Matching for parameterised symmetric golden maps

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One World Numeration
January 31, 2023
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(i) Introduction
Symmetric golden maps

Set $\beta : = \frac{\sqrt{5} + 1}{2}$ ($\beta^2 = \beta + 1$), $J_0 := [-\sqrt{\beta}, \sqrt{\beta}]$, $J_{\pm 1} := \pm (\sqrt{\beta}, 1]$

Let $s : [-1, 1] \rightarrow \{0, \pm 1\}$, where $s(x) = i$ iff $x \in J_i$

For $\alpha \in [1, \beta]$, define $S_\alpha : [-1, 1] \rightarrow [-1, 1]$ by

$$S_\alpha(x) := \beta x - s(x) \alpha$$

$$= \begin{cases} 
\beta x + \alpha & , x \in J_{-1} \\
\beta x & , x \in J_0 \\
\beta x - \alpha & , x \in J_1 
\end{cases}$$

Note $S_\alpha(J_1) = S_\alpha((-\sqrt{\beta}, 1])$

$= (1 - \alpha, \beta - \alpha]$

$\subset (-1/\beta, 1/\beta]$

$\subset J_0$
For \( j \geq 1 \), set \( s_j = s_{\alpha,j}(x) := s(\beta^{-1} s_{\alpha,j-1}(x)) \).

Let \( \nu : \{0, \pm 1 \}^* \rightarrow \mathbb{R} \)

\[ u = u_1 \ldots u_n \mapsto \nu(u) = \sum_{j=1}^{n} u_j / \beta^j \]

Via induction, \( S^k_\alpha(x) = \beta^k (x - \alpha \nu(s_1 \ldots s_k)) \) \( \forall k \geq 0 \). Hence

\[ x = \alpha \nu(s_1 \ldots s_k) + S^k_\alpha(x) / \beta^k , \]

and letting \( k \rightarrow \infty \),

\[ x = \alpha \sum_{j=1}^{\infty} s_j / \beta^j . \]

Call \( (s_j)_{j \geq 1} = (S_{\alpha,j}(x))_{j \geq 1} \) the \( S_\alpha \)-expansion of \( x \).

Since \( S_\alpha(J_{\pm 1}) \subseteq J_0 \), \( \pm 1 \) is followed by 0.
Main results

Thm. 1: For each $\alpha \in (1/\beta)$, $\exists! \lambda$ abs. cont. $\sigma_{\lambda}$-invariant measure $\mu_{\lambda}$. In fact, $\mu_{\lambda} \sim \lambda$, and $\exists$ a countable collection $\{I_d\}_{d \in M}$ of disjoint, open subintervals of $[1, \beta]$ with full $\lambda$-measure s.t. for fixed $d \in M$, the density of each $\mu_{\lambda}, \alpha \in I_d$, is a step function with (at most) the same number of jumps

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{1}_{I_d}(S_x(j))
\]

Thm. 2: The asymptotic relative frequency of $0$ in $\lambda$-a.e. $\sigma_{\lambda}$-expansion depends continuously on $\alpha$ and attains its max. $4/5$ on the maximal interval $[\sqrt[4]{2} + \sqrt[4]{\beta}, 1 + \sqrt[4]{\beta^2}]$
Prior work

Dajani–Kalle (2020) obtain similar results for a family of maps defined analogously with constant slope 2.

Thm. 1: For each \( x \in (1, \beta) \), \( \exists! \) abs. cont. \( S_x \)-invariant measure \( \mu_x \). In fact, \( \mu_x \sim \lambda \), and \( \exists \) a countable collection \( \{ I_d \}_{d \in \mathbb{M}} \) of disjoint, open subintervals of \( [1, \beta] \) with full \( \lambda \)-measure s.t. for fixed \( d \in \mathbb{M} \), the density of each \( \mu_x \), \( x \in I_d \), is a step function with (at most) the same number of jumps.

Thm. 2: The asymptotic relative frequency of 0 in \( \lambda \)-a.e. \( S_x \)-expansion depends continuously on \( x \) and attains its max. \( \frac{4}{5} \) on the maximal interval \( \left[ \frac{2}{3}, \frac{3}{2} \right] \).
<table>
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<th>Similarities</th>
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<td>· Kopf (1990)</td>
<td>· Dynamics of golden maps more subtle</td>
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<td>· Matching</td>
<td>· DK20 relate binary maps to $\alpha$-continued fractions</td>
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<td>· Birkhoff’s Ergodic Thm.</td>
<td>· Our analysis more intrinsic to golden maps</td>
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(ii) Matching
Motivation and definitions

- Fix $\alpha \in (1, \beta]$. Results of K90 imply the existence of a unique—hence ergodic—abs. cont. $S_\alpha$-invariant probability measure with density

$$f_\alpha(x) = \frac{1}{c} \sum_{j=20} \sum_{a \in \{ \pm \sqrt{\beta} \}} \frac{1}{2^{\frac{j}{12}}} \left( \mathbb{1}_{[-1, \frac{1}{2}\alpha]}(x) - \mathbb{1}_{[-1, \frac{1}{2}\alpha]}(x) \right)$$

with $c \in \mathbb{R}$ a normalising constant.

- Note that $\lim_{x \to \beta^-} S_\alpha(x) = 1$ and $\lim_{x \to \beta^+} S_\alpha(x) = 1 - \alpha$.

- Defn.: The matching index of $S_\alpha$ is $m = m(\alpha) := \inf \{ j \mid \exists \xi \text{ such that } S_\alpha(1) = S_\alpha(1-\alpha) \}^j$; $S_\alpha$ has matching if $m < \infty$.

Let $(d_j), (e_j)$ denote the $S_\alpha$-expansions of $1, 1-\alpha$, resp., and set $d = d_1 \cdots d_m$, $e = e_1 \cdots e_m$. When $m < \infty$, call $d$ the matching word corresponding to $\alpha$. A maximal subinterval of $[1, \beta]$ on which matching words equal $d$ is a matching interval.
### First examples

Note $\alpha \in J_1$, $1-\alpha \in J_0$ $\forall \alpha \in [1/\beta^3]$, so $\begin{pmatrix} d_i \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$ (throughout, $\overline{u} := -u$)

<table>
<thead>
<tr>
<th>$\alpha \in (1+\sqrt{\beta^2}, \beta]$</th>
<th>$\alpha = 1+\sqrt{\beta^2}$</th>
<th>$\alpha \in (1+\sqrt{\beta^3}, 1+\sqrt{\beta^2})$</th>
<th>$\alpha = 1+\sqrt{\beta^3}$</th>
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<tr>
<td>$S_\alpha(1) = \beta - \alpha \in [0,1/\beta^3] \subset J_0$, $S_\alpha^2(1) = \beta^2 - \beta^3 \alpha \in J_0$, $S_\alpha^3(1) = 1$</td>
<td>$S_\alpha(1) = \beta - \alpha = 1/\beta^3 \in J_0$, $S_\alpha^2(1) = 1/\beta^2 \in J_0$, $S_\alpha^3(1) = 1/\beta \in J_0$, $S_\alpha^4(1) = 1 \in J_1$</td>
<td>$S_\alpha(1) = \beta - \alpha \in (1/\beta^3, 1/\beta^2) \subset J_0$, $S_\alpha^2(1) = \beta^2 - \beta^3 \alpha \in J_0$, $S_\alpha^3(1) = (\beta^3 + 1) \alpha \in J_0$</td>
<td>$S_\alpha(1) = \beta - \alpha = 1/\beta^2 \in J_0$, $S_\alpha^2(1) = 1/\beta \in J_0$, $S_\alpha^3(1) = 1 \in J_1$, $S_\alpha^4(1) = 1 \in J_1$ and $S_\alpha^5(1) = (\beta^3 + 1) \alpha \in J_0$, $S_\alpha(1-\alpha) = 1-\alpha \in J_0$</td>
<td>$S_\alpha(1) = 1/\beta \in J_0$, $S_\alpha^2(1) = 1 \in J_1$ and $S_\alpha(1-\alpha) = 0 = 1-\alpha \in J_0$.</td>
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### Matching

- No matching (but a Markov partition)
- $\begin{pmatrix} d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$
- No matching (MP)
- $\begin{pmatrix} d \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$
- No matching (MP)
Prop. 1: For every $\alpha \in [1, \beta]$ and $j \geq 0$,

$$
\gamma_j := S^j_{\alpha} (1) - S^j_{\alpha} (1-\alpha) \in \{0, \alpha/\beta, \alpha, \beta \alpha, \beta^3 \}
$$

- Sketch: Suppose $\alpha \in (1, 1+\sqrt{\beta^3})$. Note $\gamma_0 = \alpha$. Suppose $\gamma_{j-1} \in \{0, \alpha/\beta, \alpha, \beta \alpha, \beta^3 \}$.

If $\gamma_{j-1} = 0$, done, so suppose $\gamma_{j-1} \neq 0$. Note

$$
\gamma_j = \beta S^j_{\alpha} (1) - d_j \alpha - (\beta S^j_{\alpha} (1-\alpha) - e_j \alpha)
$$

$$
= \beta \gamma_{j-1} - \alpha (d_j - e_j).
$$

Consider cases:

(i) $\gamma_{j-1} = \alpha/\beta$: Since $1/\beta < \alpha/\beta < 2/\beta$, $(d_j, e_j) \in \{(0, 1), (1, 0), (0, \frac{1}{2}) \}$.

Hence $\gamma_j \in \{0, \alpha, \beta \alpha, \beta^3 \}$.
Note \( (d_m 2 \ d_m 1 \ d_m) \in \{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \} \).

Conversely, if \( (e_{j-2} \ d_j \ e_j) \in \{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \} \), then \( j = m \).
Matching almost everywhere

Note that \( \text{diam}(\mathcal{J}_{\pm}) = 1 - \frac{1}{\beta} = \frac{1}{\beta^2} < \frac{\alpha}{\beta} \). By Prop. 1, \( S_{\alpha}^j(1-\alpha) \in \mathcal{J} \), or \( S_{\alpha}^j(1) \in \mathcal{J}_{-1} \), implies matching. But \( S_{\alpha}(x) \mid_{\mathcal{J}_0} = \beta x \), so \( S_{\alpha}^j(1-\alpha) > 0 \) or \( S_{\alpha}^j(1) < 0 \) implies matching. Hence

\[
S_{\alpha}^j(1-\alpha) \in S_{\alpha}^{-1}((-\alpha, \beta) \cap \mathbb{Z}) = (-\alpha/\beta, -\frac{\alpha}{\beta}) \quad \text{or} \quad S_{\alpha}^j(1) \in S_{\alpha}^{-1}([-\alpha, 0]) \cap (0, \mathbb{Z}) = (\frac{\alpha}{\beta}, \frac{\alpha}{\beta})
\]

implies matching.

Prop.: \( S_{\alpha} \) has matching for \( \lambda \)-a.e. \( x \in [1, \beta] \).

Idea: Let \( \ell_{\alpha} = \inf \{ j \mid (\star) \text{ holds} \} \), then \( \alpha > 1 \). Show for \( x \in [1, \alpha-1] \) and \( 0 \leq j \leq \ell_{\alpha} \), \( S_{\alpha}^j(x) = \alpha B^j(x/\alpha), \quad B(x) = \beta x \pmod{1} \).

Use ergodicity of \( B \).
Matching words/intervalss

Set \( w_0 := 00, \ w_1 := 001, \ w_2 := 01 \)

Defn.: Say \( d \in \{0,1\}^* \) is in admissible block form if \( d = 10, 1001, \) or \( d = 1w_1 \ldots w_{i-1} (1 - i/2), \) \( i, \ldots, i \in \{0,1,2\}, \ i_j = 2, \ i_n \neq 1. \)

Set \( B := \{ d \text{ in adm. block form}\} \)

Ex.: \( 101001 \notin B \) whereas \( 101001 \in B \)

Defn.: Let \( \phi : B \to \{0,1\}^* \), where \( \phi(\overline{0}) := \overline{11}, \ \phi(1001) := \overline{0010}, \) and \( \phi(1w_1 \ldots w_{i-1} (1 - i/2)) := \overline{0}w_{2-i} \ldots w_{2-i} (i/2). \)

Say \( d \in B \) satisfies Property \( M \) if \( V_j \geq 0, \ \sigma^{-j}(d) \leq d \) and \( \sigma^{-j}(\phi(d)) \leq d. \)

Set \( M := \{ d \text{ sat. Prop. } M \} \)
Prop. 2: Let $d$ be a matching word. Write $\phi(d) = (d_1 \ldots d_m)$, $m \geq 2$. Recall

$$(d_i) = (1) \quad \text{and} \quad (d_{m-2}d_{m-1}d_m)e \in \{(0 \ 1 \ 0), (0 \ 0 \ 1)\} = \{(\frac{w_1}{w_0}), (\frac{w_0}{w_1})\}.$$ 

All other digits are determined by cycles at $a/b:

$$(d_jd_{j+1})e \in \{(\frac{w_2}{w_0}), (\frac{w_0}{w_2})\}, \quad (d_jd_{j+1}d_{j+2})e = (\frac{w_1}{w_1}).$$

Thus $d \in B$ (and $e = \phi(d)$).

Note $x < y$ iff $\sigma_x(y) < \sigma_y(y)$. Since $-1 \leq \sigma_x(1-a), \sigma_x(1) \leq 1,

-d \leq \sigma_x(e), \sigma_x(d) \leq d \quad \forall x \geq 0$. 

Vertices: $y_{x} = \frac{w_0}{a} + \frac{w_1}{b}$, $v_{x} = \frac{w_0}{a} + \frac{w_1}{b}$

Edges: $\phi(d) = (d_1 \ldots d_m)$

Recall: $d \in B$, i.e.

- $d = 10, 001, \text{ or }$
- $d = 1w_1 \ldots w_{n-2}(1-in)$, $i = 2, n \neq 1$

and

- $\forall j \geq 0, \quad \sigma_x(\phi(d)), \sigma_x(\phi(d)) \leq d,$

where $\phi(10) = \phi(1000) = \phi(0010)$,

$\phi(1\ w_1 \ldots w_{n-2}(1-in)) = \overline{0\ w_{n-2} \ldots w_2 \ w_1 \ v_{n-2} \ldots v_1}$
Defn.: Set $I_{10} := (1 + \sqrt{\beta}, \beta]$, and for $d \in M \setminus \{10\}$,

$$I_d := \left( \frac{\beta^{n} + \beta^{d-1}}{\beta^{n}(d) + \beta^{d-1}}, \frac{\beta^{n} - \beta^{1-d}}{\beta^{n}(d) - \beta^{1-d}} \right)$$

Prop. 3: For each $d \in M$, $\emptyset \neq I_d \subset [1, \beta]$. Further, for each $a \in I_d$, the $Sa$-expansions of $1$ and $1-a$ begin with $d$, $\phi(d)$, resp., and

$$m(a) = \text{len}(d).$$

-Idea: For $\emptyset \neq I_d \subset [1, \beta]$, compare endpoints explicitly.

For second statement, again use for $x \in [1, \alpha-1]$, $0 \leq j \leq k$ that

$$S_{\alpha}^{j}(x) = a \beta^{j}(x/a), \quad \beta(x) = \beta x \pmod{1}.$$

Perform careful analysis of $B$-expansions of $V_{\alpha}$, $(\alpha-1)/\alpha$.

That $m(a) = \text{len}(d)$ follows from the final three digits of $d$, $\phi(d)$ and

the orbit-differences graph.

\[ \square \]

Prop's 2, 3 imply $M = \{ \text{matching words} \}$. Further, Prop. 3 implies

each $I_d \subset \{ \text{matching interval corresponding to } d' \}$.
Prop. 4: If \( S_x \) has matching then \( \alpha \in \text{Id} \), where \( d \in \mathcal{M} \) is the matching word corresponding to \( \alpha \).

Idea: Assume \( m > 2 \). From the orbit-differences graph,
\[
(d_m-2, d_m-1, d_m) \in \left\{ (0,1,0), (0,0,1) \right\}.
\]
Suppose \( d_m = 0 \). Either
\[
S_{\alpha}^{-1}(1-\alpha) \in (-\frac{1}{2}, -\frac{1}{3}) \quad \text{or} \quad S_{\alpha}^{-2}(1) \in (\frac{1}{3}, \frac{2}{3}).
\]
Use these and \( S_{x}^{j}(x) = \beta^{j}(x-s_{1}v(s_{1},...,s_{j})) \) to obtain bounds on \( \alpha \) and show \( \alpha \in \text{Id} \).

\[ \square \]

Prop's 2, 3, 4 classify matching words and intervals as \( \mathcal{M} \) and \( \{ \text{Id}, \text{Id} \} \), respectively.
(iii) Invariant measures and frequencies of digits
Invariant measures

Recall, for $\alpha \in (1, \beta]$, there exists a unique — hence ergodic — absolutely continuous $S_\alpha$-invariant probability measure $\mu_\alpha$ with density

$$f_\alpha(x) = \frac{1}{C} \sum_{j=0}^{m-1} \sum_{\alpha \in \{\pm \beta\}} \frac{1}{x^{\alpha}} \left( \mathbb{1}_{[-1, \frac{\alpha}{\alpha^2 - \alpha}]}(x) - \mathbb{1}_{[-1, \frac{\alpha}{\alpha^{2}+\alpha}]}(x) \right)$$

when matching occurs.

Thm. 1: For each $\alpha \in (1, \beta)$, $\exists$! abs. cont. $S_\alpha$-invariant measure $\mu_\alpha$. In fact, $\mu_\alpha \sim \lambda$, and $\exists$ a countable collection $\{I_d\}_{d \in M}$ of disjoint, open subintervals of $[1, \beta]$ with full $\lambda$-measure s.t. for fixed $d \in M$, the density of each $\mu_\alpha$, $\alpha \in I_d$, is a step function with (at most) the same number of jumps.
Frequencies of digits

By Birkhoff’s Ergodic Theorem, for $a \in (1, \beta^2)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \left( \frac{1}{J_i} \int_{J_i} S_n^i(x) \, dx \right) = \int_{J_i} \frac{d\mu_a}{J_i} = \mu_a(J_i)$$

for $a$-a.e. $x \in [1, \beta^2 - 1]$. 

$1 = \sum_{i=1}^{+} \mu_a(J_i)$ and $\mu_a(J_{i-1}) = \mu_a(J_i)$

Let $freqs: (1, \beta^2] \to [0, 1]

\alpha \mapsto \mu_a(J_0)$

Arguments analogous to $DU \not\equiv 0$ imply continuity.

Since matching $a$ are dense in $(1, \beta^2)$, consider $freqs|_{Id}$
For $\alpha \in I_d$, let $d \in \mathcal{U}$,
\[
\text{freqs}(\alpha) = 1 - \frac{1}{\beta \nu(\Xi(d))} \left( \frac{n(d)-1}{\alpha} - k(d) \right)
\]
where $\Xi(\cdot)$, $n(\cdot)$, and $k(\cdot)$ depend only on $d$.

- $n(d) := \sum_{j=1}^{\infty} d_j + \sum_{j=1}^{\infty} e_j$

- Note $\text{freqs}|_{I_d}$ is
\[
\begin{cases}
\text{strictly increasing, } & n(d) > 1 \\
\text{constant, } & n(d) = 1 \\
\text{strictly decreasing, } & n(d) < 1
\end{cases}
\]

Thm. 2: The asymptotic relative frequency of 0 in $\lambda$-a.e. $S_a$-expansion depends continuously on $\alpha$ and attains its max. 4/5 on the maximal interval $[\frac{1}{2} + \frac{1}{2}, 1 + \frac{1}{2}]$.
(iv) Jump transformations
For \( \alpha \in [1, \beta] \), define \( \tau_{\alpha} : [-1, 1] \to [-1, 1] \) by

\[
\tau_{\alpha}(x) := \beta + 1 \text{sgn}(x) \beta \alpha =
\begin{cases}
\beta^2 x + \beta \alpha, & \alpha \in J_1 \\
\beta x, & \alpha \in J_0 \\
\beta^2 x - \beta \alpha, & \alpha \in J_1
\end{cases}
\]

\( \tau_{\alpha} \) is the jump transformation of \( \mathcal{S}_\alpha \) wrt \( J_0 \).

Prop. Each \( \tau_{\alpha} \), \( \alpha \in (1, \beta) \) has a unique - hence ergodic - \( \tau_{\alpha} \)-invariant measure \( \nu_{\alpha} \) given by

\[
\nu_{\alpha}(A) = \frac{\mu_{\alpha}(A / \beta)}{\text{freq}_{\beta}(\alpha)}.
\]

Moreover, \( \text{freq}_{\beta}(\alpha) : = \nu_{\alpha}(J_0) = 2 - 1 / \text{freq}_{\beta}(\alpha) \).
Analogous results

Thm. 1: For each $\alpha \in (1/\beta]$, $\exists!$ abs. cont. $T_\alpha$-invariant measure $\mu$. In fact, $T_\alpha \mu = \lambda$, and $\exists$ a countable collection $\{I_d\}_{d \in \mathcal{A}}$ of disjoint, open subintervals of $[1, \beta]$ with full $\lambda$-measure s.t. for fixed $d \in \mathcal{A}$, the density of each $T_\alpha$, $\alpha \in I_d$, is a step function with (at most) the same number of jumps.

Thm. 2: The asymptotic relative frequency of 0 in $\lambda$-a.e. $T_\alpha S_\alpha$-expansion depends continuously on $\alpha$ and attains its max. on the maximal interval $[1/2 + 1/\beta, 1 + 1/\beta^2]$. 

$4/5$ $3/4$
Open questions

- Is there an explicit (useful) bijection

\[ \{ \text{matching words for } \} \leftrightarrow \{ \text{matching words for } \} \]

- Can we obtain analogous results for symmetric maps of different slopes \( m \in (1, 2) \),

\[ S_{m, \alpha}(x) := \begin{cases} 
mx + \alpha & , \ x \in [-1, -1/m) \\
mx & , \ x \in [-1/m, m] \\
mx - \alpha & , \ x \in (m, 1] 
\end{cases} \]
Thank you!

References
