The carry propagation of the successor function

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Joint work with

Valérie Berthé, Christiane Frougny, and Michel Rigo

One World Numeration Seminar, 17 November 2020
Based on the paper

The carry propagation of the successor function,

*Advances in Applied Mathematics* **120** (2020)
Part I

What is the carry propagation?
Adding machine

The Pascaline (1642)

featured the first carry propagation mechanism
Carry propagation

1 1 0 0 1 0 1 1 1

1 1 0 0 1 1 1 0

203

102
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
& & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad 203 \\
+ & & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & \quad 102 \\
\hline
& & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & 1 \\
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\downarrow & & & & & & & \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 1
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
& & & & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & & & & 203 \\
+ & & & & & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & & & & & 102 \\
\hline
& & & & & 0 & 0 & 0 & 0 & 0 & 1 & & & & & & &
\end{array}
\]
Carry propagation

\[ \begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 1 \\
\end{array} \]

203 + 102 = 100001
Carry propagation

\[
\begin{array}{cccccccc}
& & & & 1 & 1 & 0 & 0 \\
\text{+} & & & & 1 & 1 & 0 & 0 \\
\hline
& & & & 1 & 1 & 0 & 0
\end{array}
\]
Carry propagation

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}
\]
Carry propagation

1 1 0 0 1 0 1 1
+ 1 1 0 0 1 1 1 0

1 0 0 1 1 0 0 0 1
Carry propagation prevents addition to be parallelizable
Theorem (von Neumann et al. 63, Knuth 78, Pippenger 02)

Average carry propagation length for addition of two uniformly distributed $n$-digit binary numbers =

$$\log_2(n) + O(1)$$
Carry propagation for successor function in base 2

1 1 0 0 1 0 1 1

203
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & 1 & & \\
\hline
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 203
\end{array}
\]
Carry propagation for successor function in base 2

\[ \begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{array} \]

\[ + \]

\[ \begin{array}{c}
1 \\
\end{array} \]

\[ \hline \]

\[ \begin{array}{c}
0 \\
\end{array} \]

203
Carry propagation for successor function in base 2

\[ \begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & 1 & \\
\hline
   &   &   &   &   &   & 0 & 0 \\
\end{array} \]

203
Carry propagation for successor function in base $2$

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\downarrow & & & & & & & \\
1 & 0 & 0 & 0
\end{array}
\]

$203 + 1 = 204$
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\hline
+ & & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 0 & & &
\end{array}
\]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
\]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
\]

\[\text{cp}_2(203) = 3\]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
\]

\[\text{cp}_2(203) = 3\]

Amortized carry propagation (in base 2)

\[\text{CP}_2 = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_2(i)\]
Carry propagation for successor function in base 2

\[
\begin{array}{cccccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad \text{203} \\
+ & & & & & & & & \quad 1 \\
\hline
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \quad \text{204}
\end{array}
\]

\[c_{p2}(203) = 3\]

Amortized carry propagation (in base 2)

\[C_{P2} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p2}(i)\]

if it exists!
Carry propagation for successor function in base 2

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Carry propagation for successor function in base 2

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Carry propagation for successor function in base 2

\[
CP_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2
\]
Carry propagation for successor function in base $p$:

$$\text{CP}_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2$$

$$\text{CP}_p = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{p}{p-1}$$
Carry propagation for successor function in base Fibonacci

1 0 0 1 0 1 0 1
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\end{array}
\]
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & & 1 \\
\hline
& & & & & & & & 46 \\
\end{array}
\]
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & & 1 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & & 1 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

46

47
Carry propagation for successor function in base Fibonacci

\[
\begin{array}{cccccccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
+ & & & & & & & & & & \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 47 \\
\end{array}
\]

\[cp_F(46) = 6\]
Carry propagation for successor function in base Fibonacci

$$\begin{array}{cccccccccc}
55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\hline
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

$$cp_F(46) = 6$$

Amortized carry propagation in base Fibonacci

$$CP_F = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_F(i)$$

if it exists!
Carry propagation for successor function in base Fibonacci

. 0
  1 1
   10 2
    100 3
     101 4
      1000 5
       1001 6
        1010 7
         10000 8
          10001 9
           10010 10
            10100 11
             10101 12
              100000 13
               100001 14
                100010 15
                 100100 16
                  100101 17
Carry propagation for successor function in base Fibonacci

```
  .  0  1
  1  1  2
 10  2  3
100  3  1
101  4  4
1000  5  1
1001  6  2
1010  7  5
10000  8  1
10001  9  2
10010 10  3
10100 11  1
10101 12  6
100000 13  1
100001 14  2
100010 15  3
100100 16  1
100101 17  4
```
Carry propagation for successor function in base Fibonacci

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<td>14</td>
<td>2</td>
</tr>
<tr>
<td>1 0 0 0 1 0</td>
<td>15</td>
<td>3</td>
</tr>
<tr>
<td>1 0 0 1 0 0</td>
<td>16</td>
<td>1</td>
</tr>
<tr>
<td>1 0 0 1 0 1</td>
<td>17</td>
<td>4</td>
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</tbody>
</table>
Carry propagation for successor function in base Fibonacci

\[ \text{CP}_F = \frac{\varphi}{\varphi - 1} \]
Part II

A first observation and its 3 consequences
A primal observation

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]
\[ cp_F(10) = 3 \]

The Fibonacci tree
A primal observation

The Fibonacci tree

\[ \text{cp}_F(8) = 1 \]
\[ \text{cp}_F(9) = 2 \]
\[ \text{cp}_F(10) = 3 \]
\[ \text{cp}_F(11) = 1 \]
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]
\[ cp_F(10) = 3 \]
\[ cp_F(11) = 1 \]
\[ cp_F(12) = 6 \]

The Fibonacci tree
A primal observation

\[
\begin{align*}
    cp_F(8) &= 1 \\
    cp_F(9) &= 2 \\
    cp_F(10) &= 3 \\
    cp_F(11) &= 1 \\
    cp_F(12) &= 6
\end{align*}
\]

\[
\sum_{i=8}^{12} cp_F(i) = 13
\]

The Fibonacci tree
A primal observation

\[ cp_F(8) = 1 \]
\[ cp_F(9) = 2 \]
\[ cp_F(10) = 3 \]
\[ cp_F(11) = 1 \]
\[ cp_F(12) = 6 \]

\[ \sum_{i=8}^{i=12} cp_F(i) = 13 \]

The Fibonacci tree
What we learn from the primal observation

- A framework: the Abstract Numeration System model

- A general working hypothesis: Prefix-closed Extendable Languages

- An essential parameter: The local growth rate
What we learn from the primal observation: the ANS model

A framework:
the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet e.g. $A = \{0, 1\}$

$\Rightarrow A^*$ equipped with the *radix ordering*

i.e. ordered first *by length*, and then, for words of equal length, ordered *lexicographically*

e.g. $A^* = \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$
What we learn from the primal observation: the ANS model

- A framework:
  the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- $A$ finite *totally ordered* alphabet  \( \text{e.g. } A = \{0, 1\} \)
  \( \Rightarrow \) $A^*$ equipped with the *radix ordering*

- $L \subseteq A^*$ any *language* over $A^*$ ordered by radix ordering

  \[ F = \varepsilon \cup 1 A^* \setminus A^* 11 A^* \]

  \( \text{e.g. } F = \varepsilon, 1, 10, 100, 101, 1000, \ldots \)
What we learn from the primal observation: the ANS model

- A framework: the Abstract Numeration System model

**Definition (Lecomte & Rigo 2001)**

- A finite *totally ordered* alphabet e.g. \( A = \{0, 1\} \)

\( \Rightarrow \) \( A^* \) equipped with the *radix ordering*

- \( L \subseteq A^* \) any *language* over \( A^* \) ordered by radix ordering

\( \Rightarrow \) Natural integers are given *representations* by means of words of \( L \)

\( i.e. \langle n \rangle_L = (n + 1)\)-th word of \( L \) in the radix ordering

*e.g.* \( \langle 6 \rangle_F = 1001 \)
What we learn from the primal observation: the ANS model

► A framework:
   the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

• \( A \) finite \textit{totally ordered} alphabet \( \text{e.g. } A = \{0, 1\} \)
\( \Rightarrow \) \( A^* \) equipped with the \textit{radix ordering}

• \( L \subseteq A^* \) any \textit{language} over \( A^* \)
   ordered by radix ordering
\( \Rightarrow \) Natural integers are given \textit{representations}
   by means of words of \( L \)

\( L \subseteq A^* \) (together with the order on \( A \)) defines an \textit{ANS}

Fact: \textit{All ‘classical’ numeration systems are ANS}
What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

\begin{align*}
L_2 &= 1(0, 1)^* \\
. &= 0 \\
1 &= 1 \\
1 0 &= 2 \\
1 1 &= 3 \\
1 0 0 &= 4 \\
1 0 1 &= 5 \\
1 1 0 &= 6 \\
1 1 1 &= 7 \\
1 0 0 0 &= 8 \\
1 0 0 1 &= 9 \\
1 0 1 0 &= 10 \\
1 0 1 1 &= 11 \\
1 1 0 0 &= 12 \\
1 1 0 1 &= 13 \\
1 1 1 0 &= 14 \\
1 1 1 1 &= 15 \\
1 0 0 0 0 &= 16
\end{align*}
What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

\[ L_F = 1(0, 1)^* \setminus (0, 1)^*11(0, 1)^* \]
What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language $O$ (for *Oscillating*)

\[
\langle 10 \rangle_O = \text{a, b, c, d}
\]

\[
\text{dada}
\]

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\text{d} & \text{d a} & \text{d b} & \text{d c} & \text{d d} & \text{d a d} & \text{d b d} \\
\text{d b} & \text{d c d} & \text{d d d} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} \\
\text{d c} & \text{d d d} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} & \text{d b d a} \\
\text{d d} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} & \text{d b d a} & \text{d b d b} \\
\text{d a} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} & \text{d b d a} & \text{d b d b} \\
\text{d b} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} & \text{d b d a} & \text{d b d b} \\
\text{d c} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} & \text{d b d a} & \text{d b d b} \\
\text{d d} & \text{d a d a} & \text{d a d b} & \text{d a d c} & \text{d a d d} & \text{d b d a} & \text{d b d b} \\
\end{array}
\]
What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language \( \mathcal{V} \) (for \textit{Vibrating})

\[ \langle 10 \rangle_{\mathcal{V}} = bda \]
What we learn from the primal observation: an hypothesis

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
What we learn from the primal observation: an hypothesis

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]
What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model

- A general working hypothesis:
  Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]
What we learn from the primal observation: an hypothesis

- A framework:
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- A general working hypothesis:
  Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]

Notation

\[ u_L(\ell) = \text{card} \ (L \cap A^\ell) \]
\[ v_L(\ell) = \text{card} \ (L \cap A^{\leq \ell}) = \sum_{i=0}^{\ell} u_L(i) \]
What we learn from the primal observation: an hypothesis

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The formula we want:

\[ \sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell) \]
What we learn from the primal observation: an hypothesis

- A framework: 
  the Abstract Numeration System model

- A general working hypothesis: 
  Prefix-Closed Extendable Languages

\[ L \subseteq A^* \text{ an ANS} \]

**Notation**

\[
\begin{align*}
  u_L(\ell) &= \text{card} \ (L \cap A^\ell) \\
  v_L(\ell) &= \text{card} \ (L \cap A^{\leq \ell}) = \sum_{i=0}^{\ell} u_L(i)
\end{align*}
\]

The formula we want:

\[
\sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} \text{cp}_L(i) = v_L(\ell)
\]

requires \( L \) *prefix-closed* and *extendable*

i.e. to be a **PCE language**
What we learn from the primal observation: an hypothesis

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages

$L \subseteq A^*$ an ANS

Notation

\[ u_L(\ell) = \text{card} \ (L \cap A^\ell) \]
\[ v_L(\ell) = \text{card} \ (L \cap A^{\leq \ell}) = \sum_{i=0}^{\ell} u_L(i) \]

The formula we want:
\[ \sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell) \]

Fact: ‘All’ ‘classical’ ANS are PCE
What we learn from the primal observation: an hypothesis

‘All’ ‘classical’ ANS are PCE

$L_2$
What we learn from the primal observation: an hypothesis

The ANS we consider are PCE
What we learn from the primal observation: an hypothesis

The ANS we consider are PCE
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model

- A general working hypothesis: Prefix-Closed Extendable Languages

- An essential parameter: the local growth rate
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model

- A general working hypothesis: Prefix-Closed Extendable Languages

- An essential parameter: the local growth rate

From \[ \sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell) \]
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

From

$$\sum_{i=\nu_L(\ell-1)}^{\nu_L(\ell)-1} c_{\nu_L}(i) = \nu_L(\ell)$$

follows

$$\sum_{i=0}^{\nu_L(\ell)-1} c_{\nu_L}(i) = \sum_{j=0}^{\ell} \nu_L(j)$$
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

From
\[ \sum_{i=v_L(\ell-1)}^{v_L(\ell)-1} cp_L(i) = v_L(\ell) \]

follows
\[ \sum_{i=0}^{v_L(\ell)-1} cp_L(i) = \sum_{j=0}^{\ell} v_L(j) \]

hence, if
\[ CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \] exists
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

From

\[ \sum_{i=\nu_L(\ell-1)}^{\nu_L(\ell)-1} cp_L(i) = \nu_L(\ell) \]

follows

\[ \sum_{i=0}^{\nu_L(\ell)-1} cp_L(i) = \sum_{j=0}^{\ell} \nu_L(j) \]

hence, if \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \) exists

then \( CP_L = \lim_{\ell \to \infty} \frac{1}{\nu_L(\ell)} \sum_{j=0}^{\ell} \nu_L(j) \) exists.
Lemma

\[(x_\ell)_{\ell \in \mathbb{N}} \quad x_\ell \in \mathbb{R}_+ \quad \forall \ell \quad y_\ell = \sum_{j=0}^{\ell-1} x_j \quad \gamma > 1\]

TFAE

(i) \[\lim_{\ell \to \infty} \frac{x_{\ell+1}}{x_\ell} = \gamma\]

(ii) \[\lim_{\ell \to \infty} \frac{y_{\ell+1}}{y_\ell} = \gamma\]

(iii) \[\lim_{\ell \to \infty} \frac{y_\ell}{x_\ell} = \frac{\gamma}{\gamma - 1}\]
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

**Proposition**

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_L}(i) \) exists,
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_L}(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists.
What we learn from the primal observation: a new parameter

- A framework: the Abstract Numeration System model
- A general working hypothesis: Prefix-Closed Extendable Languages
- An essential parameter: the local growth rate

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_L(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists,

and \( CP_L = \frac{\gamma_L}{\gamma_L - 1} \)
What we learn from the primal observation: a new parameter

\[ L_2 \lim_{\ell \to \infty} \frac{u_2(\ell + 1)}{u_2(\ell)} = \gamma_2 = 2 \]
What we learn from the primal observation: a new parameter

\[ L_F \lim_{\ell \to \infty} \frac{u_F(\ell + 1)}{u_F(\ell)} = \gamma_F = \varphi \]
What we learn from the primal observation: a new parameter

\[
\frac{u_{O}(2\ell + 1)}{u_{O}(\ell)} = 1 \quad \frac{u_{O}(2\ell + 2)}{u_{O}(2\ell + 1)} = 4
\]
What we learn from the primal observation: a new parameter

\[ V \quad \mathbf{u}_V(\ell) = 32^{\ell-1} \quad \gamma_V = 2 \]
A natural question

**Proposition**

If \( \text{CP}_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_L}(i) \) exists,

then the **local growth rate** \( \lim_{\ell \to \infty} \frac{u_L(\ell+1)}{u_L(\ell)} = \gamma_L \) exists

and \( \text{CP}_L = \frac{\gamma_L}{\gamma_L - 1} \).
A natural question

Proposition

If \( CP_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_L}(i) \) exists,

then the local growth rate \( \lim_{\ell \to \infty} \frac{u_L(\ell + 1)}{u_L(\ell)} = \gamma_L \) exists

and \( CP_L = \frac{\gamma_L}{\gamma_L - 1} \)

Question

Is the existence of the local growth rate sufficient
to insure the existence of the carry propagation?
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]

\[ u_U(\ell) = 2^\ell \quad \frac{u_U(\ell + 1)}{u_U(\ell)} = \gamma_U = 2 \]
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
An unbalanced tree

$U \subseteq \{a, b, c\}^*$
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]

\[ \lim_{\ell \to \infty} \frac{1}{v_U(\ell)} \sum_{j=0}^{v_U(\ell)-1} cp_{U(j)} = 2 \]
An unbalanced tree

$U \subseteq \{a, b, c\}^*$
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
An unbalanced tree

$U \subseteq \{a, b, c\}^*$
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]

\[ \lim_{\ell \to \infty} \frac{1}{v'_U(\ell)} \sum_{j=0}^{v'_U(\ell)-1} cp_U(j) = \frac{11}{6} \neq 2 \]
An unbalanced tree

\[ U \subseteq \{a, b, c\}^* \]

\[
\text{CP}_U = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_U(i)
\]
does not exist
The *existence* of the carry propagation is more difficult to prove than the *computation* of the carry propagation itself.
Roadmap

Os

Vi

Ib

Rb

Un

Fb
Part III

Algebra
Ib  Rb  Un

Rational A N S

Vi  Ib  Fb

Os

Algebra
Surprise!

\[ u_V(\ell) = 3 \cdot 2^{\ell-1} \]
Surprise!

\[ u_V(\ell) = 3 \cdot 2^{\ell-1} \]

\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_V(i)} \leq \frac{28}{15} < \frac{13}{6} \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_V(i)}
\]
\( \lim\inf_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_V(i) \leq \frac{28}{15} < \frac{13}{6} \leq \lim\sup_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_V(i) \)

\( \gamma_V \) exists but \( CP_V \) does not exist
Rational A N S

Algebra
Generating functions

Definition

\[ L \subseteq A^* \quad g_L(z) \quad \text{generating function of} \quad L \]

\[ g_L(z) = \sum_{\ell=0}^{\infty} u_L(\ell) z^\ell \]
Generating functions

Definition

\[ L \subseteq A^* \quad g_L(z) \text{ generating function of } L \]

\[ g_L(z) = \sum_{\ell=0}^{\infty} u_L(\ell) z^\ell \]

\( L \) rational language \( \Rightarrow \) \( g_L(z) \) rational function

\[ g_L(z) = \frac{R(z)}{Q(z)} \quad R(z), Q(z) \in \mathbb{Z}[z] \]
Generating functions of rational languages

$L$ rational language $\implies g_L(z)$ rational function

$g_L(z)$ uniquely written as

$$g_L(z) = T(z) + \frac{S(z)}{Q(z)}$$

$T(z), S(z), Q(z) \in \mathbb{Z}[z]$ with $\deg S < \deg Q$ and $Q(0) \neq 0$
Generating functions of rational languages

\( L \) rational language \( \implies g_L(z) \) rational function

\[ g_L(z) \text{ uniquely written as} \]

\[ g_L(z) = T(z) + \frac{S(z)}{Q(z)} \]

\( T(z), S(z), Q(z) \in \mathbb{Z}[z] \)

with \( \deg S < \deg Q \) and \( Q(0) \neq 0 \)

\( P_L \) is the \textit{reciprocal polynomial} of \( Q \):

\[ P_L(z) = Q\left(\frac{1}{z}\right)z^{\deg Q} \]
Generating functions of rational languages

\[ L \text{ rational language} \implies g_L(z) \text{ rational function} \]

\[ g_L(z) \text{ uniquely written as} \]

\[ g_L(z) = T(z) + \frac{S(z)}{Q(z)} \quad T(z), S(z), Q(z) \in \mathbb{Z}[z] \]

with \( \deg S < \deg Q \) and \( Q(0) \neq 0 \)

\( P_L \) is the \textit{reciprocal polynomial} of \( Q \) : \( P_L(z) = Q(\frac{1}{z}) z^{\deg Q} \)

The \textit{eigenvalues} of \( L \) are the zeroes \( \lambda_1, \lambda_2, \ldots, \lambda_t \) of \( P_L \) and

\[ \forall \ell \in \mathbb{N} \quad u_L(\ell) = \sum_{j=1}^{t} \lambda_j^\ell P_j(\ell) \]

where \( \deg P_j = \text{multiplicity of } \lambda_j \text{ in } P_L \) minus 1
Theorem (Berstel 71)

\( f(z) \)  \( \mathbb{R}_+ \)-rational function (not a polynomial)
\( \lambda \) maximum of the moduli of its eigenvalues.

(i) \( \lambda \) is an eigenvalue of \( f(z) \) (hence an eigenvalue in \( \mathbb{R}_+ \))

(ii) Every eigenvalue of \( f(z) \) of modulus \( \lambda \)
is of the form \( \lambda e^{i\theta} \), where \( e^{i\theta} \) is a root of the unity

(iii) The multiplicity of any eigenvalue of modulus \( \lambda \)
is at most that of \( \lambda \)
Positive rational functions

Theorem (Berstel 71)

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(iii) The multiplicity of any eigenvalue of modulus \( \lambda \)

is at most that of \( \lambda \)

Definition

(i) \( f(z) \) is DEV if \( \lambda \) is the only eigenvalue of modulus \( \lambda \)

(ii) \( f(z) \) is ADEV if the multiplicity of \( \lambda \) is greater

than the multiplicity of the other eigenvalues of modulus \( \lambda \)
Some examples

\[ M_O = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \quad P_O = X^2 - 4 \]

\[ u_O(\ell) = \frac{3}{4} 2^\ell + \frac{1}{4} (-2)^\ell \]

\[ M_C = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \quad P_C = X^2 - 4 \]

\[ u_C(\ell) = 2^\ell \quad P_C = X - 2 \]

\[ M_D = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad P_D = (X^2 - 4)(2 - X) \]

\[ u_D(\ell) = \left( \frac{1}{4} \ell + \frac{7}{8} \right) 2^\ell + \frac{1}{8} (-2)^\ell \]
Some examples

- $O$ is neither DEV nor ADEV
  \[ u_O(\ell) = \frac{3}{4} 2^\ell + \frac{1}{4} (-2)^\ell \]

- $V$ is DEV
  \[ u_V(\ell) = \frac{3}{2} 2^\ell \]

- $D$ is ADEV but not DEV
  \[ u_D(\ell) = \left(\frac{1}{4} \ell + \frac{7}{8}\right) 2^\ell + \frac{1}{8} (-2)^\ell \]
Theorem
A rational language \( L \) is ADEV iff the local growth rate \( \gamma_L \) exists. In this case, the modulus of \( L \) is equal to \( \gamma_L \).
Theorem

$L$ ADEV rational PCE and $\lambda$ its modulus.

If every quotient of $L$ whose modulus is equal to $\lambda$ is ADEV, then $CP_L$ exists and $CP_L = \frac{\lambda}{\lambda - 1}$
Theorem

\( L \text{ ADEV rational PCE and } \lambda \text{ its modulus.} \)

If every quotient of \( L \) whose modulus is equal to \( \lambda \) is ADEV, then \( CP_L \) exists and \( CP_L = \frac{\lambda}{\lambda - 1} \).
Part IV

Ergodic Theory
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c p_L(i) \) exist?
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_L}(i) \) exist?

A rewriting

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c_{p_L}(\text{Succ}_L^i(\varepsilon)) \) exist?
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(Succ_i^L(\varepsilon)) \) exist?
An unmistakable fit

Our problem

Does \( \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} c \text{p}_L(\text{Succ}_L^i(\varepsilon)) \) exist?

The Ergodic Theorem

Theorem (Birkhoff 31)

Let \( (\mathcal{K}, \tau) \) be a dynamical system, \( \mu \) a \( \tau \)-invariant measure on \( \mathcal{K} \) and \( f: \mathcal{K} \to \mathbb{R} \) in \( L^1(\mu) \) (\( f \) is absolutely \( \mu \)-integrable). If \( (\mathcal{K}, \tau) \) is ergodic, then, for \( \mu \)-almost all \( s \) in \( \mathcal{K} \),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu \quad (*).
\]

If \( (\mathcal{K}, \tau) \) is uniquely ergodic and if \( f \) and \( \tau \) are continuous, then (*) holds for every \( s \) in \( \mathcal{K} \).
A bunch of definitions

- Dynamical system \((\mathcal{K}, \tau)\) = compact set \(\mathcal{K}\) equipped with \(\tau: \mathcal{K} \rightarrow \mathcal{K}\)
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- Probability measure \(\mu\) on \(\mathcal{K}\) is \(\tau\)-invariant if \(\tau\) measurable and \(\forall B\) measurable, \(\mu(\tau^{-1}(B)) = \mu(B)\)
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- \((\mathcal{K}, \tau)\) is ergodic if \(\tau^{-1}(B) = B\) implies \(\mu(B) = 0\) or \(1\) for every \(\tau\)-invariant measure \(\mu\)
A bunch of definitions

- Dynamical system \((\mathcal{K}, \tau) = \text{compact set } \mathcal{K}\) equipped with \(\tau : \mathcal{K} \to \mathcal{K}\)

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- \((\mathcal{K}, \tau)\) is uniquely ergodic if it admits a unique \(\tau\)-invariant measure
A bunch of definitions

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Theorem (Birkhoff 31)

Let \((\mathcal{K}, \tau)\) be a dynamical system, \(\mu\) a \(\tau\)-invariant measure on \(\mathcal{K}\) and \(f: \mathcal{K} \to \mathbb{R}\) in \(L^1(\mu)\) . If \((\mathcal{K}, \tau)\) is ergodic, then

for \(\mu\)-almost all \(s \in \mathcal{K}\) \[\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu\] \((*)\)

If \((\mathcal{K}, \tau)\) is uniquely ergodic and if \(f\) and \(\tau\) are continuous, then \((*)\) holds for every \(s\) in \(\mathcal{K}\).
Turning a numeration system into a dynamical system
Turning a numeration system into a dynamical system

- \( A = \{0, 1, \ldots, r - 1\} \)
Turning a numeration system into a dynamical system

- \( A = \{0, 1, \ldots, r - 1\} \)

- \( L \subseteq (A \setminus \{0\}) A^* : \) no word of \( L \) ‘begins’ with 0
Turning a numeration system into a dynamical system

- $A = \{0, 1, \ldots, r - 1\}$

- $L \subseteq (A \setminus \{0\}) A^*: \text{no word of } L \text{ ‘begins’ with } 0$

- $\omega A = \text{set of left infinite words over } A$
  
  $s = \cdots s_2 s_1 s_0 \quad \text{and} \quad s[\ell,j] = s_\ell s_{\ell-1} \cdots s_j$

  $w \mapsto \omega 0 w$ induces a bijection between $L$ and $\omega 0 L$
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- $\omega A$ with the ‘right factor distance’ topology is a compact set
  
  Basis: cylinders $[w] = \omega A w$

- Compactification of $L$: $K_L = \overline{\omega 0 L}$

  $K_L = \left\{ s \in \omega A \mid \forall j \in \mathbb{N} \exists w^{(j)} \in 0^* L \quad s_{[j,0]} \text{ right factor of } w^{(j)} \right\}$
Turning a numeration system into a dynamical system

Definition of the odometer
Definition of the odometer

- Since $L \subseteq (A \setminus \{0\})A^*$, $Succ_L$ defined on $\omega^0 L$ by
  $$Succ_L(\omega^0 w) = \omega^0 Succ_L(w)$$
Definition of the odometer

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Extension of the carry propagation
Turning a numeration system into a dynamical system

Definition of the odometer

- Since $L \subseteq (A \setminus \{0\})A^*$, $\text{Succ}_L$ defined on $\omega_0L$ by $\text{Succ}_L(\omega_0w) = \omega_0 \text{Succ}_L(w)$

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Extension of the carry propagation

$$\Delta(s, t) = \begin{cases} \min \{j \in \mathbb{N} \mid s[\infty, j] = t[\infty, j]\} & \text{if such } j \text{ exist} \\ +\infty & \text{otherwise} \end{cases}$$
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\})A^* \), \( \text{Succ}_L \) defined on \( \omega_0 L \) by \( \text{Succ}_L(\omega_0 w) = \omega_0 \text{Succ}_L(w) \)

- **Odometer** \( \tau_L \) on \( K_L = \) any extension of \( \text{Succ}_L \)

- **Succ}_L continuous** \( \Rightarrow \) \( \tau_L \) unique continuous extension of \( \text{Succ}_L \)

Extension of the carry propagation

\[
\Delta(s, t) = \begin{cases} 
\min \{ j \in \mathbb{N} \mid s_{[\infty,j]} = t_{[\infty,j]} \} & \text{if such } j \text{ exist} \\
+\infty & \text{otherwise}
\end{cases}
\]

\[
\forall s \in \omega A \quad \text{cp}_L(s) = \Delta(s, \tau_L(s))
\]
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\})A^* \), \( \text{Succ}_L \) defined on \( \omega 0 L \) by
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- **Succ\(_L\)** continuous \( \Rightarrow \) \( \tau_L \) unique continuous extension of \( \text{Succ}_L \)

Extension of the carry propagation

**Proposition**

If \( \tau_L \) is continuous,
then \( \text{cp}_L \) is continuous at any point where it takes finite values.
Turning a numeration system into a dynamical system

Definition of the odometer

- Since $L \subseteq (A \setminus \{0\})A^*$, $\text{Succ}_L$ defined on $\omega 0 L$ by
  $$\text{Succ}_L(\omega 0 w) = \omega 0 \text{Succ}_L(w)$$

- **Odometer** $\tau_L$ on $K_L$ = any extension of $\text{Succ}_L$

- $\text{Succ}_L$ continuous $\implies$ $\tau_L$ unique continuous extension of $\text{Succ}_L$

Where we are

We write $0 = \omega 0$

$$\text{CP}_L = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(\tau_L^i(0))$$
Turning a numeration system into a dynamical system

Definition of the odometer

- Since \( L \subseteq (A \setminus \{0\})A^* \), \( \text{Succ}_L \) defined on \( \omega 0 L \) by
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  \text{Succ}_L(\omega 0 w) = \omega 0 \text{Succ}_L(w)
  \]

- **Odometer** \( \tau_L \) on \( \mathcal{K}_L = \text{any extension of } \text{Succ}_L \)

- \( \text{Succ}_L \) continuous \( \implies \)
  \( \tau_L \) unique continuous extension of \( \text{Succ}_L \)

Where Birkhoff Theorem leads us

We want to show that \( 0 \) is a point such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \text{cp}_L(\tau^i_L(0)) = \int_{\mathcal{K}_L} \text{cp}_L \, d\mu
\]
Greedy numeration systems
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers

\[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]
Greedy numeration systems

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- *Greedy algorithm* yields *greedy G-expansion* of integer \( n \)
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers
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- **Greedy algorithm** yields **greedy** \( G \)-expansion of integer \( n \)
  - \( k \) defined by \( G_k \leq n < G_{k+1} \)
  - set \( x_k = q(n, G_k) \) and \( r_k = r(n, G_k) \)
Greedy numeration systems

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    set \( x_k = q(n, G_k) \) and \( r_k = r(n, G_k) \)
  - for every \( i, \ k - 1 \geq i \geq 0 \), set
    \( x_i = q(r_{i+1}, G_i) \) and \( r_i = r(r_{i+1}, G_i) \)
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  - for every \( i, k - 1 \geq i \geq 0 \), set
    
    \( x_i = q(r_{i+1}, G_i) \) and \( r_i = r(r_{i+1}, G_i) \)
  
  - \( n = x_k G_k + x_{k-1} G_{k-1} + \cdots + x_0 G_0 \)
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers
  \[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]

- **Greedy algorithm** yields greedy $G$-expansion of integer $n$
  
  - $k$ defined by $G_k \leq n < G_{k+1}$
    set $x_k = q(n, G_k)$ and $r_k = r(n, G_k)$
  
  - for every $i$, $k - 1 \geq i \geq 0$, set
    $x_i = q(r_{i+1}, G_i)$ and $r_i = r(r_{i+1}, G_i)$
  
  - $n = x_k G_k + x_{k-1} G_{k-1} + \cdots + x_0 G_0$

- $L_G = \{ \langle n \rangle_G \mid n \in \mathbb{N} \}$
Greedy numeration systems

- Basis = strictly increasing sequence of integers
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Greedy numeration systems

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  \[ G = (G_{\ell})_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]

- \[ L_G = \{ \langle n \rangle_G \mid n \in \mathbb{N} \} \]

- If \[ r = \limsup \left\lceil \frac{G_{\ell+1}}{G_{\ell}} \right\rceil \] is finite
  \[ L_G \subseteq A^*_G \text{ with } A_G = \{0, 1, \ldots, r - 1\} \]
Greedy numeration systems

- **Basis** = strictly increasing sequence of integers
  \[ G = (G_\ell)_{\ell \in \mathbb{N}} \text{ with } G_0 = 1 \]

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- If \( r = \limsup \left\lceil \frac{G_{\ell+1}}{G_\ell} \right\rceil \) is finite
  \[ L_G \subseteq A_G^* \text{ with } A_G = \{0, 1, \ldots, r-1\} \]

- \( 0^* L_G \) is closed under right factor and
  \[ K_G = \overline{\omega 0 L_G} = \{ s \in \omega A \mid \forall j \in \mathbb{N} \quad s_{[j,0]} \in 0^* L_G \} \]
Theorem (Barat–Grabner 16, Grabner–Liardet–Tichy 95)

Let $G$ be a GNS.

For every $s$ in $\mathcal{K}_G$, $\lim_{j \to \infty} \text{Succ}_G(s_{[j,0]})$ exists and defines the odometer $\tau_G : \mathcal{K}_G \to \mathcal{K}_G$:

$$\forall s \in \mathcal{K}_G \quad \tau_G(s) = \lim_{j \to \infty} \text{Succ}_G(s_{[j,0]}).$$
Ergodicity of greedy numeration systems

Theorem (Barat–Grabner 16, Grabner–Liardet–Tichy 95)

Let $G$ be a GNS. For every $s$ in $\mathcal{K}_G$, $\lim_{j \to \infty} \text{Succ}_G(s_{[j,0]}^{\ell})$ exists and defines the odometer $\tau_G : \mathcal{K}_G \to \mathcal{K}_G$:

$$\forall s \in \mathcal{K}_G \quad \tau_G(s) = \lim_{j \to \infty} \text{Succ}_G(s_{[j,0]}^{\ell}).$$

Definition

A GNS $G$ is said to be exponential if there exist two real constants $\alpha > 1$ and $C > 0$ such that $G_{\ell} \sim C\alpha^{\ell}$ when $\ell$ tends to infinity.
Ergodicity of greedy numeration systems

Theorem (Barat–Grabner 16, Grabner–Liardet–Tichy 95)
Let $G$ be a GNS.
For every $s$ in $\mathcal{K}_G$, \( \lim_{j \to \infty} \text{Succ}_G(s_{[j,0]}) \) exists and defines the odometer $\tau_G : \mathcal{K}_G \to \mathcal{K}_G$:
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Definition
A GNS $G$ is said to be exponential if there exist two real constants $\alpha > 1$ and $C > 0$ such that $G_\ell \sim C\alpha^\ell$ when $\ell$ tends to infinity.

Theorem (Barat–Downarowicz–Liardet 02)
If $G$ is an exponential GNS, then the dynamical system $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic.
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.

Corollary

Let $G$ be an exponential GNS with $G_\ell \sim C\alpha^\ell$.

If $L_G$ is PCE, then $CP_G$ exists and $CP_G = \frac{\alpha}{\alpha - 1}$. 
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.
Theorem

*If* $G$ *is an exponential GNS, then* $\text{CP}_G$ *exists.*

Proof essentially based on the work [Barat–Grabner 16]
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic $\mu_G$ the $\tau_G$-invariant measure.
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic
- $\mu_G$ the $\tau_G$-invariant measure.
- $0$ is a generic point
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic $\mu_G$ the $\tau_G$-invariant measure.
- $0$ is a generic point

$$\lim_{N \to \infty} \frac{1}{N} \text{card} \left( \{ i \mid \tau_G^i(0) \in [w] \} \right) = \mu_G([w])$$
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic $\mu_G$ the $\tau_G$-invariant measure.
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$$\lim_{N \to \infty} \frac{1}{N} \text{card} \left( \{ i \mid \tau_G^i(0) \in [w] \} \right) = \mu_G([w]) = \int_{\mathcal{K}_G} \chi_{[w]} \, d\mu_G$$
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic
- $\mu_G$ the $\tau_G$-invariant measure.
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$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{[w]}(\tau_G^i(0)) = \mu_G([w]) = \int_{\mathcal{K}_G} \chi_{[w]} \, d\mu_G$$
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

1. $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic $\mu_G$ the $\tau_G$-invariant measure.
2. $0$ is a generic point

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{[w]}(\tau_i^G(0)) = \mu_G([w]) = \int_{\mathcal{K}_G} \chi_{[w]} \, d\mu_G$$

3. Implies $\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau_i^G(0)) = \int_{\mathcal{K}_G} f \, d\mu_G$ for Riemann-integrable $f$
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

Proof essentially based on the work [Barat–Grabner 16]

$G$ exponential GNS implies

- $(\mathcal{K}_G, \tau_G)$ is uniquely ergodic 
  $\mu_G$ the $\tau_G$-invariant measure.

- 0 is a generic point

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_{[w]}(\tau_G^i(0)) = \mu_G([w]) = \int_{\mathcal{K}_G} \chi_{[w]} \ d\mu_G
\]

- Does not imply $\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau_G^i(0)) = \int_{\mathcal{K}_G} f \ d\mu_G$ for any $f$
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.

How the proof goes
Theorem

*If $G$ is an exponential GNS, then $\text{CP}_G$ exists.*

How the proof goes

- $\text{cp}_G$ is not *Riemann-integrable*

  should be treated as an *improper integral*
Carry propagation in greedy numeration systems

Theorem

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How the proof goes

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- $\forall k \quad f_k(s) = \begin{cases} \text{cp}_G(s) & \text{if } \text{cp}_G(s) \leq k + 1 \\ 0 & \text{otherwise} \end{cases}$
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.

How the proof goes

- $CP_G$ is not Riemann-integrable should be treated as an improper integral

- $\forall k \quad f_k(s) = \begin{cases} \text{cp}_G(s) & \text{if } \text{cp}_G(s) \leq k + 1 \\ 0 & \text{otherwise} \end{cases}$

- $\int_{\mathcal{K}_G} \text{cp}_G \, d\mu_G$ exists and $\lim_{k \to \infty} \int_{\mathcal{K}_G} f_k \, d\mu_G = \int_{\mathcal{K}_G} \text{cp}_G \, d\mu_G$
Carry propagation in greedy numeration systems

Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

How the proof goes

- $\text{cp}_G$ is in $L^1(\mu_G)$
Theorem

If $G$ is an exponential GNS, then $\text{CP}_G$ exists.

How the proof goes

- $\text{cp}_G$ is in $L^1(\mu_G)$

- $\forall N < G_{k+1}$
  \[
  \sum_{i=0}^{N-1} f_k(\tau_G^i(0)) = \sum_{i=0}^{N-1} f_{k-1}(\tau_G^i(0)) + \left\lfloor \frac{N}{G_k} \right\rfloor (k + 1)
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- Key result
  \[
  \forall N \in \mathbb{N}
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Carry propagation in greedy numeration systems

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- $cp_G$ is in $L^1(\mu_G)$

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- Key result
  \[ \forall N \in \mathbb{N} \quad \sum_{i=0}^{N-1} f_k(\tau_G^i(0)) = \sum_{i=0}^{N-1} f_{k-1}(\tau_G^i(0)) + \left\lfloor \frac{N}{G_k} \right\rfloor (k + 1) \]

- $\int_{K_G} f_k \, d\mu_G \leq \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} cp_G(\tau_G^i(0)) \leq \int_{K_G} f_k \, d\mu_G + M_{k+1}$
Theorem

If $G$ is an exponential GNS, then $CP_G$ exists.

Corollary

Let $G$ be an exponential GNS with $G_\ell \sim C\alpha^\ell$.

If $L_G$ is PCE, then $CP_G$ exists and $CP_G = \frac{\alpha}{\alpha - 1}$. 
Comme il y a une infinité de choses sages
qui sont menées de manière très folle,
il y a aussi des folies qui sont menées de manière très sage.

Montesquieu

Just as wise ends are oftentimes sought
in the most foolish way,
so foolishness is sometimes sought with great wisdom.

Translation by Reuben Thomas