Rauzy's One World

Tuesday, September 13, 2022

An upper bound on the box-counting dimension of the Rauzy gasket

Joint work with Mark Pollicott

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One World Numeration Seminar

Contents

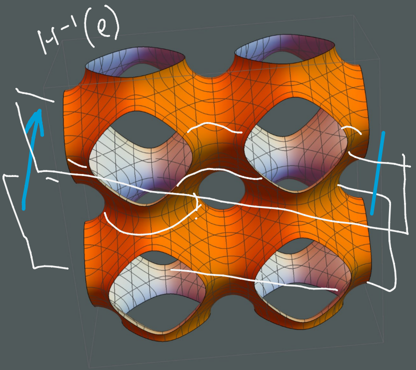
- Motivation
- Definition
- What is known about its Hausdorff dimension
- How we dominate it

What does the Rauzy gasket relate to?

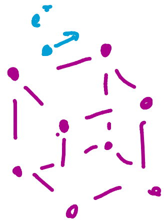
- ▷ Arnoux-Rauzy IETs
- Episturmian words on 3 symbols
- Pseudo-rotations of the circle
- ▷ Chaotic directions in a PL model of Novikov's model
- Triangular tiling billiards

Novikov's model
of electromagnetic induction

When free electrons in a metal lattice are compelled to move orthogonally to a magnetic field, how do they behave?



- $H : \mathbb{R}^3 \to \mathbb{R}$ smooth 3-ply periodic
- $e \in \mathrm{Im}(H)$
- $\Omega \in \mathbb{R}P^2$

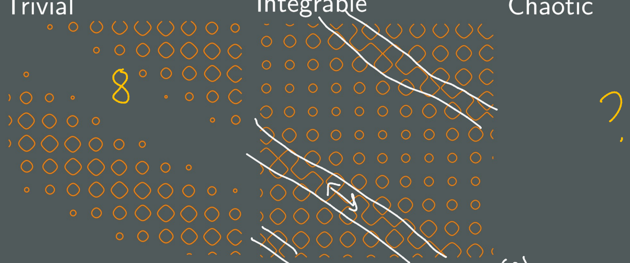


Dynnikov's theorem
Three behaviours, two typical

Fix e, Ω

Q: which Ω admit chaotic behav.?

Trivial Integrable Chaotic



$v \doteq \mathrm{soul}(\Omega) \in \mathbb{R}P^2$

tr int. tr

int/chaotic

int/chaotic

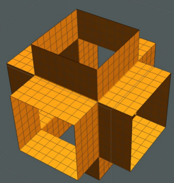
$\in \mathbb{R}P^2$

Novikov's conjecture

Novikov's conjecture is that chaotic behaviour generically never happens:

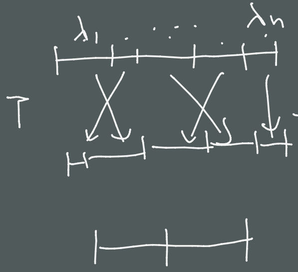
- For a generic smooth H, the set of $\Omega \in \mathbb{R}P^2$ which show chaotic behaviour has Lebesgue measure zero.
- Moreover, it has Hausdorff dimension strictly between 1 and 2.

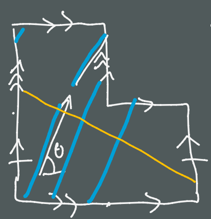
Dynnikov and De Leo investigated this for a piecewise linear H:



What was the chaotic set in this case? R

Arnoux-Rauzy I.E.Ts.





- Thm (Keane)

If

① $\{\lambda_k\}_{k=1}^{\infty}$ are lin. independent over $\mathbb{Q}$

② ρ is irreducible

Then T is minimal (and λ-generically uniquely ergodic)

Q: which $\lambda \in \Delta$ have T_λ minimal?

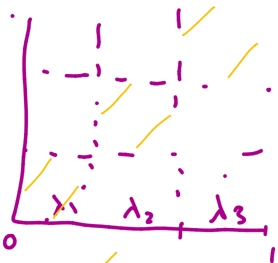
R

$(\lambda_1, \lambda_2, \lambda_3) = \lambda \in \Delta$

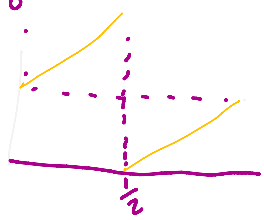
T_λ

$T_\lambda = ② \circ ①.$

①:



②:

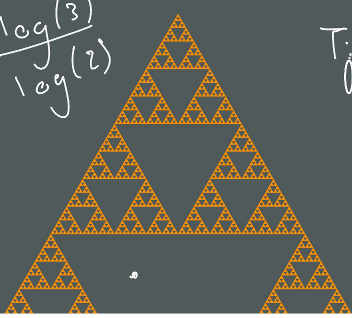


The Sierpiński gasket

A hyperbolic, self-similar attractor

$$\dim_{H, B}(\cdot) = \frac{\log(3)}{\log(2)}$$

$$T_j : x \mapsto \frac{x + e_j}{2}$$



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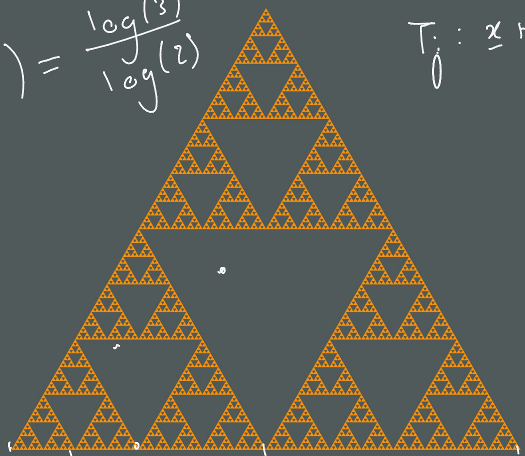


0 … 1/8 1/4 1/2 1

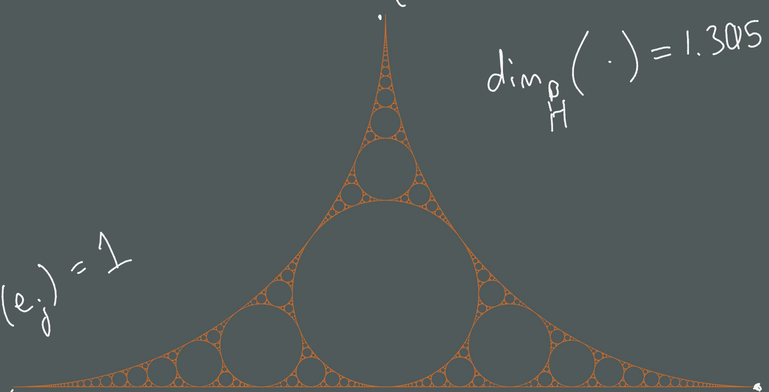
The Apollonian gasket

A parabolic, conformal attractor

i

$\dim_{B,H}(\cdot) = 1.305\ldots$

$T_j'(e_j) = 1$



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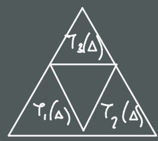
The Rauzy gasket $\mathfrak{R}$

A parabolic, self-projective fractal

$\emptyset \neq R \subset \Delta$,

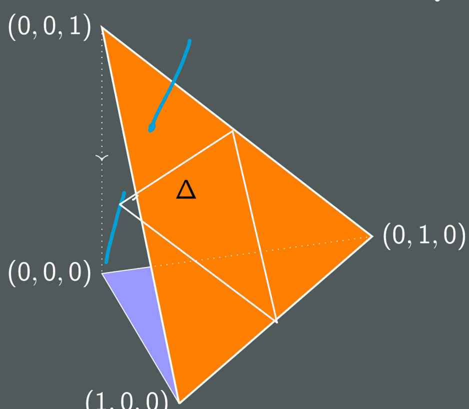
$R = T_1(R) \cup T_2(R) \cup T_3(R)$



Where $\mathfrak{R}$ lives

The standard two-simplex, Δ

Let $\Delta := \{(x,y,z) \mid x,y,z \geq 0,\ x+y+z=1\}$



The maps which preserve $\mathfrak{R}$

Letting

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

define $T_j : \Delta \to \Delta$ by

$$T_j(x) = \frac{M_j \cdot \underline{x}}{\|M_j \cdot \underline{x}\|_1},$$

E.g.,

$$T_1(x_1, x_2, x_3) = \frac{(1, x_2, x_3)}{2 - x_1}$$

We define $\mathfrak{R} \subset \Delta$ as the *attractor* of the T_j.

$$M_1 \cdot (x, y, z) = (x+y+z, y, z) = (1, y, z)$$

The dimension of $\mathfrak{R}$

Most rigorous results concern its Hausdorff dimension:

1. Avila–Hubert–Skripchenko: $\dim_H(\mathfrak{R}) < 2$
2. Fougeron: $\dim_H(\mathfrak{R}) \leq 1.825$
3. Gutiérrez-Romo–Matheus: $\dim_H(\mathfrak{R}) \geq 1.19$

Numerics of De Leo–Dynnikov give $\dim_B(\mathfrak{R}) \approx 1.72$

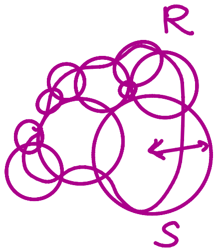
Theorem (Pollicott–S.)

$$\overline{\dim}_B(\mathfrak{R}) \leq 1.7404$$

Bounding the dimensions above

To show $\dim_H(\mathfrak{R}) \leq 1 + \delta$, it suffices to give a sequence of covers $(\mathcal{C}_n)_n$ of $\mathfrak{R}$ such that

$$\sum_{S \in \mathcal{C}_n} \mathrm{diam}(S)^{1+\delta} \to 0 \qquad (n \to \infty).$$



To improve this to $\overline{\dim}_B(\mathfrak{R}) \leq 1 + \delta$, we require also $\exists C > 0 : \forall n$

$$\frac{1}{C} \leq \frac{\max_{S \in \mathcal{C}_n}(\mathrm{diam}(S))}{\min_{S \in \mathcal{C}_n}(\mathrm{diam}(S))} \lesssim C$$

Towards a covering lemma

We write $|i| = n$ for $i \in \{1, 2, 3\}^n$, and

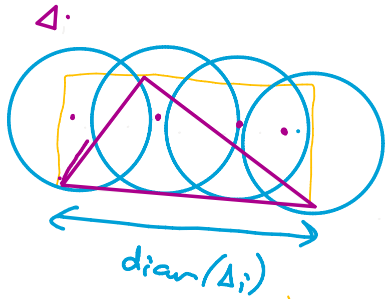
- $M_i = M_{i_1} M_{i_2} \cdots M_{i_n}$
- $T_i = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}$
- $\Delta_i = T_i(\Delta)$

From the definition of attractor,

$$R = \bigcup_{|i|=n} T_i(R) \subset \bigcup_{|i|=n} \Delta_i$$

and covering each of these level-n triangles Δ_i "efficiently" gives the nth cover $\mathcal{C}_n$ in the sequence we use to bound $\dim_H(\mathfrak{R})$.

(More care is needed re $\overline{\dim}_B(\mathfrak{R})$.)



$$\mathrm{diam} \lesssim \frac{\mathrm{area}(\Delta_i)}{\mathrm{diam}(\Delta_i)}$$

$$\# \lesssim \frac{\mathrm{diam}(\Delta_i)^2}{\mathrm{area}(\Delta_i)}$$

Lemma

If $\delta \in (0,1)$ and

$$\tilde{X}_n := \sum_{|i|=n} \text{area}(\Delta_i)^\delta \, \text{diam}(\Delta_i)^{1-\delta} \to 0$$

as $n \to \infty$, then $\dim_H(\mathfrak{R}) \leq 1 + \delta$. Furthermore, if

$$\sum_n \tilde{X}_n < \infty$$

then $\overline{\dim}_B(\mathfrak{R}) \leq 1 + \delta$.

For now, focus on areas

$X_n = \mathcal{L}_\delta^n(\mathbb{1})$

To illustrate the main ideas, consider just

$$X_n = \sum_{|i|=n} \text{area}(\Delta_i)^\delta.$$

There is a lot of structure hidden here. For example,

Lemma

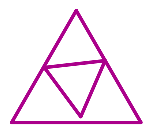
$$\frac{\text{area}(\Delta_i)}{\text{area}(\Delta)} = \nu(M_i)^{-1},$$

for any i, where $\nu : \mathbb{R}^{3,3} \to \mathbb{R}$ is given by

$$\nu \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)$$

$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & \\ & & 1 \end{pmatrix}$

$\nu(M_1) = 4$



A decomposition of $X_n = \sum_{k=1}^{n-1} X_{n,k}$ (+ rem)

For $n > k \geq 1$, let

$$A_{n,k} := \{|i| = n : i_1 = \cdots = i_k \neq i_{k+1}\}$$

and let

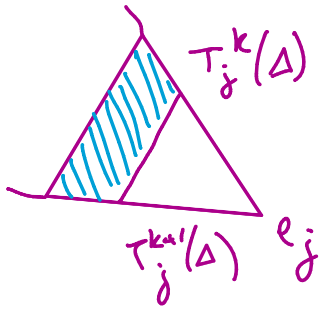
$$X_{n,k} := \sum_{i \in A_{n,k}} \text{area}(\Delta_i)^{\delta}.$$

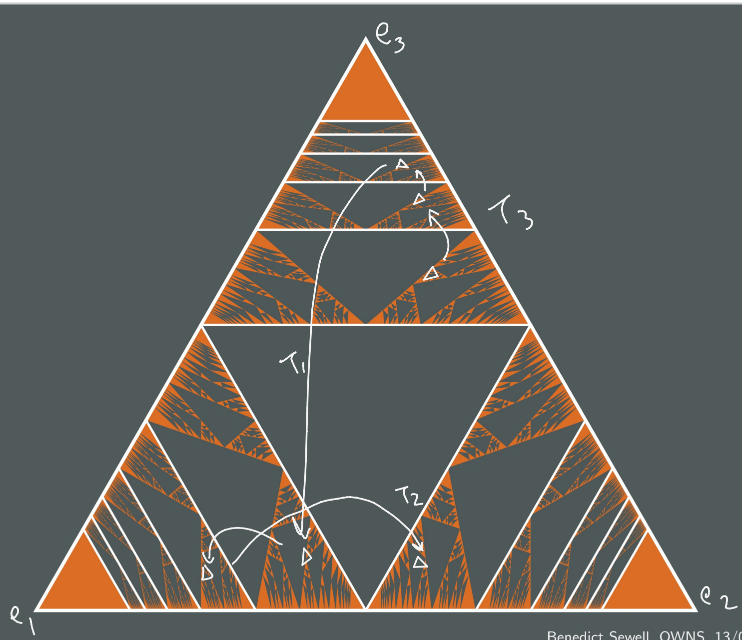
We can also define $R_k \subset \Delta$:

Lemma

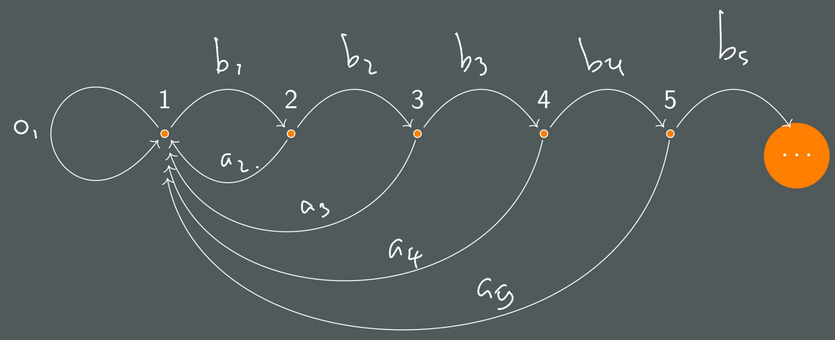
For any $|i| = n$,

$$i \in A_{n,k} \iff \Delta_i \subset R_k \quad := \bigcup_{j=1}^{3} cl\left(T_j^k(\Delta) \setminus T_j^{k+1}(\Delta)\right)$$





House of Cards Markov structure (for A_k, $X_{n,k}$, and/or R_k)



Sub-renewal inequality for $X_{n,1}$

Lemma

There exist sequences (a_k) (b_k) and (r_n) s.t. for every $n > k \geq 1$,

$$X_{n+1,k+1} \leq b_k X_{n,k}$$

and

$$X_{n+1,1} \leq \sum_{k=1}^{n-1} a_k X_{n,k} + r_n$$

i.e.,

$$X_{n+1,1} \leq \sum_{k=1}^{n-1} \left(a_k \prod_{j=1}^{n-1} b_j \right) X_{n+1-k,1} + r_n$$

$$b_k = \max_{R_k \cap \Delta_1} \left(\mathrm{Jac}(T_1)^{\delta} \right)$$
$$= \max_{x \in R_k \cap \Delta_1} \left(2 - x_1 \right)^{-3\delta}$$
$$= \left(\frac{k+2}{k+3} \right)^{3\delta}$$

$$a_k = \cdots = 2^{-3\delta} + \left(\frac{k+1}{2k+1} \right)^{3\delta}$$

Write this as

$$X_{n+1,1} \leq \sum_{k=1}^{n-1} \lambda_k X_{n+1-k,1} + r_n.$$

This is a sub-renewal equation, so we have a simple criterion for the summability $X_{n,1}$ (hence X_n, using $X_n \leq C X_{n+2,1}$).

Theorem (Renewal Theorem, after Feller)

If $\sum_n r_n < \infty$ and $\sum_{k=1}^{\infty} \lambda_k < 1$ then

$$\sum_n X_n < \infty. \quad \Rightarrow \overline{\dim}_B(R) \leq 1+\delta$$

Best upper bound from this: $\overline{\dim}_B(\mathfrak{R}) \leq 1.893\ldots$.

How to improve upon this?

To get more "competitive" upper bounds we

- re-introduce the diameter factors, and
- give more refined decompositions/partitions for $A_n/X_n/\Delta$.

These two, up to computing limitations, give us our main result:

Theorem

$$\overline{\dim}_B(\mathfrak{R}) \leq 1.7404.$$

Limitations and questions

The method presented here is simple and general, but we implicitly relied upon implicit symmetry and simplicity, since the upper bounds obtained are very sensitive to the values of a_k and b_k.

Some starter questions for future development:

1. Can $\tilde{X}_n$ be expressed via iterations of a transfer operator? (e.g., acting on 1-forms)
2. Is there an analogous method for lower bounds on $\dim_H(\mathfrak{R})$?
3. Can we obtain statistical results on the geometry of the Δ_i? (e.g., a limiting distribution for area(Δ_i) for $|i| = n \to \infty$)
4. What, if any, is the connection with eigenvalues/singular values of the M_i?

Thank you very much!