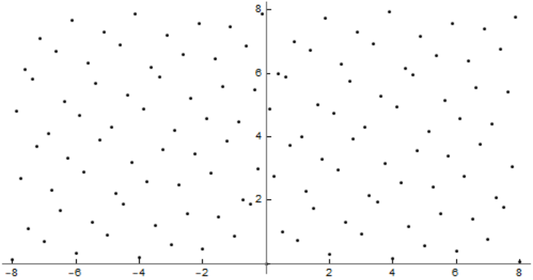
In fact, given $x, y \in \mathbb{Z}[\frac{1}{2}]$ we have that

$$\max\{|x - y|, |x - y|_2\} \geqslant 1.$$

Consider the embedding

$$\varphi : \mathbb{Q} \to \mathbb{R} \times \mathbb{Q}_2, \quad x \mapsto (x, x).$$

It turns out that, under the metric **d**, the set $\varphi(\mathbb{Z}[\frac{1}{2}])$ is a discrete subgroup of $\mathbb{R} \times \mathbb{Q}_2$ (it is a lattice).



Note that the space $\mathbb{Q}_2$ is isomorphic to the space $\mathbb{Z}((-\frac{2}{3}))$ of series of the form

$$y = \left(-\tfrac{2}{3}\right)^{\nu(y)} y_{\nu(y)} + \cdots + \left(-\tfrac{2}{3}\right)^{1} y_1 + \left(-\tfrac{2}{3}\right)^{2} y_2 + \ldots,$$

$y_j \in \mathbb{Z}$, where $\nu(y)$ is the 2-adic valuation. (More specificaly, $-\frac{2}{3}$ is a *uniformizer* for $\mathbb{Q}_2$.)

Definitions

Back to the general case, consider the space $\mathbb{Z}^n((B))$ of B-adic series together with the B-adic metric $\mathbf{d}_B$ defined before.

Definition (The representation space)

Given an expanding matrix $A \in \mathbb{Q}^{n\times n}$ and $B = A^{-1}$, define the *representation space* $\mathbb{K}_A$ as

$$\mathbb{K}_A := \mathbb{R}^n \times \mathbb{Z}^n((B))$$

with component wise addition.

We define the metric

$$\mathbf{d}((x,y),(x',y')) := \max\{\|x-x'\|, \mathbf{d}_B(y,y')\},$$

which turns $\mathbb{K}_A$ into a locally compact group.

Note that $\mathbb{Z}^n((B))$ is an ultrametric space, meaning it satisfies the strong triangle inequality:

$$\mathbf{d}_B(y, y') \leqslant \max\{\mathbf{d}_B(y, y''), \mathbf{d}_B(y'', y')\}$$

for every $y, y', y'' \in \mathbb{Z}^n((B))$.

We can interpret this by thinking that the elements of $\mathbb{Z}^n((B))$ have different "orders of magnitude" (given by the valuation).

Hence, the product $\mathbb{K}_A = \mathbb{R}^n \times \mathbb{Z}^n((B))$ can be seen as piling up layers of $\mathbb{R}^n$.

Also, every ball in $\mathbb{K}_A$ can be decomposed as a product of balls in each respective space

$$\mathbf{B}_r(x, y) = \mathbf{B}_r(x) \times \mathbf{B}_r(y).$$

We define the embedding

$$\begin{aligned}\varphi :&\mathbb{Z}^n[A, A^{-1}] \to \mathbb{K}_A\\ &x \mapsto (x, x).\end{aligned}$$

It turns out that $\varphi(\mathbb{Z}^n[A])$ is a **lattice** in $\mathbb{K}_A$.

We can define on $\mathbb{Z}^n((B))$ a Haar measure μ_B that is compatible with the topology, meaning that if M is a measurable set, then $\mu_B(B\,M) = \frac{1}{b}\,\mu_B(M)$.

Let λ be the Lebesgue measure in $\mathbb{R}^n$.

We consider on $\mathbb{K}_A$ the product measure $\mu := \lambda \times \mu_B$.
Let $M = M_1 \times M_2 \subset \mathbb{K}_A$ be a measurable set. We have

$$\lambda(AM_1) = \tfrac{a}{b}\lambda(M_1), \qquad \mu_B(A\,M_2) = b\mu_B(M_2),$$

This yields

$$\mu(A\,M) = \lambda(A\,M_1)\,\mu_B(A\,M_2) = a\,\mu(M).$$

Now our number system has "enough space" for a digits.

Given $(x_1, x_2) \in \mathbb{K}_A$, we seek expansions of the form

$$(x_1, x_2) = A^k\varphi(d_k) + \cdots + \varphi(d_0) + A^{-1}\varphi(d_{-1}) + \cdots$$

where $d_j \in \mathcal{D}$, $d_k \neq \mathbf{0}$.

The first coordinate is a convergent sequence in $\mathbb{R}^n$ and the second one is a convergent sequence in $\mathbb{Z}^n((B))$.

Definition (Rational self-affine tile)

Define $\mathcal{F} = \mathcal{F}(A, \mathcal{D}) \subset \mathbb{K}_A$ as the unique nonempty compact set satisfying the set equation

$$A\mathcal{F} = \bigcup_{d \in \mathcal{D}} (\mathcal{F} + \varphi(d)).$$

The set $\mathcal{F}$ is given explicitly by

$$\mathcal{F} = \left\{ \sum_{j=1}^{\infty} A^{-j} \varphi(d_j) \mid d_j \in \mathcal{D} \right\}.$$

If $\mu(\mathcal{F}) > 0$ then $\mathcal{F}$ is called a *rational self-affine tile*.

Let $\mathcal{H}(\mathbb{K}_A)$ be the family of nonempty compact subsets of $\mathbb{K}_A$, and consider the maps

$$\Psi_d : \mathcal{H}(\mathbb{K}_A) \to \mathcal{H}(\mathbb{K}_A); \quad X \mapsto A^{-1}(X + \varphi(d)).$$

Then $\mathcal{F}$ is the attractor of the iterated function system

$$\{\Psi_d \mid d \in \mathcal{D}\}.$$

When $\mathcal{F}$ has zero measure, it is a (generalization of a) Cantor set.

Example

The tile $\mathcal{F}$ related to the digit system with base $\frac{4}{3}$ and digits $\{0, 1, 8, 9\}$. We represent the space $\mathbb{R} \times \mathbb{Q}_3$ in $\mathbb{R}^2$.

A natural question is: how do we know that $\mathcal{F}(\frac{4}{3}, \mathcal{D})$ has positive measure for $\mathcal{D} = \{0, 1, 8, 9\}$?

Note that $\mathcal{D}$ can be decomposed as a direct sum $\mathcal{D} = \{0, 1\} \oplus 4\,\{0, 2\}$.

So

$$\begin{aligned}\mathcal{F}(\tfrac{4}{3}, \mathcal{D}) &= \sum_{j=1}^{\infty} (\tfrac{4}{3})^{-j} \varphi(\{0, 1\} \oplus 4\,\{0, 2\}) \\ &= \sum_{j=1}^{\infty} (\tfrac{4}{3})^{-j} \varphi(\{0, 1\}) + 4 \sum_{j=1}^{\infty} (\tfrac{4}{3})^{-j} \varphi(\{0, 2\})\end{aligned}$$

Note that

$$
\begin{aligned}
4\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,2\}) &= 3\,.\tfrac{4}{3}\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,2\}) \\
&= 3\sum_{j=0}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,2\}) \\
&= 3\left(\varphi(\{0,2\}) + \sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,2\})\right) \\
&= \varphi(\{0,6\}) + \sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,6\}).
\end{aligned}
$$

Then

$$\begin{aligned}\mathcal{F}(\tfrac{4}{3},\mathcal{D}) &= \sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,1\}) + 4\sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,2\}) \\ &= \varphi(\{0,6\}) + \sum_{j=1}^{\infty}(\tfrac{4}{3})^{-j}\varphi(\{0,1\}\oplus\{0,6\}) \\ &= \varphi(\{0,6\}) + \mathcal{F}(\tfrac{4}{3},\{0,1,6,7\}).\end{aligned}$$

But $(\frac{4}{3},\{0,1,6,7\})$ is a standard digit system, so the set $\mathcal{F}(\frac{4}{3},\{0,1,6,7\})$ has positive measure, and hence so does $\mathcal{F}(\frac{4}{3},\mathcal{D})$.

Example

Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & \frac{5}{3} \end{pmatrix} \in \mathbb{Q}^{2\times 2}.$$

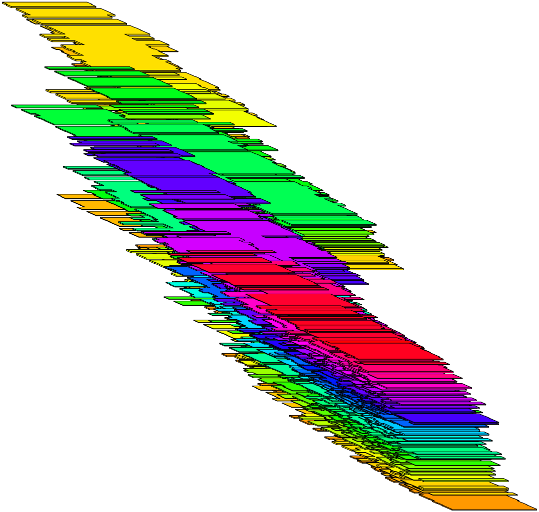
Then $\det A = \frac{10}{3}$, and it holds that $a = 10$ and $b = 3$.

It can be shown that the space $\mathbb{Z}^n((B))$ is isomorphic to $\mathbb{Q}_3$. Consider

$$\mathcal{D} := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 9 \end{pmatrix} \right\}.$$

Then $(A, \mathcal{D})$ is a nonstandard digit system.

We regard $\mathcal{F}(A, \mathcal{D})$ as a subset of $\mathbb{K}_A \simeq \mathbb{R}^2 \times \mathbb{Q}_3$. It turns out that $\mathcal{F}(A, \mathcal{D})$ has positive measure, that is, it is a **rational self-affine tile**.



The tile $\mathcal{F}(A, \mathcal{D})$ represented in $\mathbb{R}^3$.

Challenges of this theory

- We consider the n-dimensional setting, meaning we need tools of linear algebra when working with lattices or quotients. In particular, we use the Frobenius normal form of A to compute $|\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|$.
- Our representation space in non-Euclidean, which makes it harder to visualize the tilings and to draw pictures.
- When working with nonstandard digit systems, we loose the structure of the set of integer expansions and it is harder to define tilings.

Computing $|\mathcal{D}|$

We now show how to to compute $a = |\mathbb{Z}^n[A]/A\mathbb{Z}^n[A]|$.

Every $A \in \mathbb{Q}^{n\times n}$ is similar to a block diagonal matrix $F = \mathrm{diag}(C_1, \dots, C_k)$ called the **Frobenius normal form** of A (also known as rational canonical form), where each C_i is the companion matrix of a polynomial p_i $(1 \leqslant i \leqslant k)$.

These polynomials $p_i \in \mathbb{Q}[t]$ are assumed to be monic and to have the divisibility properties $p_1 \mid p_2 \mid \dots \mid p_k \mid \chi_A$.

With this assumption they are unique, and are the so-called *invariant factors* of A.

Proposition

Let $A \in \mathbb{Q}^{n\times n}$ be given, let p_i $(1 \leqslant i \leqslant k)$ be the corresponding invariant factors, and consider the integer polynomials $q_i = c_i p_i \in \mathbb{Z}[t]$, where each $c_i \in \mathbb{Z}$ is chosen so that q_i has coprime coefficients. Then

$$a = \prod_{i=1}^{k} |q_i(0)|.$$

Main results

We generalize the results obtained by Lagarias and Wang for integral self-affine tiles to the rational case.

Let $A \in \mathbb{Q}^{n\times n}$ be an expanding matrix and $(A, \mathcal{D})$ a digit system. Consider the associated set of fractional parts $\mathcal{F} = \mathcal{F}(A, \mathcal{D}) \subset \mathbb{K}_A$.

Theorem

The following assertions are equivalent:

- *$\mathcal{F}$ has postitive measure.*
- *$\mathcal{F}$ has nonempty interior.*
- *$\mathcal{F}$ is the closure of its interior and its boundary $\partial\mathcal{F}$ has measure zero.*

For $k \geqslant 1$, let $\mathcal{D}_k$ be the set of strings of k digits in base A, namely

$$\mathcal{D}_k := \left\{ d_0 + Ad_1 + \cdots + A^{k-1} d_{k-1} \mid d_j \in \mathcal{D} \right\}$$

and let $\mathcal{D}_\infty := \bigcup_{k \geqslant 1} \mathcal{D}_k$.

So, $\varphi(\mathcal{D}_\infty)$ is the set of integer parts.

Definition (Uniform discreteness)

We say that a set $M \subset \mathbb{K}_A$ is *uniformly discrete* if there exists $r > 0$ such that every open ball of radius r in $\mathbb{K}_A$ contains at most one point of M.

Recall that $\mathcal{F}(A,\mathcal{D})$ is a rational self-affine tile if it has positive measure.

Theorem

$\mathcal{F}(A,\mathcal{D})$ is a rational self-affine tile if and only if, for every $k \geqslant 1$, all a^k expansions in $\varphi(\mathcal{D}_k)$ are distinct and $\varphi(\mathcal{D}_\infty)$ is a uniformly discrete set.

Corollary

If $(A,\mathcal{D})$ is a standard digit system, then $\mathcal{F}(A,\mathcal{D})$ has positive measure.

For any $k \geqslant 1$ consider the difference sets

$$\mathcal{D}_k - \mathcal{D}_k = \{\bar{d} - \bar{d}' \mid \bar{d}, \bar{d}' \in \mathcal{D}_k\}$$

and define

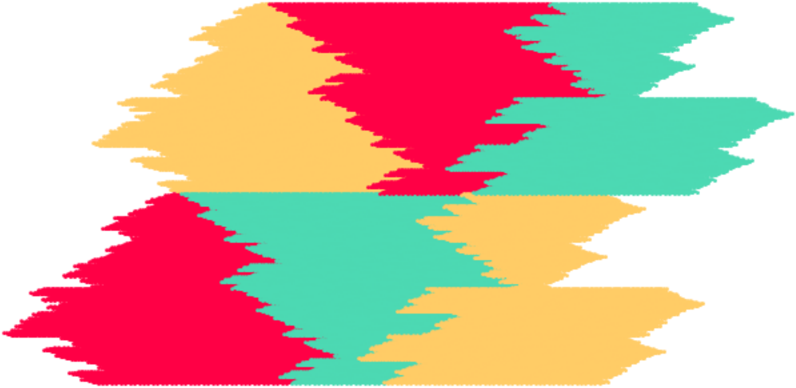
$$\Delta := \bigcup_{k=1}^{\infty} \varphi(\mathcal{D}_k - \mathcal{D}_k).$$

Theorem

Suppose $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ is a rational self-affine tile. Then:

- *There exists a set of translations $\Gamma \subset \Delta$ such that $\mathcal{F} + \Gamma$ tiles $\mathbb{K}_A$.*
- *If $\mathbf{0}$ is an interior point of $\mathcal{F}$ we can consider $\Gamma = \varphi(\mathcal{D}_\infty)$.*
- *If Δ is a lattice then Δ is a tiling set for $\mathcal{F}$.*

For the example $\alpha = -\frac{3}{2}$ and $\mathcal{D} = \{0, 1, 2\}$, the set of integer parts $\varphi(\mathcal{D}_\infty) = \varphi(\mathbb{Z}[\frac{1}{2}])$ is a lattice, and the collection $\mathcal{F}(\alpha, \mathcal{D}) + \varphi(\mathbb{Z}[\frac{1}{2}])$ gives a tiling of $\mathbb{R} \times \mathbb{Q}_2$.



Our final result is the existence of a multiple tiling. It is fairly easy to prove for standard digit systems, but in our setting we made use of character theory of locally compact abelian groups.

Theorem

If $\mathcal{F} = \mathcal{F}(A, \mathcal{D})$ is a rational self-affine tile, then $\mathcal{F} + \varphi(\mathbb{Z}^n[A])$ is a multiple tiling of $\mathbb{K}_A$.

Idea of proof:

Consider the torus $\mathbb{T} := \mathbb{K}_A/\varphi(\mathbb{Z}^n[A])$ and let π be the canonical projection. Define

$$\Phi(x, y) := |\pi^{-1}(x, y) \cap \mathcal{F}|.$$

meaning Φ counts the points on $\mathcal{F}$ that are congruent to (x, y) modulo $\varphi(\mathbb{Z}^n[A])$.

To prove that $\mathcal{F} + \varphi(\mathbb{Z}^n[A])$ is a multiple tiling, it suffices to show that $\Phi(x,y)$ is constantly equal to some $k \geqslant 1$ almost everywhere. The value k is the order of the multiple tiling.

Let $\bar{\mu}$ be the Haar measure on the torus. It suffices to show that every $S \subset \mathbb{T}$ satisfies

$$\int_S \Phi(x,y)\, d\bar{\mu}(x,y) = k\, \bar{\mu}(S).$$

Denote the multiplication by A on $\mathbb{T}$ as

$$\tau_A : \mathbb{T} \to \mathbb{T}; \quad (x,y) \mapsto A(x,y) \mod \varphi(\mathbb{Z}^n[A]).$$

Using the set equation of $\mathcal{F}$ on doing some change of variables we see that

$$\int_S \Phi(x,y)d\bar{\mu}(x,y) = \int_{\tau_A^{-1}(S)} \Phi(x,y)\, d\bar{\mu}(x,y).$$

It holds that Φ is constant a.e. if τ_A is ergodic.

To prove that τ_A is ergodic, we give an explicit formula for all the characters of $\mathbb{K}_A$ and of $\mathbb{T}$ and show that the trivial character $\chi = 1$ is the only character of $\mathbb{T}$ that satisfies $\chi \circ \tau_A^k = \chi$ for some $k \geqslant 1$. (This implies ergodicity).

Open questions

- Let $a \in \mathbb{Z}$ and $\mathcal{D} \subset \mathbb{Z}$. Suppose $\mathcal{F}(a, \mathcal{D})$ is a self-affine tile in $\mathbb{R}$. Let $2 \leqslant b < a$ such that a and b are coprime. Is $\mathcal{F}(\frac{a}{b}, \mathcal{D})$ a rational self-affine tile in $\mathbb{R} \times \mathbb{Q}_b$?

- Given $\frac{a}{b}$, can we classify all digit sets $\mathcal{D}$ such that $\mathcal{F}(\frac{a}{b}, \mathcal{D})$ is a rational self-affine tile when a is a prime power, a product of two primes, etc.?

- Can we give an analytic criterion in terms of $\mathcal{D}$ to determine whether $\mathcal{F}(A, \mathcal{D})$ is a rational self-affine tile?

Thank you!