An upper bound on the box-counting dimension of the Rauzy gasket

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Novikov’s model of electromagnetic induction

When free electrons in a metal lattice are compelled to move orthogonally to a magnetic field, how do they behave?

\[ H : \mathbb{R}^3 \rightarrow \mathbb{R} \text{ smooth } 3-\text{ply} \]
\[ e \in \text{Im}(H) \]
\[ \Omega \in \mathbb{R}P^2 \]

Dynnikov’s theorem

Three behaviours, two typical

Fix \( e, \Omega \)

Trivial

Integrable

Chaotic

Q: Which \( \Omega \) admik chaotic behaviour?
Novikov’s conjecture

Novikov’s conjecture is that chaotic behaviour generically never happens:
- For a generic smooth $H$, the set of $\Omega \in \mathbb{R}P^2$ which show chaotic behaviour has Lebesgue measure zero.
- Moreover, it has Hausdorff dimension strictly between 1 and 2.

Dynnikov and De Leo investigated this for a piecewise linear $H$:

What was the chaotic set in this case? $\mathbb{R}$

The Sierpiński gasket

A hyperbolic, self-similar attractor

$$\dim_H(\cdot) = \frac{\log(3)}{\log(2)}$$

$$T_0: z \mapsto \frac{z + e_i}{2}$$
The Sierpiński gasket
A hyperbolic, self-similar attractor

\[ \dim_H(\cdot) = \frac{\log(3)}{\log(2)} \]

\[ T_0: z \rightarrow \frac{z + \sqrt{3}i}{2} \]

The Apollonian gasket
A parabolic, conformal attractor

\[ \dim_H(\cdot) = 1.305 \ldots \]

\[ T_0 z \rightarrow 1 \]
The Rauzy gasket $\mathcal{R}$
A parabolic, self-projective fractal

$\emptyset \neq \mathcal{R} \subseteq \Delta$

$\mathcal{R} = T_1(R) \cup T_2(R) \cup T_3(R)$

Where $\mathcal{R}$ lives
The standard two-simplex, $\Delta$

Let $\Delta := \{ (x,y,z) \mid x, y, z \geq 0, \ x+y+z = 1 \}$

(0, 0, 0) (0, 0, 1) (1, 0, 0) (0, 1, 0)
The maps which preserve $\mathfrak{R}$

Letting

\[
M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},
\]

define $T_j : \Delta \to \Delta$ by

\[
T_j(x) = \frac{M_j \cdot x}{\|M_j \cdot x\|_1}.
\]

E.g.,

\[
T_1(x_1, x_2, x_3) = \left( 1, \frac{x_1 + x_2}{2} - x_3 \right).
\]

We define $\mathfrak{R} \subset \Delta$ as the attractor of the $T_j$.

The dimension of $\mathfrak{R}$

Most rigorous results concern its Hausdorff dimension:

1. Avila–Hubert–Skripchenko: $\dim_H(\mathfrak{R}) < \frac{2}{1.825}$
2. Fougéron: $\dim_H(\mathfrak{R}) \leq 1.825$
3. Gutiérrez-Romo-Mateus: $\dim_H(\mathfrak{R}) \geq 1.78$

Numerics of De Leo–Dynnikov give $\dim_B(\mathfrak{R}) \approx 1.72$.

**Theorem (Pollicott–S.)**

$\overline{\dim}_B(\mathfrak{R}) \leq 1.7804$. 
Bounding the dimensions above

To show $\dim_H(\mathcal{R}) \leq 1 + \delta$, it suffices to give a sequence of covers $(C_n)_n$ of $\mathcal{R}$ such that

$$\sum_{S \in C_n} \text{diam}(S)^{1+\delta} \to 0 \quad (n \to \infty).$$

To improve this to $\overline{\dim}_B(\mathcal{R}) \leq 1 + \delta$, we require also $\exists C > 0 : \forall n$

$$\frac{1}{C} \leq \frac{\max_{S \in C_n}(\text{diam}(S))}{\min_{S \in C_n}(\text{diam}(S))} \leq C.$$

Towards a covering lemma

We write $|i| = n$ for $i \in \{1, 2, 3\}^n$, and
- $M_i = M_{i_1}M_{i_2} \cdots M_{i_n}$,
- $T_i = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}$,
- $\Delta_i = T_i(\Delta)$.

From the definition of attractor,

$$\mathcal{R} = \bigcup_{|i| = n} T_i(\mathcal{R}) \subset \bigcup_{|i| = n} \Delta_i,$$

and covering each of these level-$n$ triangles $\Delta_i$ "efficiently" gives the $n$th cover $C_n$ in the sequence we use to bound $\dim_H(\mathcal{R})$.

(More care is needed re $\overline{\dim}_B(\mathcal{R})$.)
Lemma

If $\delta \in (0, 1)$ and

$$\check{X}_n := \sum_{i=1}^{\infty} \frac{\text{area}(\Delta_i)^{\delta}}{\text{dim}(\Delta_i)} \rightarrow 0$$

as $n \rightarrow \infty$, then $\dim_H(\Omega) \leq 1 + \delta$. Furthermore, if

$$\sum_{i=1}^{\infty} \frac{\text{area}(\Delta_i)}{\text{dim}(\Delta_i)} < \infty$$

then $\dim_B(\Omega) \leq 1 + \delta$.

For now, focus on areas

$$X_n = \mathcal{L}^n(\Omega)$$

To illustrate the main ideas, consider just

$$X_n = \sum_{|i|=n} \text{area}(\Delta_i)^{\delta}.$$

There is a lot of structure hidden here. For example,

**Lemma**

$$\frac{\text{area}(\Delta_i)}{\text{area}(\Delta)} = \nu(M_i)^{-1},$$

for any $i$, where $\nu : \mathbb{R}^{3,3} \rightarrow \mathbb{R}$ is given by

$$\nu \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)(c_1 + c_2 + c_3)$$

$$M_i = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\nu(M_i) = 4$$
A decomposition of $X_n = \sum_{k=1}^{n-i} X_{n,k}$

For $n > k \geq 1$, let

$$A_{n,k} := \{i| = n: i_1 \neq \ldots \neq i_k \neq i_{k+1} \}$$

and let

$$X_{n,k} := \sum_{i \in A_{n,k}} \text{area}(\Delta_i)^5$$

We can also define $R_k \subset \Delta$:

**Lemma**

For any $|i| = n$,

$$i \in A_{n,k} \iff \Delta_i \subset R_k := \left( T_{\frac{k}{\delta}}(\Delta) \setminus T_{\frac{k}{\delta}}(\Delta) \right)$$
House of Cards Markov structure (for $A_k$, $X_{n,k}$, and/or $R_k$)

Sub-renewal inequality for $X_{n,1}$

**Lemma**
There exist sequences $(a_k)$ $(b_k)$ and $(r_n)$ s.t. for every $n > k \geq 1$,

$$X_{n+1,k+1} \leq b_k \cdot X_{n,k}$$

and

$$X_{n+1,1} \leq \sum_{k=1}^{n-1} \frac{a_k}{b_k} \cdot X_{n+1-k,1} + r_n$$

i.e.,

$$X_{n+1,1} \leq \sum_{k=1}^{n-1} \frac{h_k}{b_k} \cdot X_{n-k,1} + r_n$$

$$b_n = \max_{R_k} \left( \frac{2 - x}{k+3} \right)$$

$$a_n = \cdots = 2^{-3n} + \frac{(k+1)/3}{(2k+1)}$$
Write this as
\[ X_{n+1,1} \leq \sum_{k=1}^{n-1} \lambda_k X_{n+1-k,1} + r_n. \]

This is a sub-renewal equation, so we have a simple criterion for
the summability \( X_{n,1} \) (hence \( X_n \), using \( X_n \leq CX_{n+2,1} \)).

**Theorem (Renewal Theorem, after Feller)**

If \( \sum_{k=1}^{n} \lambda_k < \infty \) and \( \sum_{k=1}^{n} X_{k,1} < \infty \), then

\[ \sum_{n} X_n < \infty. \quad \Rightarrow \quad \overline{\text{dim}}_{\text{B}}(\mathcal{R}) \leq 1.893 \ldots \]

Best upper bound from this: \( \overline{\text{dim}}_{\text{B}}(\mathcal{R}) \leq 1.893 \ldots \)

How to improve upon this?

To get more "competitive" upper bounds we
- re-introduce the diameter factors, and
- give more refined decompositions/partitions for \( A_n/X_n/\Delta \).

These two, up to computing limitations, give us our main result:

**Theorem**

\[ \overline{\text{dim}}_{\text{B}}(\mathcal{R}) \leq 1.7404. \]
Limitations and questions

The method presented here is simple and general, but we implicitly relied upon implicit symmetry and simplicity, since the upper bounds obtained are very sensitive to the values of \( a_k \) and \( b_k \).

Some starter questions for future development:

1. Can \( \tilde{X}_k \) be expressed via iterations of a transfer operator? (e.g., acting on 1-forms)
2. Is there an analogous method for lower bounds on \( \dim_H(\mathfrak{M}) \)?
3. Can we obtain statistical results on the geometry of the \( \Delta_i \)? (e.g., a limiting distribution for \( \text{area}(\Delta_i) \) for \( |i| = n \to \infty \))
4. What, if any, is the connection with eigenvalues/singular values of the \( M_i \)?

Thank you very much!