The carry propagation of the successor function

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Joint work with

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One World Numeration Seminar, 17 November 2020

Based on the paper

The carry propagation of the successor function,

Advances in Applied Mathematics **120** (2020)

Part I

What is the carry propagation?

Adding machine

The Pascaline (1642)

featured the first carry propagation mechanism

Carry propagation

1	1	0	0	1	0	1	1		203
	1	1	0	0	1	1	0		102

Carry propagation

$$
\begin{array}{r l}
\; 1\;1\;0\;0\;1\;0\;1\;1 & 203 \\
+\qquad 1\;1\;0\;0\;1\;1\;0 & 102 \\
\hline
\end{array}
$$

Carry propagation

$$
\begin{array}{rccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad 203 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \quad 102 \\
\hline
 & & & & & & & & 1 &
\end{array}
$$

Carry propagation

$$\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & 102 \\
\hline
 & & & & & & & 0 & 1 & &
\end{array}$$

Carry propagation

$$
\begin{array}{rcccccccc}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
 & & & & & & 0 & 0 & 1
\end{array}
\qquad
\begin{array}{l}
203 \\
102 \\
\\
\end{array}
$$

Carry propagation

$$
\begin{array}{r r r r r r r r r l}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad 203 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & \quad 102 \\
\hline
 & & & & & 0 & 0 & 0 & 1 &
\end{array}
$$

Carry propagation

	1	1	0	0	1	0	1	1	203
+		1	1	0	0	1	1	0	102
				1	0	0	0	1	

Carry propagation

$$
\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & 102 \\
\hline
 & & & & 1 & 1 & 0 & 0 & 0 & 1 &
\end{array}
$$

Carry propagation

$$
\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & 102 \\
\hline
 & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & &
\end{array}
$$

Carry propagation

$$
\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & 102 \\
\hline
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & &
\end{array}
$$

Carry propagation

		1	1	0	0	1	0	1	1	203
+			1	1	0	0	1	1	0	102
	1	0	0	1	1	0	0	0	1	305

Carry propagation

		1	1	0	0	1	0	1	1	203
+			1	1	0	0	1	1	0	102
	1	0	0	1	1	0	0	0	1	305

Carry propagation prevents addition to be parallelable

Carry propagation

$$
\begin{array}{rccccccccccl}
 & & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & & 1 & 1 & 0 & 0 & 1 & 1 & 0 & & 102 \\
\hline
 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & & 305
\end{array}
$$

Theorem (von Neumann et al. 63, Knuth 78, Pippenger 02)

Average carry propagation length for addition of two uniformly distributed n-digit binary numbers =

$$\log_2(n) + \mathrm{O}(1)$$

Carry propagation for successor function in base 2

1 1 0 0 1 0 1 1   203

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & & & & & & & 1 & & \\
\hline
\end{array}
$$

Carry propagation for successor function in base 2

$$
\begin{array}{r}
1\ 1\ 0\ 0\ 1\ 0\ 1\ 1 \quad 203 \\
+ \qquad\qquad\qquad\qquad 1 \\
\hline
0
\end{array}
$$

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccc}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & & 1 \\
\hline
 & & & & & & & 0 & 0
\end{array}
\qquad 203
$$

Carry propagation for successor function in base 2

$$\begin{array}{rr} & 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1 \\ + & 1 \\ \hline & 1\ 0\ 0 \end{array}$$

203

Carry propagation for successor function in base 2

$$
\begin{array}{rr}
 & 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1 \\
+ & 1 \\
\hline
 & 1\ 1\ 0\ 0
\end{array}
\qquad 203
$$

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccc l}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \quad 203 \\
+ & & & & & & & & 1 & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & \color{red}{1} & \color{red}{0} & \color{red}{0} & \quad 204
\end{array}
$$

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & & & & & & & 1 & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & & 204
\end{array}
$$

$$\mathrm{cp}_2(203) = 3$$

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccccl}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & 203 \\
+ & & & & & & & & 1 & & \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & & 204
\end{array}
$$

$$\mathrm{cp}_2(203) = 3$$

Amortized carry propagation (in base 2)

$$\mathrm{CP}_2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_2(i)$$

Carry propagation for successor function in base 2

$$
\begin{array}{rcccccccc}
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
+ & & & & & & & & 1 \\
\hline
 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
$$

203

204

$$\mathrm{cp}_2(203) = 3$$

Amortized carry propagation (in base 2)

$$\mathrm{CP}_2 = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_2(i)$$

if it exists!

Carry propagation for successor function in base 2

·	0
1	1
1 0	2
1 1	3
1 0 0	4
1 0 1	5
1 1 0	6
1 1 1	7
1 0 0 0	8
1 0 0 1	9
1 0 1 0	10
1 0 1 1	11
1 1 0 0	12
1 1 0 1	13
1 1 1 0	14
1 1 1 1	15
1 0 0 0 0	16
1 0 0 0 1	17

Carry propagation for successor function in base 2

·	0	1
1	1	
1 0	2	
1 1	3	
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	
1 1	3	
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	
1 0 0	4	
1 0 1	5	
1 1 0	6	
1 1 1	7	
1 0 0 0	8	
1 0 0 1	9	
1 0 1 0	10	
1 0 1 1	11	
1 1 0 0	12	
1 1 0 1	13	
1 1 1 0	14	
1 1 1 1	15	
1 0 0 0 0	16	
1 0 0 0 1	17	

Carry propagation for successor function in base 2

·	0	1
1	1	2
1 0	2	1
1 1	3	3
1 0 0	4	1
1 0 1	5	2
1 1 0	6	1
1 1 1	7	4
1 0 0 0	8	1
1 0 0 1	9	2
1 0 1 0	10	1
1 0 1 1	11	3
1 1 0 0	12	1
1 1 0 1	13	2
1 1 1 0	14	1
1 1 1 1	15	5
1 0 0 0 0	16	1
1 0 0 0 1	17	2

Carry propagation for successor function in base 2

·	0	1
1	1	1 1
1 0	2	1
1 1	3	1 1 1
1 0 0	4	1
1 0 1	5	1 1
1 1 0	6	1
1 1 1	7	1 1 1 1
1 0 0 0	8	1
1 0 0 1	9	1 1
1 0 1 0	10	1
1 0 1 1	11	1 1 1
1 1 0 0	12	1
1 1 0 1	13	1 1
1 1 1 0	14	1
1 1 1 1	15	1 1 1 1 1
1 0 0 0 0	16	1
1 0 0 0 1	17	1 1

Carry propagation for successor function in base 2

$$\mathrm{CP}_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2$$

·	0	1
1	1	1 1
1 0	2	1
1 1	3	1 1 1
1 0 0	4	1
1 0 1	5	1 1
1 1 0	6	1
1 1 1	7	1 1 1 1
1 0 0 0	8	1
1 0 0 1	9	1 1
1 0 1 0	10	1
1 0 1 1	11	1 1 1
1 1 0 0	12	1
1 1 0 1	13	1 1
1 1 1 0	14	1
1 1 1 1	15	1 1 1 1 1
1 0 0 0 0	16	1
1 0 0 0 1	17	1 1

Carry propagation for successor function in base p

$$\mathrm{CP}_2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{2}{2-1} = 2$$

$$\mathrm{CP}_p = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{p}{p-1}$$

·	0	1
1	1	1 1
1 0	2	1
1 1	3	1 1 1
1 0 0	4	1
1 0 1	5	1 1
1 1 0	6	1
1 1 1	7	1 1 1 1
1 0 0 0	8	1
1 0 0 1	9	1 1
1 0 1 0	10	1
1 0 1 1	11	1 1 1
1 1 0 0	12	1
1 1 0 1	13	1 1
1 1 1 0	14	1
1 1 1 1	15	1 1 1 1 1
1 0 0 0 0	16	1
1 0 0 0 1	17	1 1

Carry propagation for successor function in base Fibonacci

1 0 0 1 0 1 0 1

Carry propagation for successor function in base Fibonacci

55	34	21	13	8	5	3	2	1	
	1	0	0	1	0	1	0	1	46

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

$$\mathrm{cp}_F(46) = 6$$

Carry propagation for successor function in base Fibonacci

	55	34	21	13	8	5	3	2	1	
		1	0	0	1	0	1	0	1	46
+									1	
		1	0	1	0	0	0	0	0	47

$$\mathrm{cp}_F(46) = 6$$

Amortized carry propagation in base Fibonacci

$$\mathrm{CP}_F = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_F(i)$$

if it exists!

Carry propagation for successor function in base Fibonacci

·	0
1	1
1 0	2
1 0 0	3
1 0 1	4
1 0 0 0	5
1 0 0 1	6
1 0 1 0	7
1 0 0 0 0	8
1 0 0 0 1	9
1 0 0 1 0	10
1 0 1 0 0	11
1 0 1 0 1	12
1 0 0 0 0 0	13
1 0 0 0 0 1	14
1 0 0 0 1 0	15
1 0 0 1 0 0	16
1 0 0 1 0 1	17

Carry propagation for successor function in base Fibonacci

·	0	1
1	1	2
1 0	2	3
1 0 0	3	1
1 0 1	4	4
1 0 0 0	5	1
1 0 0 1	6	2
1 0 1 0	7	5
1 0 0 0 0	8	1
1 0 0 0 1	9	2
1 0 0 1 0	10	3
1 0 1 0 0	11	1
1 0 1 0 1	12	6
1 0 0 0 0 0	13	1
1 0 0 0 0 1	14	2
1 0 0 0 1 0	15	3
1 0 0 1 0 0	16	1
1 0 0 1 0 1	17	4

Carry propagation for successor function in base Fibonacci

CP_F ?

·	0	1
1	1	2
1 0	2	3
1 0 0	3	1
1 0 1	4	4
1 0 0 0	5	1
1 0 0 1	6	2
1 0 1 0	7	5
1 0 0 0 0	8	1
1 0 0 0 1	9	2
1 0 0 1 0	10	3
1 0 1 0 0	11	1
1 0 1 0 1	12	6
1 0 0 0 0 0	13	1
1 0 0 0 0 1	14	2
1 0 0 0 1 0	15	3
1 0 0 1 0 0	16	1
1 0 0 1 0 1	17	4

Carry propagation for successor function in base Fibonacci

CP_F ?

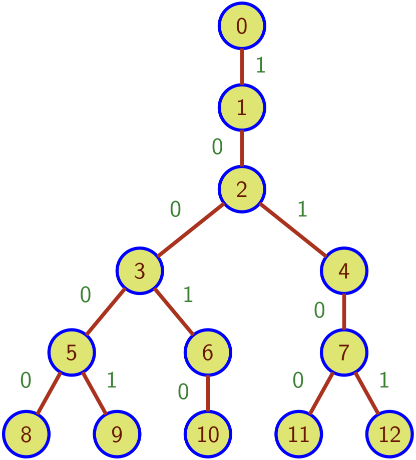
$\mathrm{CP}_F = \dfrac{\varphi}{\varphi - 1}$?

·	0	1
1	1	2
1 0	2	3
1 0 0	3	1
1 0 1	4	4
1 0 0 0	5	1
1 0 0 1	6	2
1 0 1 0	7	5
1 0 0 0 0	8	1
1 0 0 0 1	9	2
1 0 0 1 0	10	3
1 0 1 0 0	11	1
1 0 1 0 1	12	6
1 0 0 0 0 0	13	1
1 0 0 0 0 1	14	2
1 0 0 0 1 0	15	3
1 0 0 1 0 0	16	1
1 0 0 1 0 1	17	4

Part II

A first observation and its 3 consequences

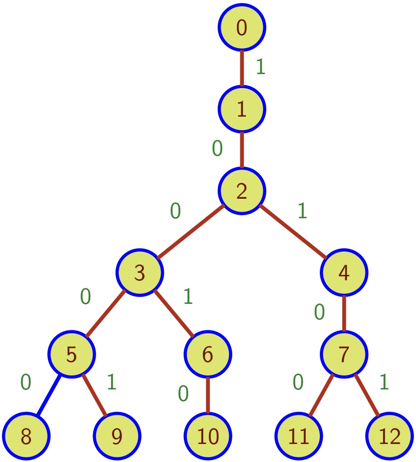
A primal observation



The Fibonacci tree

A primal observation

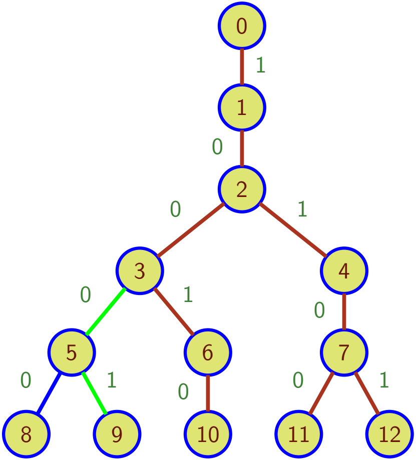
$\mathrm{cp}_F(8) = 1$



The Fibonacci tree

A primal observation

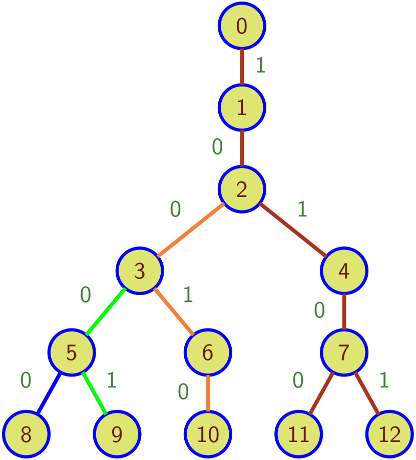
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$



The Fibonacci tree

A primal observation

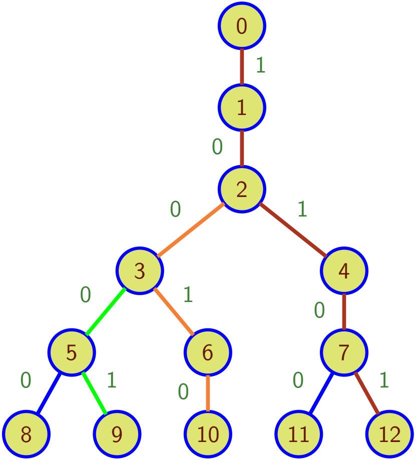
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$



The Fibonacci tree

A primal observation

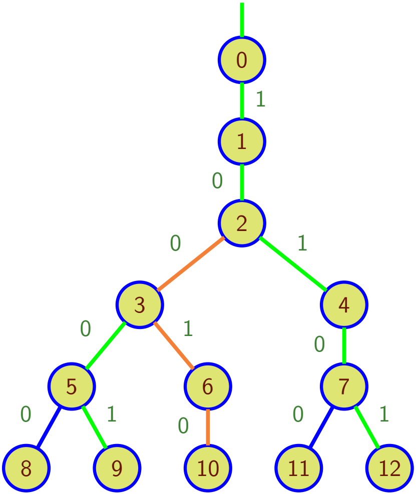
$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$
$\mathrm{cp}_F(11) = 1$



The Fibonacci tree

A primal observation

$\mathrm{cp}_F(8) = 1$
$\mathrm{cp}_F(9) = 2$
$\mathrm{cp}_F(10) = 3$
$\mathrm{cp}_F(11) = 1$
$\mathrm{cp}_F(12) = 6$

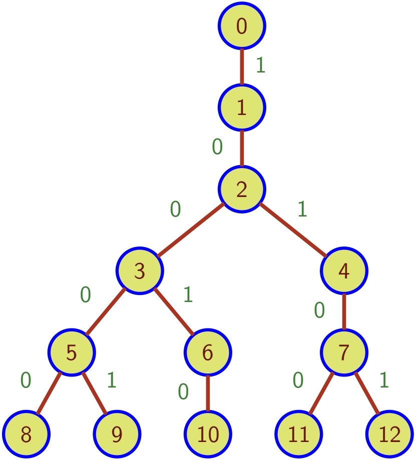


The Fibonacci tree

A primal observation

$$\mathrm{cp}_F(8) = 1$$
$$\mathrm{cp}_F(9) = 2$$
$$\mathrm{cp}_F(10) = 3$$
$$\mathrm{cp}_F(11) = 1$$
$$\mathrm{cp}_F(12) = 6$$

$$\sum_{i=8}^{i=12} \mathrm{cp}_F(i) = 13$$



The Fibonacci tree

A primal observation

$\mathrm{cp}_F(8) = 1$

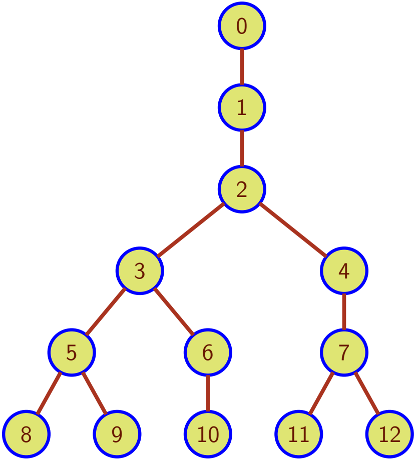
$\mathrm{cp}_F(9) = 2$

$\mathrm{cp}_F(10) = 3$

$\mathrm{cp}_F(11) = 1$

$\mathrm{cp}_F(12) = 6$

$$\sum_{i=8}^{i=12} \mathrm{cp}_F(i) = 13$$



The Fibonacci tree

What we learn from the primal observation

- A framework:

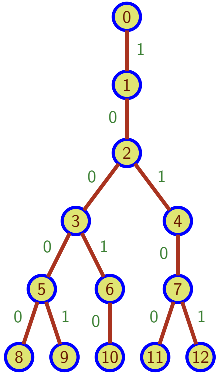
  the Abstract Numeration System model

- A general working hypothesis:

  Prefix-closed Extendable Languages

- An essential parameter:

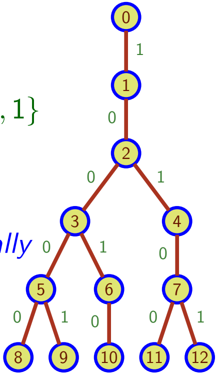
  The local growth rate



What we learn from the primal observation: the ANS model

- A framework:

  the Abstract Numeration System model



Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet *e.g.* $A = \{0, 1\}$

$\Rightarrow$ A^* equipped with the *radix ordering*

i.e. ordered first *by length*, and then,
for words of equal length, ordered *lexicographically*

e.g. $A^* = \varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$

What we learn from the primal observation: the ANS model

- A framework:

  the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet *e.g.* $A = \{0, 1\}$

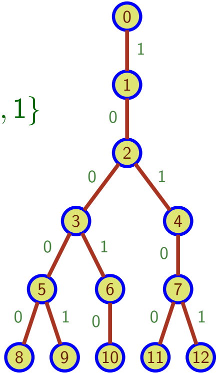
$\Rightarrow$ A^* equipped with the *radix ordering*

- $L \subseteq A^*$ any *language* over A^*

  ordered by radix ordering

e.g. $F = \varepsilon \cup 1A^* \setminus A^*11A^*$

$F = \varepsilon, 1, 10, 100, 101, 1000, \ldots$



What we learn from the primal observation: the ANS model

- A framework:
  the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet *e.g.* $A = \{0, 1\}$

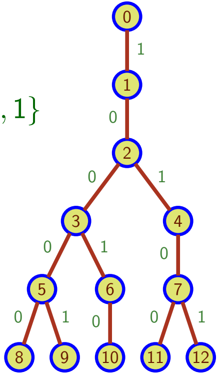
$\Rightarrow$ A^* equipped with the *radix ordering*

- $L \subseteq A^*$ any *language* over A^*
  ordered by radix ordering

$\Rightarrow$ Natural integers are given *representations*
by means of words of L

i.e. $\langle n \rangle_L =$ $(n+1)$-th word of L in the radix ordering

e.g. $\langle 6 \rangle_F = 1001$



What we learn from the primal observation: the ANS model

- A framework:

  the Abstract Numeration System model

Definition (Lecomte & Rigo 2001)

- A finite *totally ordered* alphabet *e.g.* $A = \{0, 1\}$

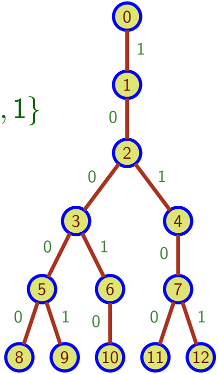
$\Rightarrow$ A^* equipped with the *radix ordering*

- $L \subseteq A^*$ any *language* over A^*

  ordered by radix ordering

$\Rightarrow$ Natural integers are given *representations*

by means of words of L



$L \subseteq A^*$ (together with the order on A) defines an ANS

Fact: *All 'classical' numeration systems are ANS*

What we learn from the primal observation: the ANS model

All 'classical' numeration systems are ANS

$$L_2 = 1\,(0,1)^*$$

·	0
1	1
1 0	2
1 1	3
1 0 0	4
1 0 1	5
1 1 0	6
1 1 1	7
1 0 0 0	8
1 0 0 1	9
1 0 1 0	10
1 0 1 1	11
1 1 0 0	12
1 1 0 1	13
1 1 1 0	14
1 1 1 1	15
1 0 0 0 0	16

What we learn from the primal observation: the ANS model

All ‘classical’ numeration systems are ANS

$L_F = 1\,(0,1)^* \setminus (0,1)^* 1\,1\,(0,1)^*$

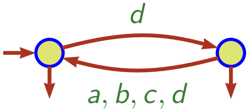
·	·	0
1	1	1
1 0	1 0	2
1 0 0	1 1	3
1 0 1	1 0 0	4
1 0 0 0	1 0 1	5
1 0 0 1	1 1 0	6
1 0 1 0	1 1 1	7
1 0 0 0 0	1 0 0 0	8
1 0 0 0 1	1 0 0 1	9
1 0 0 1 0	1 0 1 0	10
1 0 1 0 0	1 0 1 1	11
1 0 1 0 1	1 1 0 0	12
1 0 0 0 0 0	1 1 0 1	13
1 0 0 0 0 1	1 1 1 0	14
1 0 0 0 1 0	1 1 1 1	15
1 0 0 1 0 0	1 0 0 0 0	16

What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language O (for *Oscillating*)

$\mathcal{O}$



$$\langle 10 \rangle_O = d\,a\,d\,a$$

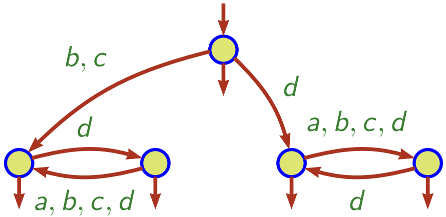
.	0
d	1
d a	2
d b	3
d c	4
d d	5
d a d	6
d b d	7
d c d	8
d d d	9
d a d a	10
d a d b	11
d a d c	12
d a d d	13
d b d a	14
d b d b	15
d b d c	16

What we learn from the primal observation: the ANS model

Any language can be seen as an ANS

Language V (for *Vibrating*)

$\mathcal{V}$

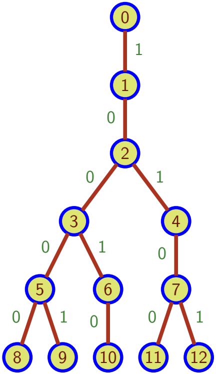


$$\langle 10 \rangle_V = b\,d\,a$$

.	0
b	1
c	2
d	3
b d	4
c d	5
d a	6
d b	7
d c	8
d d	9
b d a	10
b d b	11
b d c	12
b d d	13
c d a	14
c d b	15
c d c	16

What we learn from the primal observation: an hypothesis

- A framework:
  the Abstract Numeration System model
- A general working hypothesis:
  Prefix-Closed Extendable Languages



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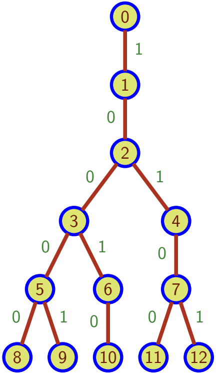
  $L \subseteq A^*$ an ANS



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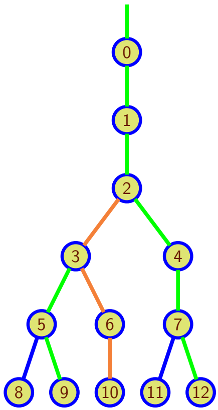

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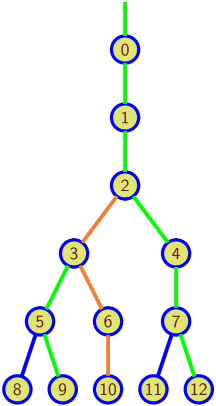
Notation

$$\mathbf{u}_L(\ell) = \mathrm{card}\left(L \cap A^{\ell}\right)$$
$$\mathbf{v}_L(\ell) = \mathrm{card}\left(L \cap A^{\leqslant \ell}\right) = \sum_{i=0}^{\ell} \mathbf{u}_L(i)$$



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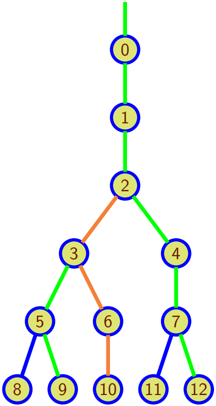
The formula we want: $$\sum_{i=\mathbf{v}_L(\ell-1)}^{\mathbf{v}_L(\ell)-1} \mathrm{cp}_L(i) = \mathbf{v}_L(\ell)$$



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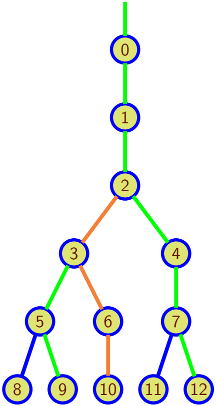

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requires L *prefix-closed* and *extendable*

i.e. to be a PCE language

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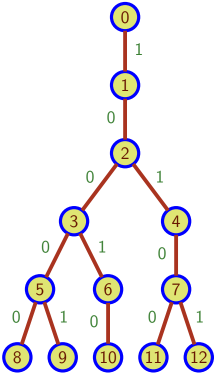
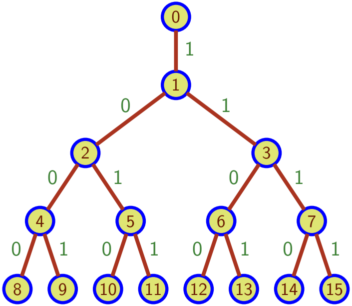


Fact: *'All' 'classical' ANS are* PCE

What we learn from the primal observation: an hypothesis

'All' 'classical' ANS are PCE

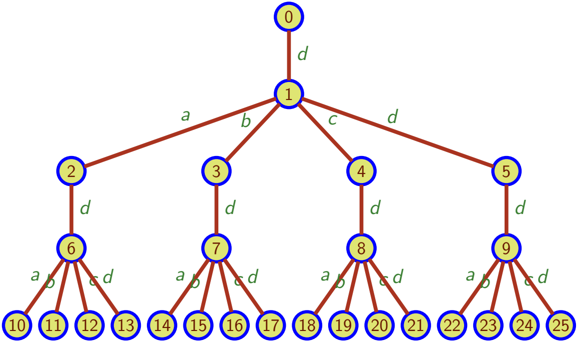


L_2

.
1
1 0
1 1
1 0 0
1 0 1
1 1 0
1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
1 0 0 0 0

What we learn from the primal observation: an hypothesis

The ANS we consider are PCE

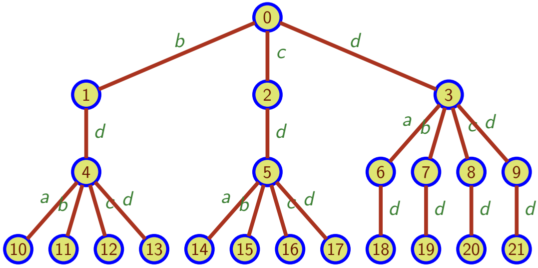


O

.
d
d a
d b
d c
d d
d a d
d b d
d c d
d d d
d a d a
d a d b
d a d c
d a d d
d b d a
d b d b
d b d c

What we learn from the primal observation: an hypothesis

The ANS we consider are PCE



V

.
b
c
d
b d
c d
d a
d b
d c
d d
b d a
b d b
b d c
b d d
c d a
c d b
c d c

What we learn from the primal observation: a new parameter

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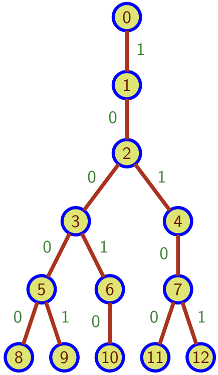
- An essential parameter: the local growth rate



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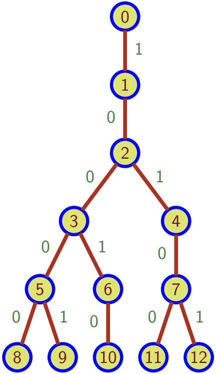

From $$\sum_{i=\mathbf{v}_L(\ell-1)}^{\mathbf{v}_L(\ell)-1} \mathsf{cp}_L(i) = \mathbf{v}_L(\ell)$$

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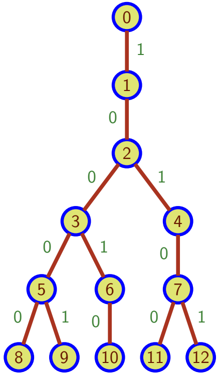
From $$\sum_{i=\mathbf{v}_L(\ell-1)}^{\mathbf{v}_L(\ell)-1} \mathrm{cp}_L(i) = \mathbf{v}_L(\ell)$$

follows $$\sum_{i=0}^{\mathbf{v}_L(\ell)-1} \mathrm{cp}_L(i) = \sum_{j=0}^{\ell} \mathbf{v}_L(j)$$



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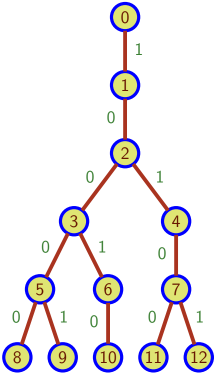

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hence, if $\mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(i)$ exists

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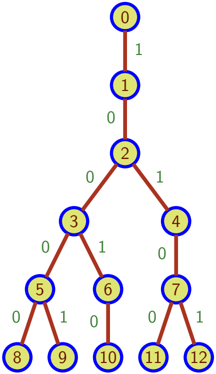

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then $$\mathrm{CP}_L = \lim_{\ell\to\infty} \frac{1}{\mathbf{v}_L(\ell)} \sum_{j=0}^{\ell} \mathbf{v}_L(j)$$ exists

Intermede: a freshperson calculus lemma

Lemma

$$(x_\ell)_{\ell\in\mathbb{N}} \quad x_\ell \in \mathbb{R}_+ \qquad \forall \ell \quad y_\ell = \sum_{j=0}^{\ell-1} x_j \qquad \gamma > 1$$

TFAE

(i) $\lim_{\ell\to\infty} \frac{x_{\ell+1}}{x_\ell} = \gamma$

(ii) $\lim_{\ell\to\infty} \frac{y_{\ell+1}}{y_\ell} = \gamma$

(iii) $\lim_{\ell\to\infty} \frac{y_\ell}{x_\ell} = \frac{\gamma}{\gamma-1}$

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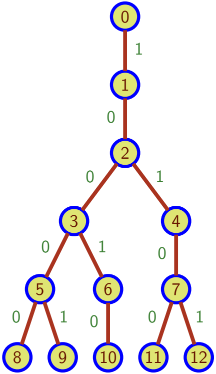

Proposition

If $\quad \mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(i) \quad$ *exists,*

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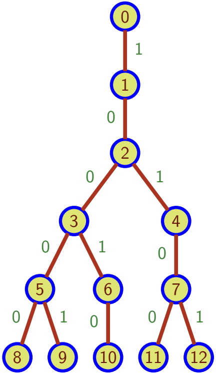

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then the local growth rate $\lim_{\ell\to\infty} \dfrac{\mathbf{u}_L(\ell+1)}{\mathbf{u}_L(\ell)} = \gamma_L$ *exists*

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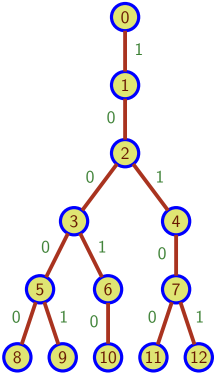

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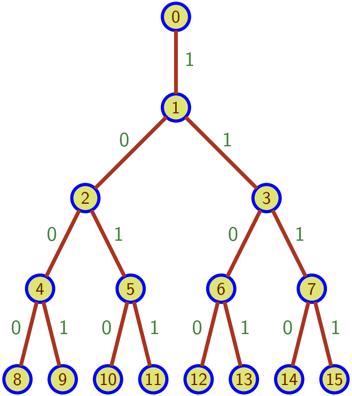
What we learn from the primal observation: a new parameter



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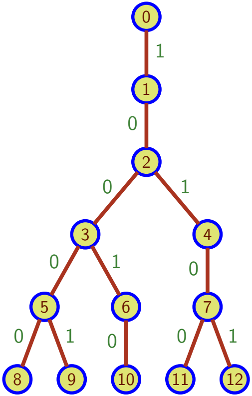
1 1 0
1 1 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
1 0 0 0 0

L_2 $\qquad$ $\lim_{\ell\to\infty} \dfrac{\mathbf{u}_2(\ell+1)}{\mathbf{u}_2(\ell)} = \gamma_2 = 2$

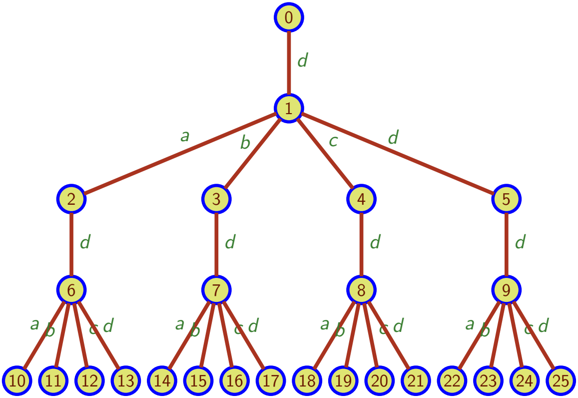
What we learn from the primal observation: a new parameter



.
1
1 0
1 0 0
1 0 1
1 0 0 0
1 0 0 1
1 0 1 0
1 0 0 0 0
1 0 0 0 1
1 0 0 1 0
1 0 1 0 0
1 0 1 0 1
1 0 0 0 0 0
1 0 0 0 0 1
1 0 0 0 1 0
1 0 0 1 0 0

L_F $\qquad \lim_{\ell\to\infty} \dfrac{\mathbf{u}_F(\ell+1)}{\mathbf{u}_F(\ell)} = \gamma_F = \varphi$

What we learn from the primal observation: a new parameter

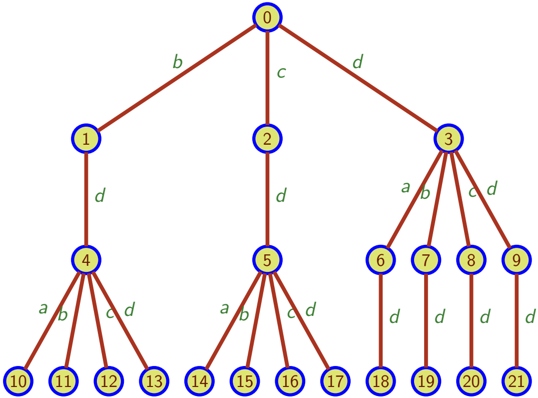


.
d
d a
d b
d c
d d
d a d
d b d
d c d
d d d
d a d a
d a d b
d a d c
d a d d
d b d a
d b d b
d b d c

O

$$\frac{\mathbf{u}_O(2\ell+1)}{\mathbf{u}_O(\ell)} = 1 \qquad \frac{\mathbf{u}_O(2\ell+2)}{\mathbf{u}_O(2\ell+1)} = 4$$

What we learn from the primal observation: a new parameter



V $\qquad$ $\mathbf{u}_V(\ell) = 3\,2^{\ell-1}$ $\qquad$ $\gamma_V = 2$

.
b
c
d
b d
c d
d a
d b
d c
d d
b d a
b d b
b d c
b d d
c d a
c d b
c d c

A natural question

Proposition

If $\mathrm{CP}_L = \lim_{N\to\infty} \dfrac{1}{N} \displaystyle\sum_{i=0}^{N-1} \mathrm{cp}_L(i)$ *exists,*

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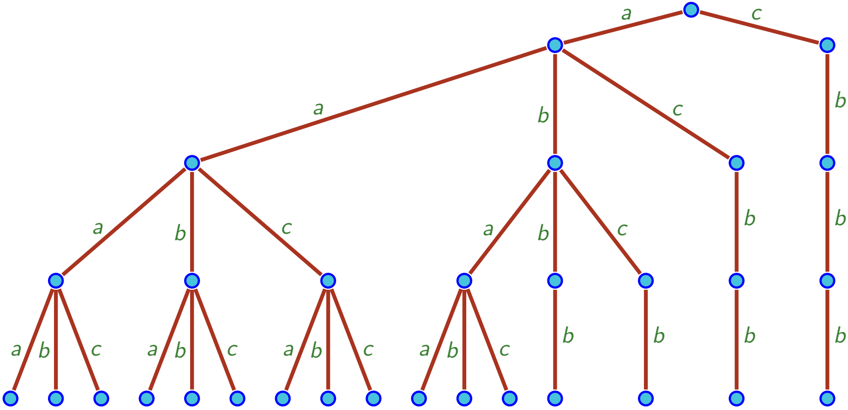
Question

Is the existence of the local growth rate sufficient
to insure the existence of the carry propagation?

An unbalanced tree

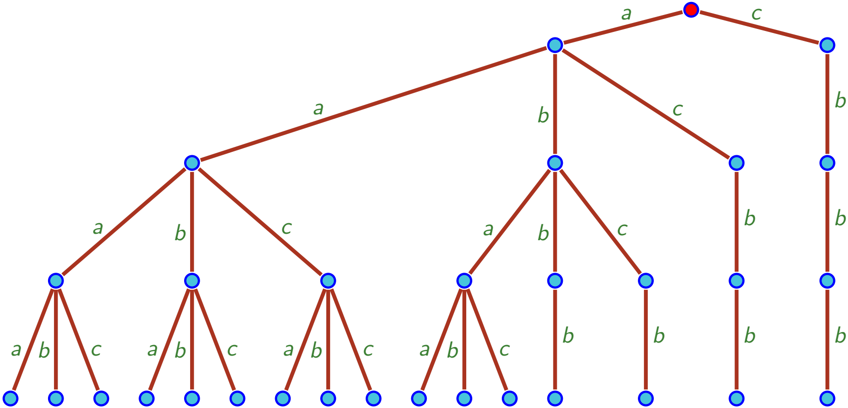
$U \subseteq \{a, b, c\}^*$



$$\mathbf{u}_U(\ell) = 2^\ell \qquad \frac{\mathbf{u}_U(\ell+1)}{\mathbf{u}_U(\ell)} = \gamma_U = 2$$

An unbalanced tree

$U \subseteq \{a, b, c\}^*$



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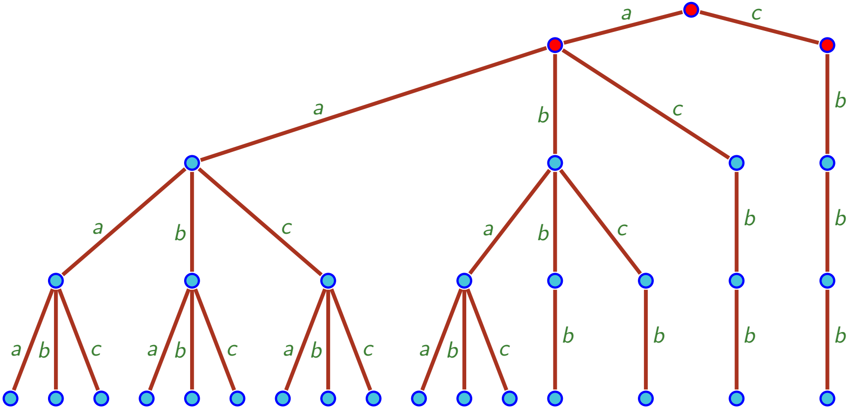

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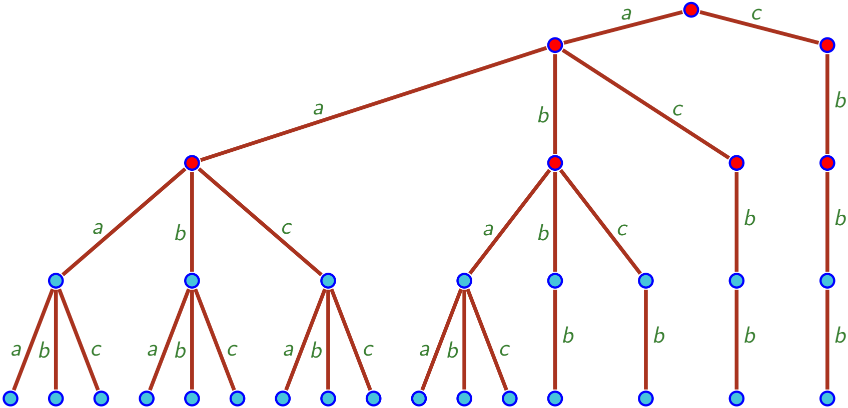

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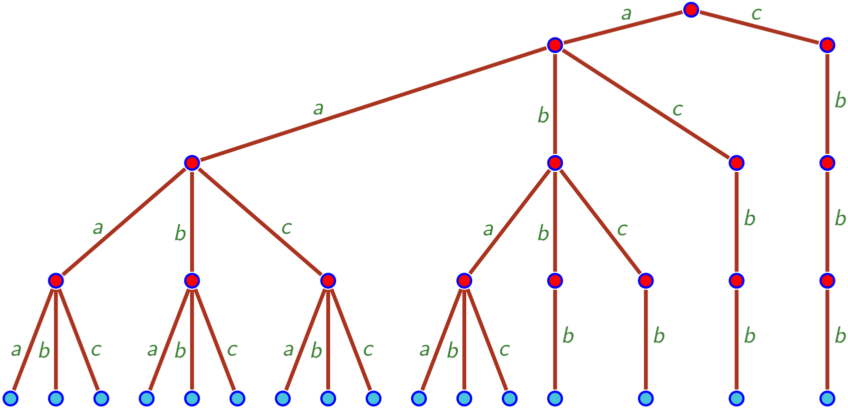

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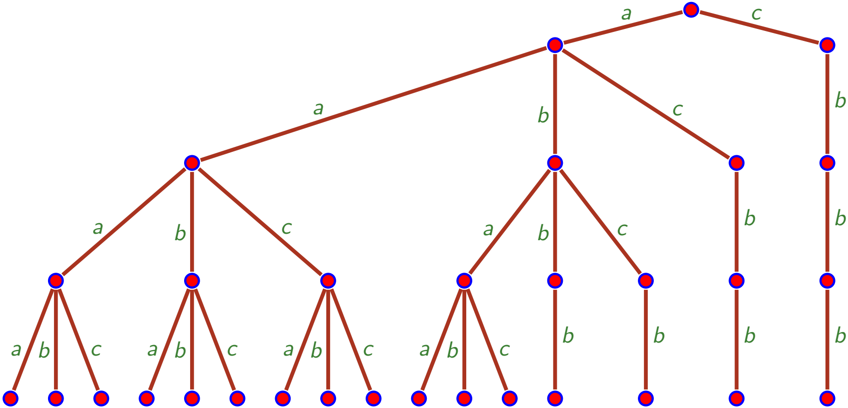
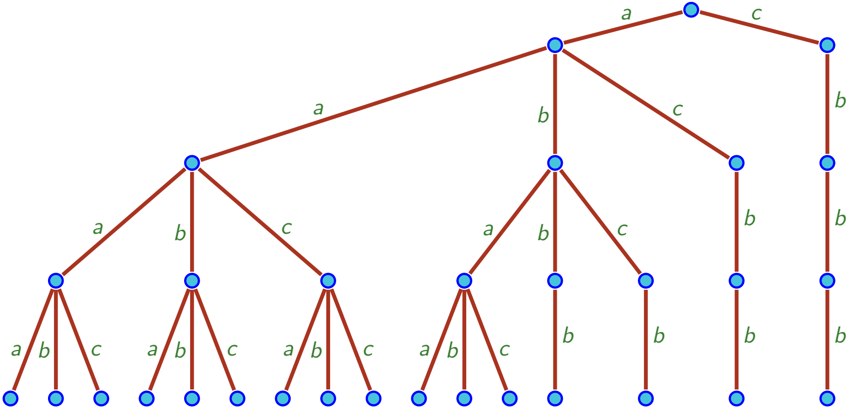


$$\lim_{\ell \to \infty} \frac{1}{\mathbf{v}_U(\ell)} \sum_{j=0}^{\mathbf{v}_U(\ell)-1} \mathrm{cp}_U(j) = 2$$

An unbalanced tree

$U \subseteq \{a, b, c\}^*$



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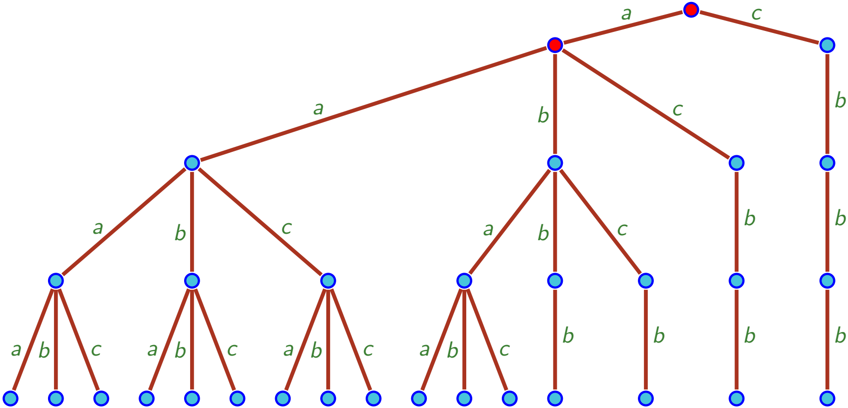

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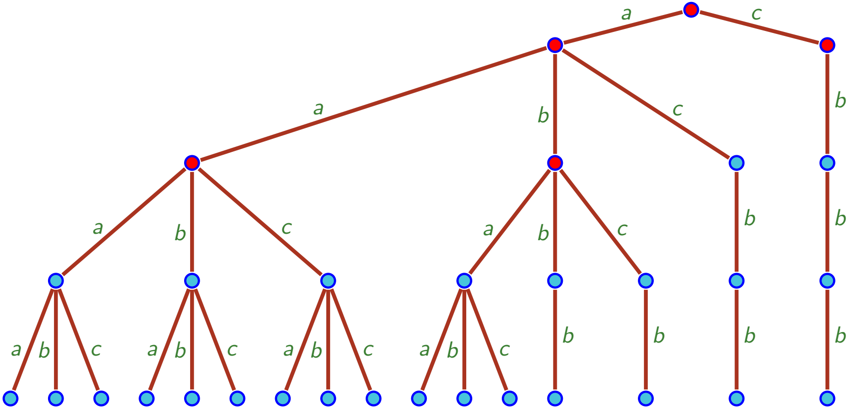

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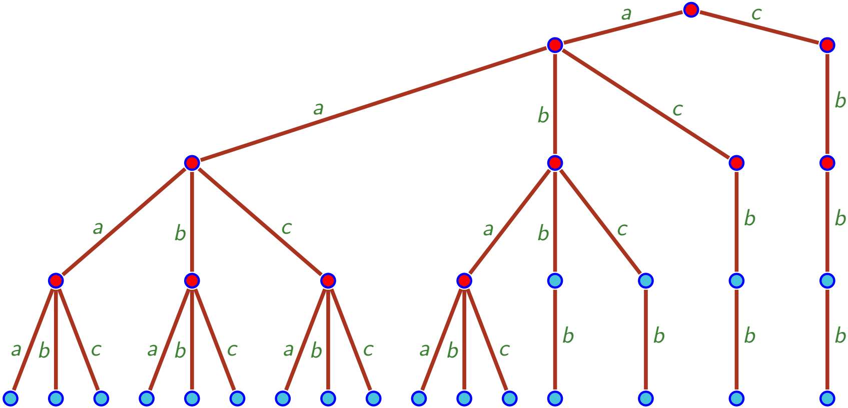

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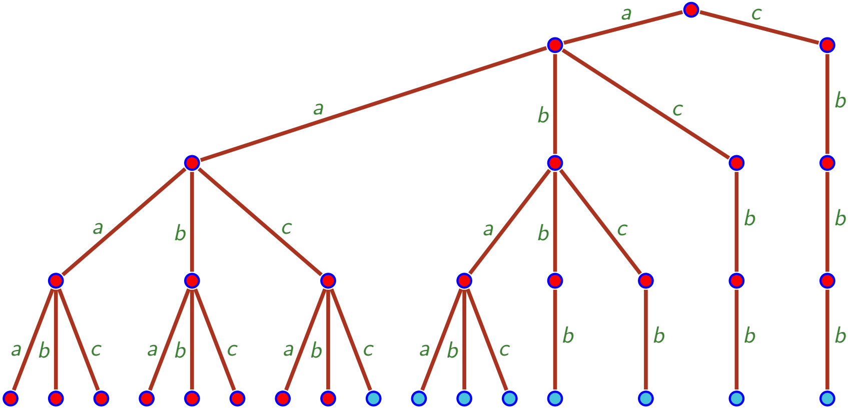
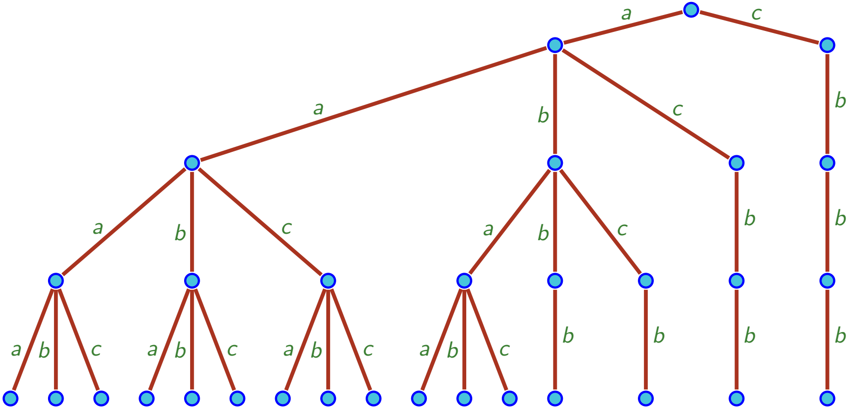


$$\lim_{\ell \to \infty} \frac{1}{\mathbf{v}'_U(\ell)} \sum_{j=0}^{\mathbf{v}'_U(\ell)-1} \mathrm{cp}_U(j) = \frac{11}{6} \neq 2$$

An unbalanced tree

$U \subseteq \{a, b, c\}^*$

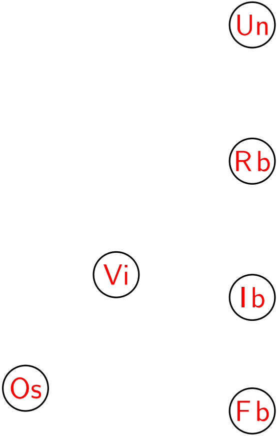


$$\mathrm{CP}_U = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_U(i)$$ does not exist

A first conclusion

The *existence* of the carry propagation
is more difficult to prove
than the *computation* of the carry propagation itself

Roadmap



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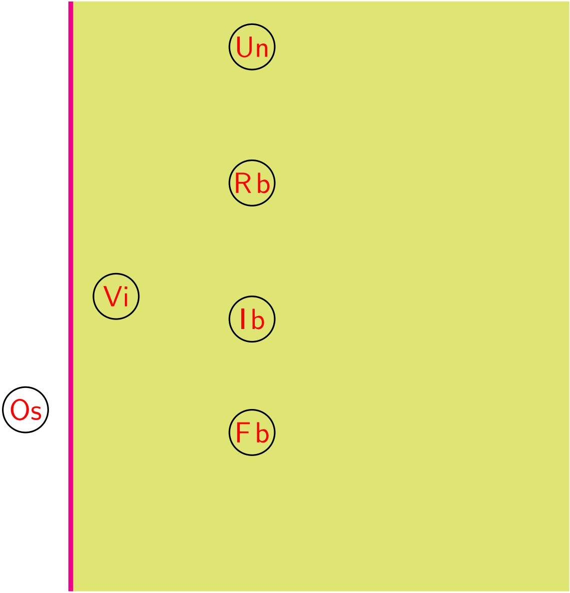

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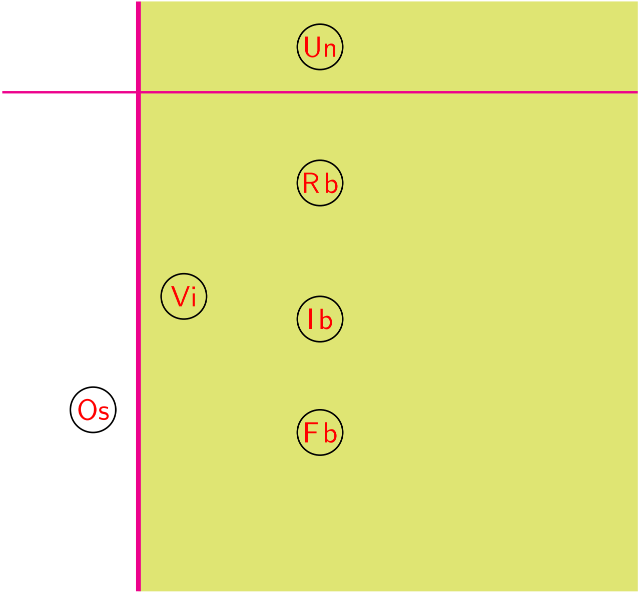

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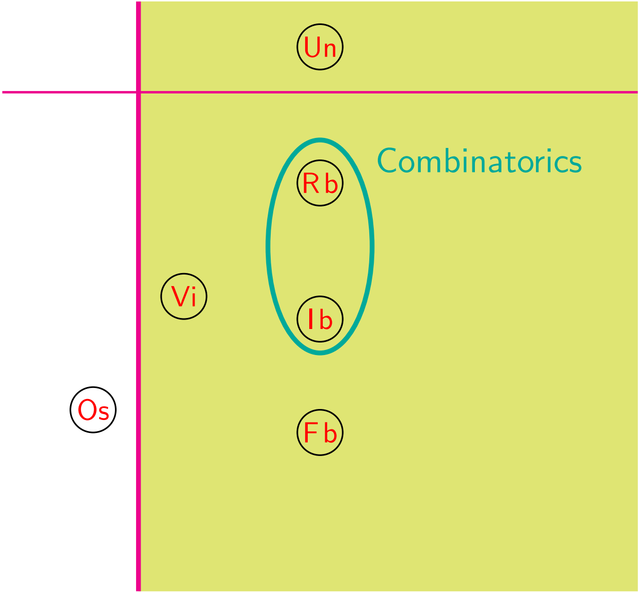

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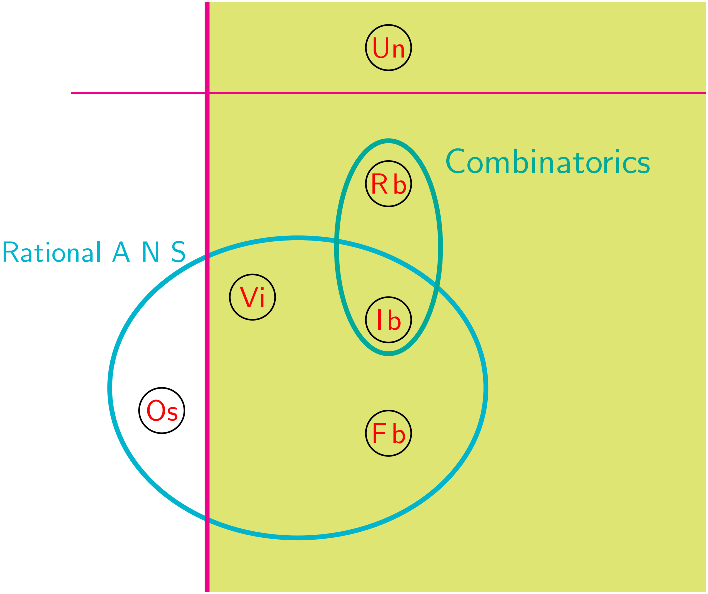

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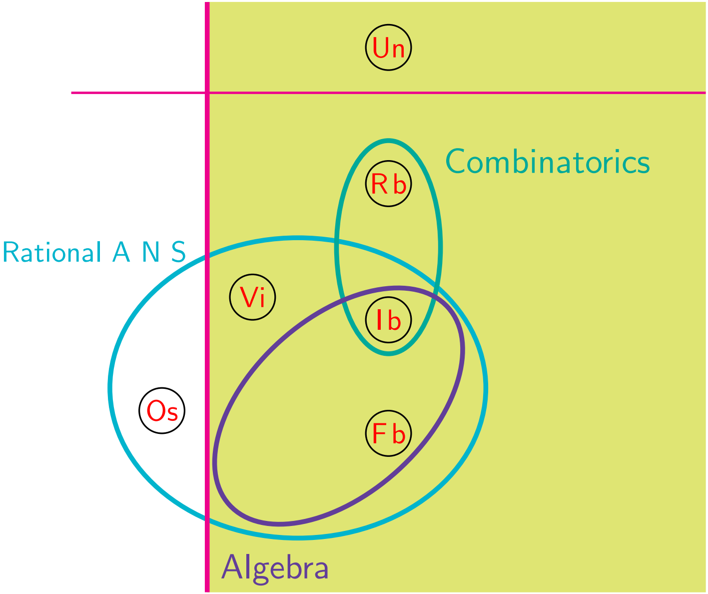

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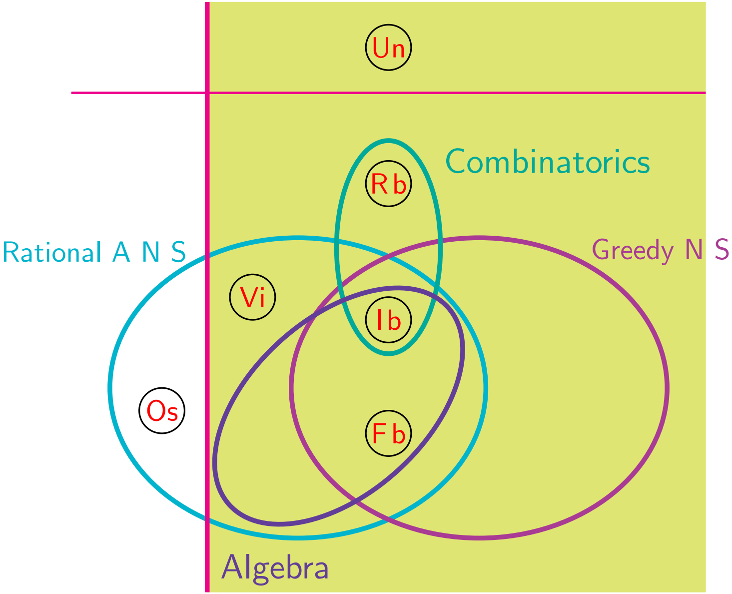

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Greedy N S
Vi
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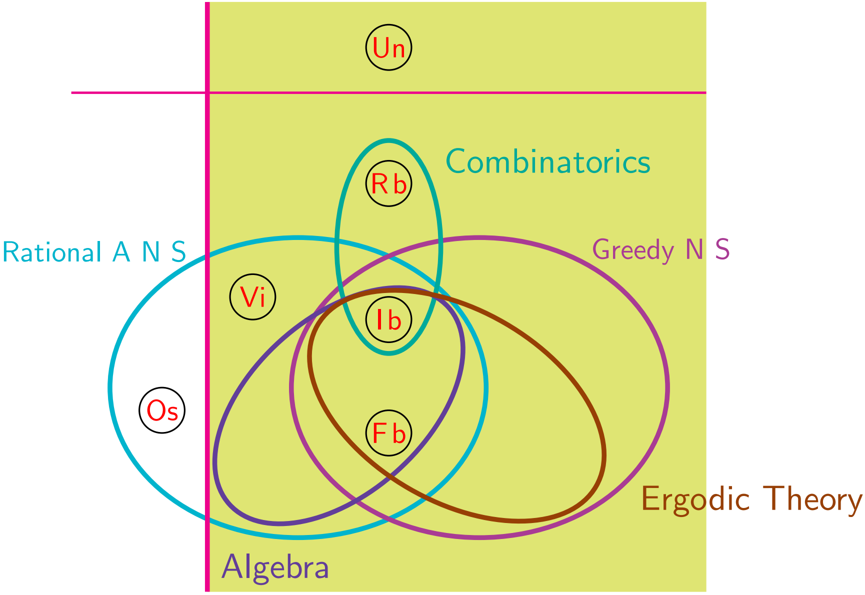
Os
Fb
Algebra

Roadmap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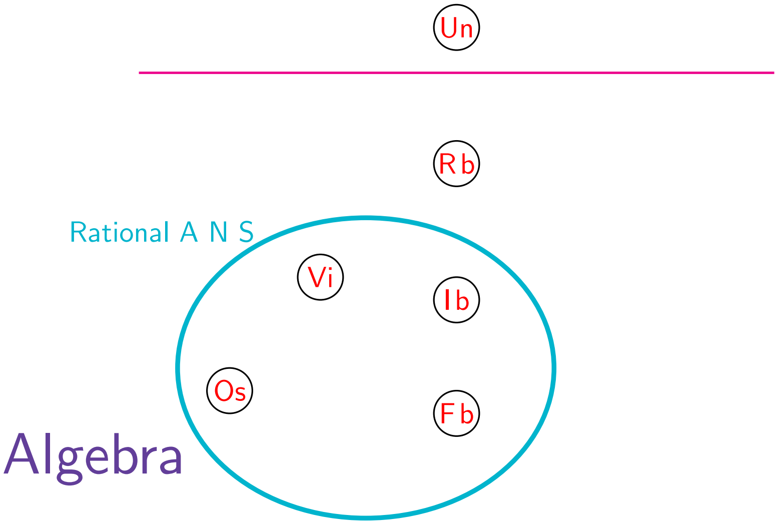
Ergodic Theory
Algebra

Part III

Algebra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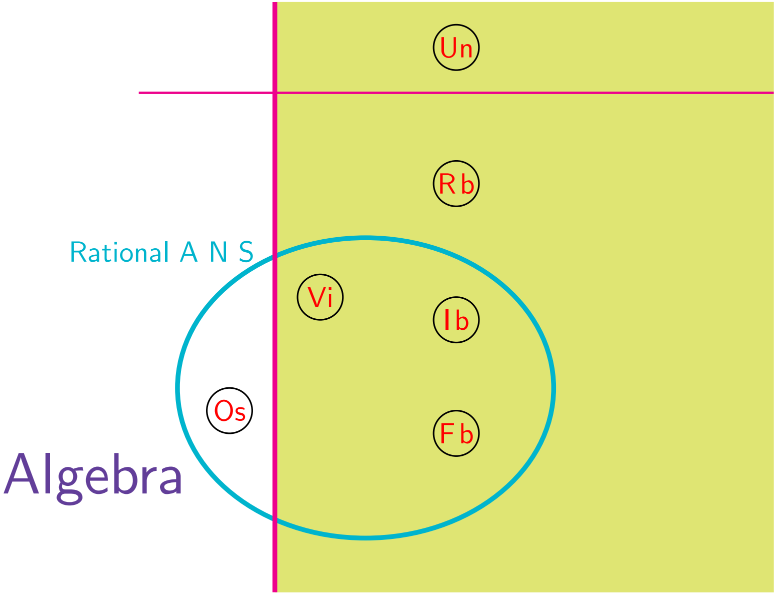
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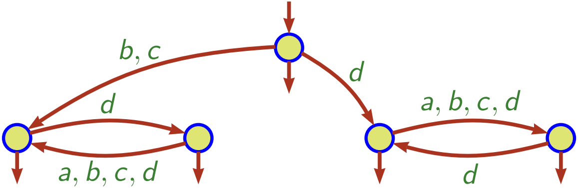
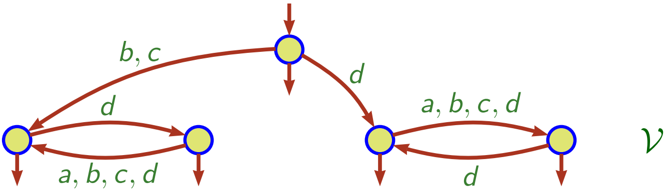
Surprise !



$\mathcal{V}$

$$\mathbf{u}_V(\ell) = 3 \, . \, 2^{\ell-1}$$

Surprise !



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$$\liminf_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \mathrm{cp}_V(i) \;\leqslant\; \frac{28}{15} \;<\; \frac{13}{6} \;\leqslant\; \limsup_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} \mathrm{cp}_V(i)$$

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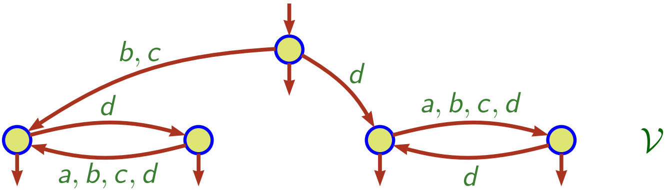

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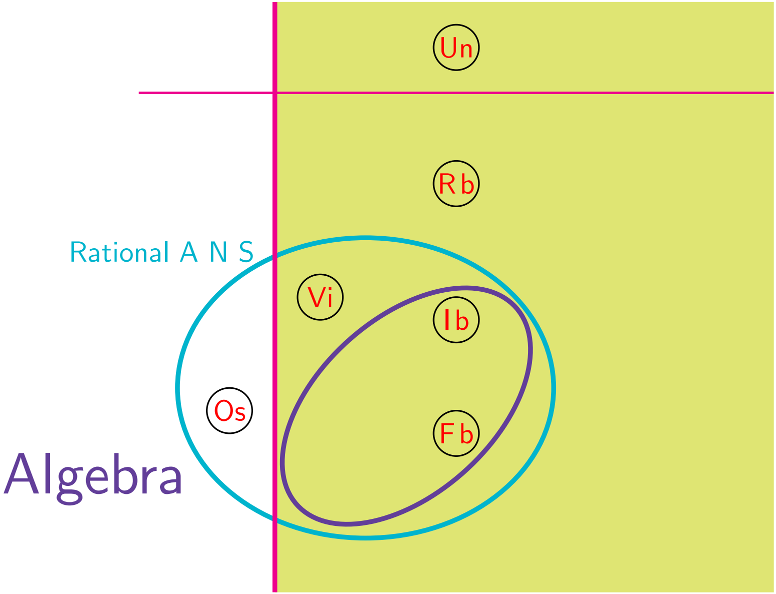
γ_V exists but CP_V does not exists



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Generating functions

Definition

$L \subseteq A^*$ $\qquad$ $\mathrm{g}_L(z)$ *generating function* of L

$$\mathrm{g}_L(z) = \sum_{\ell=0}^{\infty} \mathbf{u}_L(\ell)\, z^{\ell}$$

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L rational language $\quad \Longrightarrow \quad$ $\mathrm{g}_L(z)$ rational function

$$\mathrm{g}_L(z) = \frac{R(z)}{Q(z)} \qquad\qquad R(z), Q(z) \in \mathbb{Z}[z]$$

Generating functions of rational languages

L rational language $\Longrightarrow$ $\mathrm{g}_L(z)$ rational function

$\mathrm{g}_L(z)$ *uniquely* written as

$$\mathrm{g}_L(z) = T(z) + \frac{S(z)}{Q(z)} \qquad T(z), S(z), Q(z) \in \mathbb{Z}[z]$$

with $\deg S < \deg Q$ and $Q(0) \neq 0$

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P_L is the *reciprocal polynomial* of Q : $\mathrm{P}_L(z) = Q(\frac{1}{z})\, z^{\deg Q}$

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P_L is the *reciprocal polynomial* of Q : $\mathrm{P}_L(z) = Q(\frac{1}{z})\, z^{\deg Q}$

The *eigenvalues* of L are the zeroes λ_1, λ_2,..., λ_t of P_L and

$$\forall \ell \in \mathbb{N} \qquad \mathbf{u}_L(\ell) = \sum_{j=1}^{t} \lambda_j^\ell \, P_j(\ell)$$

where $\deg P_j$ = multiplicity of λ_j in P_L minus 1

Positive rational functions

Theorem (Berstel 71)

$f(z)$ $\mathbb{R}_+$ -rational function (not a polynomial)
λ maximum of the moduli of its eigenvalues.

(i) *λ is an eigenvalue of $f(z)$ (hence an eigenvalue in $\mathbb{R}_+$)*

(ii) *Every eigenvalue of $f(z)$ of modulus λ*
is of the form $\lambda \mathrm{e}^{i\,\theta}$, where $\mathrm{e}^{i\,\theta}$ is a root of the unity

(iii) *The multiplicity of any eigenvalue of modulus λ*
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Definition

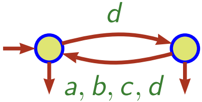
(i) $f(z)$ is DEV if λ is the *only* eigenvalue of modulus λ

(ii) $f(z)$ is ADEV if the multiplicity of λ is *greater* than the multiplicity of the other eigenvalues of modulus λ

Some examples

$$M_{\mathcal{O}} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \qquad \mathrm{P}_{\mathcal{O}} = X^2 - 4$$

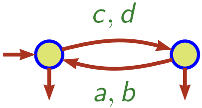
$$\mathbf{u}_O(\ell) = \frac{3}{4}\, 2^{\ell} + \frac{1}{4}\, (-2)^{\ell}$$



$\mathcal{O}$

$$M_{\mathcal{C}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \qquad \mathrm{P}_{\mathcal{C}} = X^2 - 4$$

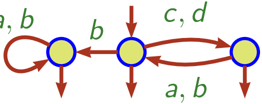
$$\mathbf{u}_C(\ell) = 2^{\ell} \qquad \mathrm{P}_C = X - 2$$



$\mathcal{C}$

$$M_{\mathcal{D}} = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \mathrm{P}_{\mathcal{D}} = (X^2-4)(2-X)$$

$$\mathbf{u}_D(\ell) = (\frac{1}{4}\,\ell + \frac{7}{8})\, 2^{\ell} + \frac{1}{8}\, (-2)^{\ell}$$



$\mathcal{D}$

Some examples

- O is neither DEV nor ADEV $\qquad \mathbf{u}_O(\ell) = \frac{3}{4}\,2^\ell + \frac{1}{4}\,(-2)^\ell$

- V is DEV $\qquad \mathbf{u}_V(\ell) = \frac{3}{2}\,2^\ell$

- D is ADEV but not DEV $\qquad \mathbf{u}_D(\ell) = (\frac{1}{4}\,\ell + \frac{7}{8})\,2^\ell + \frac{1}{8}\,(-2)^\ell$

Theorem

A rational language L *is* ADEV *iff the local growth rate* γ_L *exists.*

In this case, the modulus of L *is equal to* γ_L .

Theorem

L ADEV rational PCE and λ its modulus.

If every quotient of L whose modulus is equal to λ is ADEV,

$$\text{then } \mathsf{CP}_L \text{ exists and } \mathsf{CP}_L = \frac{\lambda}{\lambda - 1}$$

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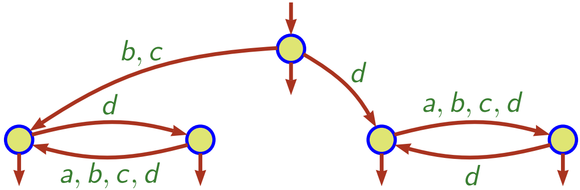
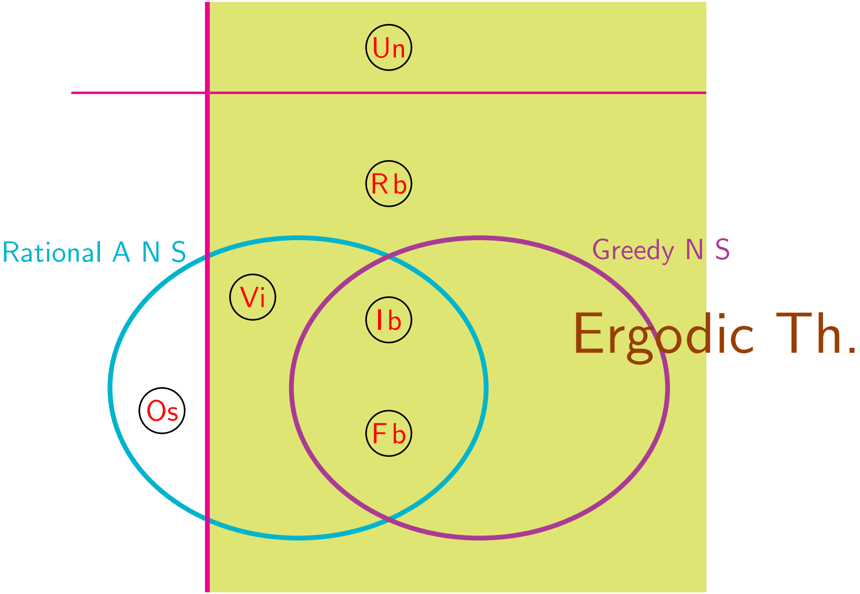
$\mathcal{V}$



Part IV

Ergodic Theory



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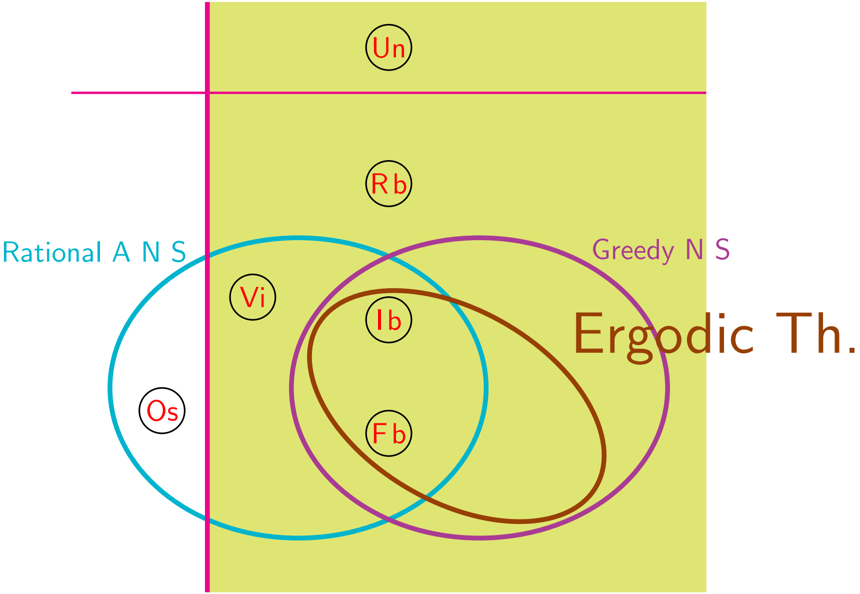

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An unmistakable fit

Our problem

$$\text{Does} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(i) \quad \text{exist ?}$$

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A rewriting

$$\text{Does} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathsf{cp}_L(\mathsf{Succ}_L^i(\varepsilon)) \quad \text{exist ?}$$

An unmistakable fit

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Does $\lim_{N\to\infty} \dfrac{1}{N} \displaystyle\sum_{i=0}^{N-1} \mathrm{cp}_L(\mathrm{Succ}_L^i(\varepsilon))$ exist ?

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The Ergodic Theorem

Theorem (Birkhoff 31)

Let $(\mathcal{K}, \tau)$ be a dynamical system, μ a τ-invariant measure on $\mathcal{K}$ and $f : \mathcal{K} \to \mathbb{R}$ in $\mathrm{L}^1(\mu)$ (f is absolutely μ-integrable).
If $(\mathcal{K}, \tau)$ is ergodic, then, for μ-almost all s in $\mathcal{K}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu \ . \qquad (*)$$

If $(\mathcal{K}, \tau)$ is uniquely ergodic and if f and τ are continuous, then (*) holds for every s in $\mathcal{K}$.

A bunch of definitions

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Let $(\mathcal{K}, \tau)$ *be a dynamical system,* μ *a* τ*-invariant measure on* $\mathcal{K}$ *and* $f : \mathcal{K} \to \mathbb{R}$ *in* $\mathrm{L}^1(\mu)$ *. If* $(\mathcal{K}, \tau)$ *is ergodic, then*

for μ*-almost all* $s \in \mathcal{K}$ $\quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(s)) = \int_{\mathcal{K}} f \, d\mu$ *(*)*

If $(\mathcal{K}, \tau)$ *is uniquely ergodic and if* f *and* τ *are continuous, then (*) holds for every* s *in* $\mathcal{K}$ *.*

Turning a numeration system into a dynamical system

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- ${}^{\omega}\!A$ = set of left infinite words over A

$$s = \cdots s_2 s_1 s_0 \quad \text{and} \quad s_{[\ell,j]} = s_\ell s_{\ell-1} \cdots s_j$$

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  Basis: *cylinders* $[w] = {}^{\omega}\!A\, w$

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- *Compactification* of L : $\mathcal{K}_L = \overline{{}^{\omega}0\, L}$

$$\mathcal{K}_L = \left\{ s \in {}^{\omega}\!A \;\middle|\; \forall j \in \mathbb{N} \quad \exists w^{(j)} \in 0^* L \qquad s_{[j,0]} \text{ right factor of } w^{(j)} \right\}$$

Turning a numeration system into a dynamical system

Definition of the odometer

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- Since $L \subseteq (A \setminus \{0\})\, A^*$, Succ_L defined on ${}^{\omega}0\, L$ by
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Extension of the carry propagation

$$\Delta(s,t) = \begin{cases} \min \left\{ j \in \mathbb{N} \;\middle|\; s_{[\infty,j]} = t_{[\infty,j]} \right\} & \text{if such } j \text{ exist} \\ +\infty & \text{otherwise} \end{cases}$$

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$$\forall s \in {}^{\omega}A \qquad \mathrm{cp}_L(s) = \Delta(s, \tau_L(s))$$

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Extension of the carry propagation

Proposition

If τ_L *is continuous,*
then cp_L *is continuous at any point where it takes finite values.*

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Where we are

We write $0 = {}^{\omega}0$

$$\mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(\tau_L^i(0))$$

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Where Birkhoff Theorem leads us

We want to show that 0 is a point such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathsf{cp}_L(\tau_L^i(0)) = \int_{\mathcal{K}_L} \mathsf{cp}_L \; d\mu$$

Greedy numeration systems

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- Basis = strictly increasing sequence of integers

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- $0^* L_G$ is closed under right factor and

$$\mathcal{K}_G = \overline{{}^\omega 0\, L_G} = \left\{s \in {}^\omega\!A \;\middle|\; \forall j \in \mathbb{N} \qquad s_{[j,0]} \in 0^* L_G\right\}$$

Ergodicity of greedy numeration systems

Theorem (Barat–Grabner 16, Grabner–Liardet–Tichy 95)

Let G *be a GNS.*

For every s *in* $\mathcal{K}_G$, $\lim_{j\to\infty} \mathsf{Succ}_G(s_{[j,0]})$ *exists and defines the odometer* $\tau_G\colon \mathcal{K}_G \to \mathcal{K}_G$ *:*

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A GNS G is said to be *exponential*

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If G *is an exponential GNS, then the dynamical system* $(\mathcal{K}_G, \tau_G)$ *is uniquely ergodic.*

Carry propagation in greedy numeration systems

Theorem

If G is an exponential GNS, then CP_G exists.

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Corollary

Let G *be an exponential GNS with* $G_\ell \sim C\,\alpha^\ell$.

If L_G *is* PCE, *then* CP_G *exists and* $\mathsf{CP}_G = \dfrac{\alpha}{\alpha - 1}$.

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- Implies $\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} f\big(\tau_G^i(0)\big) = \int_{\mathcal{K}_G} f\, d\mu_G$ for Riemann-integrable f

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- Does not imply $\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} f\big(\tau_G^i(0)\big) = \int_{\mathcal{K}_G} f\, d\mu_G$ for any f

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- $$\int_{\mathcal{K}_G} f_k \, d\mu_G \quad \leqslant \quad \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_G(\tau_G^i(0)) \quad \leqslant \quad \int_{\mathcal{K}_G} f_k \, d\mu_G + M_{k+1}$$

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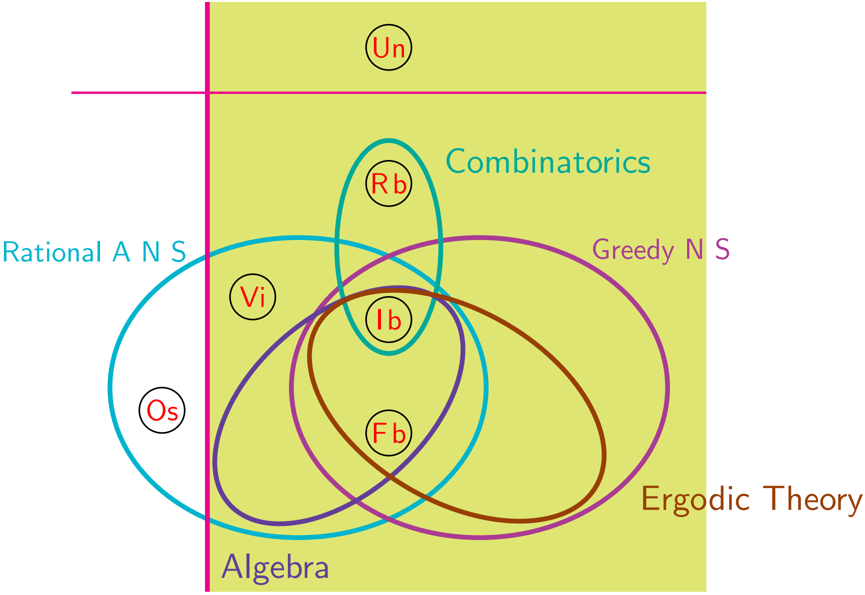
Roadmap



Un
Combinatorics
Rb
Rational A N S
Greedy N S
Vi
Ib
Os
Fb
Ergodic Theory
Algebra

Comme il y a une infinité de choses sages
qui sont menées de manière très folle,
il y a aussi des folies qui sont menées de manière très sage.

Montesquieu

Just as wise ends are oftentimes sought
in the most foolish way,
so foolishness is sometimes sought with great wisdom.

Translation by Reuben Thomas