Limit theorems for counting large continued fraction digits

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Consider the unique continued fraction (CF) expansion of an irrational, positive x given by

$$x = [a_0(x); a_1(x), a_2(x), \ldots] := a_0(x) + \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}}.$$

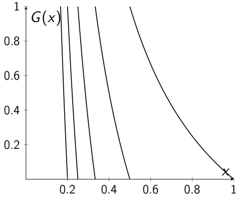
In case that $x \in [0,1)$ we write

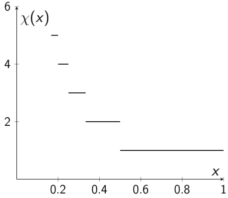
$$x = [a_1(x), a_2(x), \ldots] := \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots}}.$$

We may consider $a_1, a_2, \ldots$ as random variables obtained by a dynamical system. Define

$$G\colon\ [0,1) \to [0,1)\,, \qquad x \mapsto \begin{cases} 1/x \mod 1, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

$$\chi\colon\ [0,1) \to \mathbb{R}_{>0} \qquad x \mapsto \lfloor 1/x \rfloor\,.$$





Then $a_n(x) = \chi \circ G^{n-1}(x)$ with $G^0 := \mathrm{id}$ and $G^n := G \circ G^{n-1}$, $n \geq 1$. G is also called *Gauss map*. For rational numbers x there exists $n \in \mathbb{N}$ such that $G^n(x) = 0$ which yields a finite CF expansion.

- The following is standard material, see for example [Khinchin, 1997, Dajani and Kraaikamp, 2002, Iosifescu and Kraaikamp, 2009].
- Gauss found a G-invariant measure $\mathfrak{m}$ which is equivalent to the Lebesgue measure λ with density $h(x) = 1/((x+1)\log 2)$, $x \in [0,1)$, i.e. we have for each measurable set A that

$$\mathfrak{m}(A) = \int_A h(x) \mathrm{d}\lambda(x) = \frac{1}{\log 2} \cdot \int_A \frac{1}{x+1} \mathrm{d}x.$$

- In particular, for each measurable set A it holds that

$$\frac{\lambda(A)}{\log 2} \leq \mathfrak{m}(A) \leq \frac{2 \cdot \lambda(A)}{\log 2}.$$

- The dynamical system $([0,1), \mathcal{B}, G, \mathfrak{m})$ is in fact ergodic.
- In particular, by Birkhoff's ergodic theorem this implies for any subset $A \subset \mathbb{N}$ that

$$\lim_{n\to\infty} \frac{1}{n} \text{card}\left\{k \leq n \colon a_k \in A\right\} = \mathfrak{m}\left(A\right) \text{a.s.}$$

- We can conclude that for a.e. x every fixed digit $k \in \mathbb{N}$ will be realized in the CF expansion $(a_n(x))$ infinitely often.

Will $a_n \geq k_n$ still be realized infinitely often if (k_n) is an increasing sequence?

Theorem 1.1 (Borel-Bernstein Theorem, [Borel, 1909, Bernstein, 1911, Bernstein, 1912])

Consider a sequence of positive reals (b_n). Then $a_n \geq b_n$ holds infinitely often with Lebesgue measure 0 or 1, according as the series $\sum_{n\in\mathbb{N}} 1/b_n$ converges or diverges.

We are interested in a refinement of this theorem concerning a_n hitting more general intervals. The refinements we will give are inspired by [Galambos, 1972] who considered the independent case of entries of the classical Lüroth series.
(For those who don't know Lüroth expansions: Those are numeration systems related to the continued fraction systems, i.e. the digits obey, up to a constant, the same distribution function. But the entries behave like i.i.d. random variables.)

To state the next theorem by Philipp we let $\mathbb{E}(X)$ denote the expectation and $\mathbb{V}(X) := \mathbb{E}\big((X - \mathbb{E}(X))^2\big)$ denote the variance of a random variable X.

Theorem 1.2 (Theorem 2 of [Philipp, 1970])

Consider a sequence of positive reals $(b_n)_{n\in\mathbb{N}}$. If

$$\lim_{n\to\infty} b_n = \infty \quad \textit{and} \quad \sum_{n\in\mathbb{N}} \frac{1}{b_n} = \infty,$$

then for $S_n := \sum_{k=1}^{n} \mathbb{1}_{\{a_k > b_k\}}$ we have

$$\lim_{n\to\infty} \mathfrak{m}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} < z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \mathrm{d}t.$$

We remember that by the Borel-Bernstein theorem $\sum_{n\in\mathbb{N}} \frac{1}{b_n} = \infty$ already implied that $a_n \geq b_n$ holds infinitely often with Lebesgue 1. We are intrigued to ask if the condition $\lim_{n\to\infty} b_n = \infty$ is indeed a necessary condition for Philipp's CLT to hold.

- Compare the assumptions of Theorem 1.2 with the necessary conditions for the CLT to hold in the case that $(X_i) := (\mathbb{1}_{\{a_i > b_i\}})$ is a sequence of independent random variables with the same distribution function.
- In this case the statement could be proven using Lindeberg's condition, i.e. we assume that for all $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\mathbb{V}(S_n)} \cdot \sum_{i=1}^{n} \mathbb{E}\left((X_i - \mathbb{E}(X_i))^2 \cdot \mathbb{1}_{\left\{|X_i - \mathbb{E}(X_i)| > \epsilon \cdot \sqrt{\mathbb{V}(S_n)}\right\}} \right) = 0.$$

- For the distribution function of the CF digits this condition is in fact equivalent to $\lim_{n\to\infty} \mathbb{V}(S_n) = \infty$.
- In the i.i.d. case this is equivalent to $\lim_{n\to\infty} \sum_{k=1}^{n} \mathfrak{m}(a_k > b_k) \cdot \mathfrak{m}(a_k \leq b_k) = \infty$.
- Since $\mathfrak{m}(a_k \leq b_k) \leq \mathfrak{m}(a_k \leq 1) = \mathfrak{m}(a_k = 1)$, we have that $\lim_{n\to\infty} \mathbb{V}(S_n) = \infty$ is in the i.i.d. case equivalent to $\sum_{n\to\infty} \mathfrak{m}(a_k > b_k) = \infty$.
- This is equivalent to $\sum_{k=1}^{\infty} 1/b_k = \infty$.
- We will show that despite of the lack of independence the additional condition $\lim_{n\to\infty} b_n = \infty$ is not necessary.

Theorem 2.1 ([Kesseböhmer and S., 2020])

Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers and $(d_n)_{n\in\mathbb{N}}$ be a sequence of positive integers both tending to infinity. Then

$$d_n \leq a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right) \tag{1}$$

holds infinitely often with Lebesgue measure 0 or 1, according as

$$\max\left\{\sum_{n\in\mathbb{N}} \frac{1}{c_n d_n}, \sum_{n\in\mathbb{N}} \frac{1}{d_n^2}\right\} \tag{2}$$

is finite or not.

Let us now choose $c_n := 2d_n$, then (1) simplifies to $d_n \leq a_n \leq d_n + 1/2$ and (2) becomes $\sum_{n\in\mathbb{N}} 1/(2d_n^2)$.

This gives the following corollary:

Corollary 2.2

Let $(d_n)_{n\in\mathbb{N}}$ *be a sequence of positive integers tending to infinity. Then*

$$a_n = d_n$$

holds infinitely often with Lebesgue measure 0 *or* 1*, according as*

$$\sum_{n\in\mathbb{N}} \frac{1}{d_n^2}$$

is finite or not.

- For $d_n := \lfloor \sqrt{n \log(n)} \rfloor$ there are almost surely infinitely many values of n such that $a_n = d_n$.
- For $e_n := \lfloor \sqrt{n} \log(n) \rfloor$ there are almost surely only finitely many values of n such that $a_n = e_n$.

We state a slightly different version of the previous Theorem 2.1.

Theorem 2.3

Let $(c_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers and* $(d_n)_{n\in\mathbb{N}}$ *be a sequences of positive integers both tending to infinity. Then*

$$d_n < a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right)$$

holds infinitely often with Lebesgue measure 0 *or* 1*, according as*

$$\sum_{n:\ c_n \leq d_n} \frac{1}{c_n d_n}$$

is finite or not.

Definition 2.4

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{C}, \mathcal{D} \subset \mathcal{A}$ two σ-fields, then the following quantities measure the dependence of the sub-σ-fields:

$$\phi(\mathcal{C}, \mathcal{D}) := \sup_{C,D} |\mathbb{P}(D \mid C) - \mathbb{P}(D)|, C \in \mathcal{C}, D \in \mathcal{D}, \mathbb{P}(C) > 0,$$

$$\psi(\mathcal{C}, \mathcal{D}) := \sup_{C,D} \left| \frac{\mathbb{P}(C \cap D)}{\mathbb{P}(C) \cdot \mathbb{P}(D)} - 1 \right|, C \in \mathcal{C}, D \in \mathcal{D}, \mathbb{P}(C), \mathbb{P}(D) > 0.$$

Let $(X_n)_{n\in\mathbb{N}}$ be a (not necessarily stationary) sequence of random variables. For $1 \leq J \leq L \leq \infty$ we can define a σ-field by

$$\mathcal{A}_J^L := \sigma(X_k, k \in \mathbb{N} \cap [J, L]).$$

With that the dependence coefficients are defined by

$$\phi(n) := \sup_{k\in\mathbb{N}} \phi\left(\mathcal{A}_0^k, \mathcal{A}_{k+n}^\infty\right) \text{ and } \psi(n) := \sup_{k\in\mathbb{N}} \psi\left(\mathcal{A}_0^k, \mathcal{A}_{k+n}^\infty\right).$$

(X_n) is said to be ϕ-(or ψ-)mixing if $\phi(n) \to 0$ (or $\psi(n) \to 0$) as $n \to \infty$.

- ϕ-mixing is weaker than ψ-mixing, indeed $\phi(n) \leq \frac{1}{2}\psi(n)$, for all $n \in \mathbb{N}$.
- Let $\psi_{\mathfrak{m}}$ ($\phi_{\mathfrak{m}}$) be the ψ (ϕ)-mixing coefficient for the Gauss system, [Iosifescu and Kraaikamp, 2009, Chapter 2.3.4] gives the estimate $\psi_{\mathfrak{m}}(n) \leq \rho\theta^{n-2}$ for $n \geq 2$, where $\rho = \pi^2 \log 2/6 - 1$ and θ is a constant less than 0.30367, and $\psi_{\mathfrak{m}}(1) = 2\log 2 - 1$.
- In particular we have $\sum_{n=1}^{\infty} \phi_{\mathfrak{m}}(n) < \infty$.

- As the ideas are similar, we only give a proof for Theorem 2.1 for the intervals $A_n := [d_n, d_n(1 + 1/c_n)]$.
- The first direction is easy and does not need any mixing theory.
- Remember that $\lambda(A) / \log 2 \leq \mathfrak{m}(A) \leq 2 \cdot \lambda(A) / \log 2$.
- We have that

$$\lambda(A_n) = \frac{1}{d_n} - \frac{1}{d_n + \lfloor d_n/c_n \rfloor + 1}.$$

An easy calculation shows that

$$\lambda(A_n) \leq \frac{1}{d_n} - \frac{1}{d_n + d_n/c_n + 1} < \frac{1}{c_n d_n} + \frac{1}{d_n^2}.$$

Hence, if the sum of the right hand sides is finite, the first Borel-Cantelli implies $\mathfrak{m}(\limsup_{n\to\infty} A_n) = 0$.

In the i.i.d. case one would use the second Borel-Cantelli lemma to obtain the other direction which due to the dependence of the CF digits is not immediately applicable here.
However, all our zero-one laws can be proven by the following lemma which is a simplified version of [Philipp, 1967, Theorem 3].

Lemma 2.5

Let $(\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of measurable sets in any probability space $(\Omega, \mathcal{A}, \mu)$. Suppose that there exists a function $q\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ fulfilling $\sum_{n=1}^{\infty} q(n) < \infty$ such that for all integers $n > m$ we have

$$\mu\left(\Gamma_m \cap \Gamma_n\right) \leq \mu\left(\Gamma_m\right) \cdot \mu\left(\Gamma_n\right) + q\left(n-m\right) \cdot \mu\left(\Gamma_n\right).$$

Then Γ_n holds infinitely often with Lebesgue measure 0 or 1 according as $\sum_{n=1}^{\infty} \mu\left(\Gamma_n\right)$ is finite or not.

We can set q as the ϕ-mixing coefficient and use $\sum_{n=1}^{\infty} \phi_{\mathfrak{m}}\left(n\right) < \infty$.
It would also be possible to directly use a dynamical Borel-Cantelli lemma like [Kim, 2007].

For the following we denote by $\sigma(a_n)$ the σ-algebra generated by the nth continued fraction digit.
The previous calculations give the following lemma:

Lemma 2.6

Let (D_n) be a sequence of events such that $D_n \in \sigma(a_n)$, for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \mathfrak{m}(D_n) = \infty$, then $\mathfrak{m}(\limsup_{n\to\infty} D_n) = 1$.

- Remember $\lambda(A_n) = 1/d_n - 1/(d_n + \lfloor d_n/c_n \rfloor + 1)$.
- Hence, $\lambda(A_n) > \dfrac{1}{d_n} - \dfrac{1}{d_n + d_n/c_n} > \dfrac{1}{2c_n d_n}$,
  i.e. $\sum_{n\in\mathbb{N}} \lambda(A_n) = \infty$ if $\sum_{n\in\mathbb{N}} 1/(c_n d_n) = \infty$.
- On the other hand $A_n \supset \{a_n = d_n\}$ and thus

$$\lambda(A_n) \geq \lambda(a_n = d_n) = \frac{1}{d_n} - \frac{1}{d_n + 1} \geq \frac{1}{2d_n^2},$$

  i.e. $\sum_{n=1}^{\infty} \lambda(A_n)$ diverges if $\sum_{n=1}^{\infty} 1/d_n^2$ does.
- An application of Lemma 2.6 gives the desired statement.

Theorem 3.1 ([Kesseböhmer and S., 2020])

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events such that $A_n \in \sigma(a_n)$ for all $n \in \mathbb{N}$. Suppose

$$V_n := \sum_{k=1}^{n} \mathfrak{m}(A_k) \cdot \mathfrak{m}(A_k^c) \to \infty.$$

Then for $S_n := \sum_{k=1}^{n} \mathbb{1}_{A_k}$ we have

$$\lim_{n\to\infty} \mathfrak{m}\left(\frac{S_n - \mathbb{E}(S_n)}{\sqrt{\mathbb{V}(S_n)}} < z\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \mathrm{d}t. \tag{3}$$

We remark here that to provide error terms as in [Philipp, 1970] and [Zuparov, 1986] or to prove that for S_n a functional CLT as in [Philipp and Webb, 1973] holds follows along the same lines as in the original papers.

Corollary 3.2

Let (b_n) and $(c_n)_{n\in\mathbb{N}}$ be arbitrarily chosen sequences of positive real numbers and $(d_n)_{n\in\mathbb{N}}$ be a sequences of positive integers. Suppose that either

- (A) $A_n := \{a_n \geq b_n\}$ *with* $\sum_{n: b_n > 1} 1/b_n = \infty$,
- (B) $A_n := \{a_n = d_n\}$ *with* $\sum_{n\in\mathbb{N}} 1/d_n^2 = \infty$,
- (C) $A_n := \left\{d_n \leq a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right)\right\}$ *with* $\sum_{n\in\mathbb{N}} 1/(c_n d_n) = \infty$ *or* $\sum_{n: d_n > 1} 1/d_n^2 = \infty$,
- (D) $A_n := \left\{d_n < a_n \leq d_n \cdot \left(1 + \frac{1}{c_n}\right)\right\}$ *with* $\sum_{n:\ c_n \leq d_n} 1/(c_n d_n) = \infty$,

then for $S_n := \sum_{k=1}^{n} \mathbb{1}_{A_k}$ *the CLT in* (3) *holds.*

We see that indeed the condition $\lim_{n\to\infty} b_n = \infty$ in (A) is not necessary. The other results correspond to our previously proven zero-one laws.

- To prove Theorem 3.1 we can make use of [Philipp, 1970, Theorem 3].
- Beside some technical conditions one assumption of this theorem is

$$\lim_{n\to\infty} \mathbb{V}\left(\sum_{k=1}^{n} \mathbb{1}_{A_k}\right) = \infty.$$

- We will prove the following lemma:

Lemma 3.3

There exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$

$$\mathbb{V}\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}\right) > \epsilon \cdot \sum_{i=1}^{n} \mathbb{V}\left(\mathbb{1}_{A_i}\right).$$

- This lemma implies $\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{V}\left(\mathbb{1}_{A_i}\right) = \infty \Rightarrow \lim_{n\to\infty} \mathbb{V}\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}\right) = \infty$ and we thus only need the condition as in the i.i.d. case.

Notice that

$$\mathbb{V}\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}\right) = \sum_{i=1}^{n}\left(\mathbb{V}\left(\mathbb{1}_{A_i}\right) + 2\sum_{j=i+1}^{n} \operatorname{Cov}\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)\right)$$
$$\geq \sum_{i=1}^{n}\left(\mathbb{V}\left(\mathbb{1}_{A_i}\right) - 2\sum_{j>i} \left|\operatorname{Cov}\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)\right|\right)$$

For $i < j$ we have that

$$\left|\operatorname{Cov}\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)\right| = \left|\mathfrak{m}\left(A_i \cap A_j\right) - \mathfrak{m}\left(A_i\right)\cdot\mathfrak{m}\left(A_j\right)\right| \leq \phi_{\mathfrak{m}}\left(j-i\right)\cdot\mathfrak{m}\left(A_i\right)$$

and on the other hand

$$\left|\operatorname{Cov}\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)\right| = \left|\mathfrak{m}\left(A_i^c \cap A_j\right) - \mathfrak{m}\left(A_i^c\right)\cdot\mathfrak{m}\left(A_j\right)\right| \leq \phi_{\mathfrak{m}}\left(j-i\right)\cdot\mathfrak{m}\left(A_i^c\right)$$

yielding

$$2\sum_{j>i}\left|\operatorname{Cov}\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)\right| \leq 2\left(\sum_{k=1}^{\infty}\phi_{\mathfrak{m}}\left(k\right)\right)\cdot\min\left\{\mathfrak{m}\left(A_i\right), \mathfrak{m}\left(A_i^c\right)\right\}$$
$$=: \kappa\cdot\min\left\{\mathfrak{m}\left(A_i\right), \mathfrak{m}\left(A_i^c\right)\right\}.$$

On the other hand we have that

$$\mathbb{V}\left(\mathbb{1}_{A_i}\right) = \mathfrak{m}\left(A_i\right) \cdot \mathfrak{m}\left(A_i^c\right) \geq \frac{\min\left\{\mathfrak{m}\left(A_i\right), \mathfrak{m}\left(A_i^c\right)\right\}}{2}.$$

Hence,

$$\begin{aligned}
\mathbb{V}\left(\sum_{i=1}^{n} \mathbb{1}_{A_i}\right) &\geq \sum_{i=1}^{n} \left(\mathbb{V}\left(\mathbb{1}_{A_i}\right) - 2\sum_{j>i} \left|\mathrm{Cov}\left(\mathbb{1}_{A_i}, \mathbb{1}_{A_j}\right)\right|\right) \\
&\geq \sum_{i=1}^{n} \left(\frac{\min\left\{\mathfrak{m}\left(A_i\right), \mathfrak{m}\left(A_i^c\right)\right\}}{2} - \kappa \cdot \min\left\{\mathfrak{m}\left(A_i\right), \mathfrak{m}\left(A_i^c\right)\right\}\right) \\
&= \left(\frac{1}{2} - \kappa\right) \cdot \sum_{i=1}^{n} \min\left\{\mathfrak{m}\left(A_i\right), \mathfrak{m}\left(A_i^c\right)\right\} \\
&\geq \left(\frac{1}{2} - \kappa\right) \cdot \sum_{i=1}^{n} \mathbb{V}\left(\mathbb{1}_{A_i}\right).
\end{aligned}$$

To prove the lemma it suffices to prove that $\kappa < 1/2$.

- Let us now calculate $\kappa = 2\sum_{j=1}^{\infty} \phi_{\mathfrak{m}}(j)$.
- We remember that $\phi_{\mathfrak{m}}(n) \leq 1/2\psi_{\mathfrak{m}}(n)$, for all $n \in \mathbb{N}$, and
- $\psi_{\mathfrak{m}}(n) \leq \rho\,\theta^{n-2}$ for $n \geq 2$, where $\rho = \pi^2 \log 2/6 - 1$ and θ is a constant less than 0.30367, and $\psi_{\mathfrak{m}}(1) = 2\log 2 - 1$.
- This is not good enough! With this estimate $\kappa > 1/2$!
- But we can show the following:

Lemma 3.4

Let $\phi_{\mathfrak{m}}$ denote the ϕ-mixing coefficient for the Gauss system. Then we have that

$$\phi_{\mathfrak{m}}(1) = \frac{1 - \log 2 + \log\log 2}{\log 2} < 0.0861.$$

- Using this estimate yields $\kappa < 1/2$.
- It turns out that $\phi_{\mathfrak{m}}(1)$ coincides with the Erdős-Ford-Tenenbaum constant.

- The proof of this lemma can be done using the natural extension of the random variables (a_n).
- Loosely speaking this is a larger dynamical system which is invertible and there exists a projection to the original system preserving some dynamical structures.
- First define $\overline{G} : (0,1) \times [0,1] \to (0,1) \times [0,1]$ by

$$\overline{G}(\omega, \theta) := \left(G(\omega), \frac{1}{a_1(\omega) + \theta} \right).$$

- It can be easily seen that

$$\overline{G}^n(\omega, \theta) = (G^n(\omega), [a_n(\omega), \ldots, a_2(\omega), a_1(\omega) + \theta]).$$

- Then we define the bi-infinite sequence $(\overline{a}_k)_{k \in \mathbb{Z}}$, where each $\overline{a}_k : (0,1) \times [0,1] \to \mathbb{N}$ is given by

$$\overline{a}_k(\omega, \theta) := \overline{a}_1\left(\overline{G}^k(\omega, \theta)\right) \text{ with } \overline{a}_1(\omega, \theta) := a_1(\omega).$$

- There exists a unique probability measure $\overline{\mathfrak{m}}$ such that $\overline{G}$ preserves $\overline{\mathfrak{m}}$ called *extended Gauss measure*.

Lemma 3.5 ([Iosifescu and Kraaikamp, 2009, Theorem 1.3.5])

For any $x \in [0,1]$ we have the conditional probability

$$\overline{\mathfrak{m}}\left([0,x] \times [0,1] \,|\, (\overline{a}_0, \overline{a}_{-1}, \ldots)\right) = \frac{(a+1)\,x}{ax+1}\ \overline{\mathfrak{m}}\text{-a.s.},$$

for the random variable $a := [\overline{a}_0, \overline{a}_{-1}, \ldots]$.

Motivated by this lemma we also define the probability measure $\mathfrak{m}_a$ on $\mathcal{B}_{[0,1]}$ via its distribution function, for $a \in [0,1]$, by

$$\mathfrak{m}_a\left([0,x]\right) := \frac{(a+1)\,x}{ax+1}.$$

Proof of the value of $\phi_{\mathfrak{m}}(1)$, i.e. Lemma 3.4 (Sketch).

- Let

$$\eta := \sup \left|\mathfrak{m}_a(B) - \mathfrak{m}(B)\right|$$

  with the supremum taken over all $a \in [0, 1]$ and $B \in \mathcal{B}_{[0,1]}$.
- The proof of the lemma is separated into two parts, namely we show that
  - (A) $\eta = (1 - \log 2 + \log\log 2) / \log 2$ and
  - (B) $\phi_{\mathfrak{m}}(1) \leq \eta$.
- The proof of (B) is inspired by the estimate of the ψ-mixing coefficient in [Iosifescu and Kraaikamp, 2009].

Proof of (A) (Sketch).

- We want to calculate the precise value of $\eta := \sup_{a,B} |\mathfrak{m}_a(B) - \mathfrak{m}(B)|$.
- We define $f : [0,1)^2 \to \mathbb{R}$ by

$$f(a,x) := \mathfrak{m}_a([0,x]) - \mathfrak{m}([0,x]) = \frac{(a+1)x}{ax+1} - \frac{\log(x+1)}{\log 2}.$$

- $f(a,\cdot)$ is the distribution function of a signed measure with density $\partial f(a,x)/\partial x$.
- For each $a \in [0,1)$ we have that $\sup_B(\mathfrak{m}_a(B) - \mathfrak{m}(B))$ will be attained for $B = \{x \colon \partial f(a,x)/\partial x > 0\}$ and $\inf_B(\mathfrak{m}_a(B) - \mathfrak{m}(B))$ will be attained for B^c.
- For given $a \in [0,1)$ calculate $\sup_B |\mathfrak{m}_a(B) - \mathfrak{m}(B)|$, then take the supremum over $a \in [0,1)$.

Proof of (B) (Sketch).

- We want to show that $\phi_{\mathfrak{m}}(1) = \sup_{a,B} |\mathfrak{m}_a(B) - \mathfrak{m}(B)|$.
- Remember the definition of the ϕ-mixing coefficient:

$$\phi(\mathcal{C},\mathcal{D}) := \sup_{C,D} |\mathbb{P}(D \mid C) - \mathbb{P}(D)|, C \in \mathcal{C}, D \in \mathcal{D}, \mathbb{P}(C) > 0.$$

- We have that $\phi_{\mathfrak{m}}(1) = \sup\left\{\left|\overline{\mathfrak{m}}\left(\overline{B}|\overline{A}\right) - \overline{\mathfrak{m}}\left(\overline{B}\right)\right|\right\}$, where the supremum is taken over $\overline{B} \in \sigma(\overline{a}_n, \overline{a}_{n+1}, \ldots)$, $\overline{A} \in \sigma(\ldots, \overline{a}_{n-2}, \overline{a}_{n-1})$ with $\overline{\mathfrak{m}}\left(\overline{A}\right) > 0$, $n \in \mathbb{N}$.
- By the shift invariance of $\overline{\mathfrak{m}}$ it is enough to take the supremum over $\overline{B} \in \sigma(\overline{a}_1, \overline{a}_2, \ldots)$ and $\overline{A} \in \sigma(\overline{a}_0, \overline{a}_{-1}, \ldots)$ for which $\overline{\mathfrak{m}}\left(\overline{A}\right) > 0$.
- Since $\overline{B} = B \times [0,1)$ and $\overline{A} = [0,1) \times A$, for some $A, B \in \mathcal{B}_{[0,1)}$, we have

$$\phi_{\mathfrak{m}}(1) = \sup\left\{\left|\frac{\overline{\mathfrak{m}}(A \times B)}{\mathfrak{m}(A)} - \mathfrak{m}(B)\right| : A, B \in \mathcal{B}_{[0,1)}, \mathfrak{m}(A) > 0\right\}.$$

Proof of (B) (Sketch) part II.

- Estimate

$$\phi_{\mathfrak{m}}(1) = \sup\left\{\left|\frac{\overline{\mathfrak{m}}(A \times B)}{\mathfrak{m}(A)} - \mathfrak{m}(B)\right| : A, B \in \mathcal{B}_{[0,1)}, \mathfrak{m}(A) > 0\right\}.$$

- We have that $\overline{\mathfrak{m}}(A \times B) = \int_A \mathfrak{m}_a(B)\, \mathrm{d}\mathfrak{m}(a)$, for $A, B \in \mathcal{B}_{[0,1)}$.
- For given $B \in \mathcal{B}_{[0,1)}$ we have that

$$\sup_{a \in [0,1)} \mathfrak{m}_a(B) \geq \sup_{A \in \mathcal{B}_{[0,1)}} \frac{\int_A \mathfrak{m}_a(B)\, \mathrm{d}\mathfrak{m}(a)}{\mathfrak{m}(A)}$$
$$\inf_{a \in [0,1)} \mathfrak{m}_a(B) \leq \inf_{A \in \mathcal{B}_{[0,1)}} \frac{\int_A \mathfrak{m}_a(B)\, \mathrm{d}\mathfrak{m}(a)}{\mathfrak{m}(A)}.$$

- Some additional calculations give equality in the above cases.
- Thus, $\phi_{\mathfrak{m}}(1) = \sup_{a,B} |\mathfrak{m}_a(B) - \mathfrak{m}(B)|$.

□

Set $$\frac{p_n}{q_n} := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_0; a_1, \ldots, a_n].$$

Definition 4.1

The approximation coefficient $\theta\left(x, p/q\right)$ of a rational number p/q with respect to an irrational number x is defined by

$$\theta\left(x, \frac{p}{q}\right) = q \cdot \left|qx - p\right|.$$

In particular we have that $\left|x - \frac{p_k}{q_k}\right| = \frac{\theta\left(x, \frac{p_k}{q_k}\right)}{q_k^2}$.

Defining $u_n := q_{n-1}^{-2} \left|x - \frac{p_{n-1}}{q_{n-1}}\right|^{-1}$ gives us $\theta\left(x, p_n/q_n\right) = u_{n+1}^{-1}$.

- Further interesting random variables might be

$$r_n := \frac{1}{G^{n-1}} = [a_n; a_{n+1}, a_{n+2}, \ldots] \text{ and}$$
$$y_n := \frac{q_n}{q_{n-1}}.$$

- We have $q_n = y_1 \cdot \ldots \cdot y_n$ and $y_n = [a_n; a_{n-1}, \ldots, a_1] = a_n + y_{n-1}$, $n \in \mathbb{N}$.
- Recall also the well-known estimate

$$\frac{1}{q_{n-1}\left(q_n + q_{n-1}\right)} < \left| x - \frac{p_{n-1}}{q_{n-1}} \right| < \frac{1}{q_{n-1} q_n}.$$

- The differences between the above defined variables and a_n are bounded as follows

$$a_n \le r_n < a_n + 1$$
$$a_n \le y_n < a_n + 1$$
$$a_n < u_n < a_n + 2.$$

Our aim is to provide analogous results to the famous Khinchine theorem.

Theorem 4.2 (Khinchine's Theorem)

Let $k : \mathbb{N} \to (0, \infty)$ *be non-increasing. Then*

$$\left| x - \frac{p}{q} \right| \leq \frac{k(q)}{q}$$

for $p \in \mathbb{N}$ *choosen appropriately for each* $q \in \mathbb{N}$ *holds infinitely often with Lebesgue measure* 0 *or* 1*, according as*

$$\sum_{n=1}^{\infty} k(n)$$

is finite or not.

Lemma 4.3 ([Kesseböhmer and S., 2020])

Let $(c_n)_{n\in\mathbb{N}}$ *be a sequence of real numbers and* $(d_n)_{n\in\mathbb{N}}$ *be a sequence of natural numbers both tending to infinity such that* $\sum_{n=1}^{\infty} 1/(c_n d_n) = \infty$. *Then for the random variables,* r_n, y_n, *and* u_n, *associated to the CF digits we have that*

$$d_n < r_n, y_n \leq d_n \left(1 + 1/c_n\right) + 1$$

and

$$d_n < u_n \leq d_n \left(1 + 1/c_n\right) + 2$$

hold for infinitely many $n \in \mathbb{N}$ *Lebesgue almost everywhere.*

Further questions:

- Is it possible to state the 0-1 laws for (u_n) in the same precise way as for the CF digits (a_n)?
- Does the CLT holds in the same way for (u_n) as for (a_n)?
- Can one generalize the results for higher dimensional continued fractions in the same way as for example the Borel-Bernstein theorem, [Nogueira, 2001]?

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