Probabilistic Effectivity in the Subspace Theorem and the Distribution of Algebraic Projective Points

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Roth's Theorem

Theorem (Roth 1955)

Let α be a real algebraic number of degree at least 2. There exist only finitely many coprime pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ satisfying

$$\left|\alpha - \frac{p}{q}\right| < \psi(q)$$

whenever $\psi(q) = q^{-2-\varepsilon}$ for some $\varepsilon > 0$.

This theorem and its generalisations implies results about the structure of the decimal expansion of a real algebraic number.

Example

if $\alpha = 0.a_1 a_2 \ldots a_n \ldots$ is the decimal expansion of the algebraic number α, define for any $n \geq 1$ the quantity $\ell(n)$ as the smallest index $\ell \geq 1$ such that $a_{n+\ell} \neq 0$. Then, $\ell(n)/n \to 0$ as n tends to infinity.

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Example (Adamczewski and Bugeaud, 2007)

if $\alpha = 0.a_1 a_2 \ldots a_n \ldots$ is the decimal expansion of the algebraic number α, define for any $n \geq 1$ the quantity $p(n)$ as the number of distinct blocks of length n in the decimal expansion of α. Then, $p(n)/n \to \infty$ as n tends to infinity.

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whenever $\psi(q) = q^{-2-\varepsilon}$ for some $\varepsilon > 0$.

This theorem is ineffective; we have no idea how large the height of p/q may be while satisfying this inequality. This motivates us to analyse this problem through a *probabilistic* lens.

Question

What "proportion" of the algebraic numbers admit "large" height solutions to the above?

Roth's Theorem from the Probabilistic Perspective

Let $H, Q_1, Q_2 \in \mathbb{N}$ with $Q_1 < Q_2$, and let $\psi : \mathbb{N} \to \mathbb{R}_{>0}$.

$$J_\psi(Q_1, Q_2) := \bigcup_{Q_1 \le q < Q_2} \bigcup_{0 \le p \le q} B\left(\frac{p}{q}, \psi(q)\right)$$

$$\mathbb{A}_d(H) := \{\alpha \in \mathbb{R} : \alpha \text{ algebraic }, \ \deg(\alpha) = d, \ \text{height}(\alpha) \le H\}$$

$$\mathbb{A}'_d(H) := \mathbb{A}_d(H) \cap [0, 1]$$

Theorem (Adiceam, Shirandami)

$$\frac{\#(J_\psi(Q_1, Q_2) \cap \mathbb{A}'_d(H))}{\#\mathbb{A}_d(H)} \ll_d \left(1 + \cdot \frac{(\log H)^{\delta_{d,2}}}{H \min_{Q_1 \le q < Q_2} \psi(q)}\right) |J_\psi(Q_1, Q_2)|.$$

The implied constant is explicitly given.

Roth's Theorem from the Probabilistic Perspective

Theorem (Adiceam, Shirandami)

$$\limsup_{Q_1\to\infty}\ \limsup_{Q_2\to\infty}\ \limsup_{H\to\infty} \frac{\#(J_\psi(Q_1,Q_2)\cap\mathbb{A}'_d(H))}{\#\mathbb{A}_d(H)} = \begin{cases} 0, & \sum_{q\geq 1} q\psi(q) < \infty \\ 1, & \sum_{q\geq 1} q\psi(q) = \infty \ \&\ \psi \ \textit{monotonic} \end{cases}$$

The goal is to derive similar results for the Subspace Theorem, which broadly generalises Roth's Theorem. To do this it shall be necessary to analyse the distribution of algebraic projective points.

Distribution of Real Algebraic Numbers

The limiting distribution of the real algebraic numbers was derived by Koleda.

Koleda 2017

There is an explicit continuous function $\Phi_d : \mathbb{R} \to \mathbb{R}_{>0}$ such that for all intervals $I \subseteq \mathbb{R}$,

$$\left| \frac{\#(\mathbb{A}_d(H) \cap I)}{H^{d+1}} - \int_I \mathrm{d}x \, \Phi_d(x) \right| \ll_d \frac{(\log H)^{\delta_{d,2}}}{H}.$$

The error term is independent of the chosen interval I which may be infinite in either direction.

For our purposes, we shall need a broad generalisation of this result, which is of independent interest.

Distribution of Algebraic Vectors

For $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n_{\geq 2}$ and $\boldsymbol{H} = (H_1, \ldots, H_n) \in \mathbb{R}^n_{\geq 0}$, let

$$\mathbb{A}_{\boldsymbol{d}}(\boldsymbol{H}) := \mathbb{A}_{d_1}(H_1) \times \cdots \times \mathbb{A}_{d_n}(H_n).$$

Semi-algebraic family

A map $Z : \Omega \to \mathcal{P}(\mathbb{R}^n)$ such that the sets Ω and

$$\tilde{Z} := \left\{(\boldsymbol{t}, \boldsymbol{x}) : \boldsymbol{t} \in \Omega, \boldsymbol{x} \in Z(\boldsymbol{t})\right\}$$

are semi-algebraic.

Our generalisation of Koleda's Theorem is concerned with counting algebraic vectors falling within the fibres of a semi-algebraic family.

Distribution of Algebraic Vectors

Theorem (Adiceam, Shirandami)

Let $Z : \Omega \to \mathcal{P}(\mathbb{R}^n)$ be a semi-algebraic family. Then, uniformly in $\boldsymbol{t} \in \Omega$,

$$\left| \frac{\#(\mathbb{A}_{\boldsymbol{d}}(\boldsymbol{H}) \cap Z(\boldsymbol{t}))}{H_1^{d_1+1} \dots H_n^{d_n+1}} - \int_{Z(\boldsymbol{t})} dx_1 \dots dx_n \, \Phi_{d_1}(x_1) \dots \Phi_{d_n}(x_n) \right|$$

$$\ll_d \max_{1 \le i \le n} \frac{(\log H_i)^{\delta_{d_i,2}}}{H_i}.$$

The error term does not depend on the fibre parameter $\boldsymbol{t} \in \Omega$.

The Subspace Theorem

We reformulate Schmidt's Subspace Theorem in a more geometrical language.

- Given $L_1, L_2 \in \mathbb{P}_{n-1}(\mathbb{R})$, set $\delta(L_1, L_2^{\perp}) := \sin\theta$ where $\theta \in [0, \pi/2]$ is the angle between L_1 and $L_2^{\perp}$.
- Given $L_1, \dot{;} L_n \in \mathbb{P}_{n-1}(\mathbb{R})$, set

$$\xi(L_1, \dots, L_n) := \frac{||\boldsymbol{\alpha}_1 \wedge \cdots \wedge \boldsymbol{\alpha}_n||_2}{||\boldsymbol{\alpha}||_2 \dots ||\boldsymbol{\alpha}_n||_2},$$

  where each $\boldsymbol{\alpha}_i$ is a representative of L_i $(1 \leq i \leq n)$.
- Given $L \in \mathbb{P}_{n-1}(\mathbb{Q})$ let $H(L)$ denote its height.

The Subspace Theorem

Schmidt 1972

Let $L_1, \ldots, L_n \in \mathbb{P}_{n-1}(\mathbb{A})$ and set $\psi(L) = H(L)^{-\varepsilon}$ for some $\varepsilon > 0$. The set of $L \in \mathbb{P}_{n-1}(\mathbb{Q})$ satisfying

$$\prod_{i=1}^{n} \delta(L_i, L^{\perp}) \leq \frac{\xi(L_1, \ldots, L_n)}{H(L)^n} \cdot \psi(L) \qquad (\star)$$

lies in a finite union of proper rational subspaces of $\mathbb{Q}^n$.

We now search for an analogue of our probabilistic result inspired by Roth's Theorem.

To do so we must decide upon a notion of "height" for the tuples $(L_1, \ldots, L_n) \in \mathbb{P}_{n-1}(\mathbb{A})^n$. We proceed via *parametrisation*.

Parameters for Projective Space

Let

$$\boldsymbol{e}_0 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}_{n-1} = (0, \ldots, 0, 1)$$

be the standard basis of $\mathbb{R}^n$. Set

$$\mathbb{P}^+_{n-1}(\mathbb{R}) := \left\{ L \in \mathbb{P}_{n-1}(\mathbb{R}) : L \neq \boldsymbol{e}_0 \right\}.$$

This is homeomorphic to the open Euclidean ball $\mathbb{B}_{n-1} \subset \mathbb{R}^{n-1}$. This shown explicitly via the stereographic projection.

$$\begin{aligned} \varphi : \mathbb{P}^+_{n-1}(\mathbb{R}) &\to B_{n-1} \\ L &\mapsto \big(\varphi_1(L), \ldots, \varphi_{n-1}(L)\big). \end{aligned}$$

Note that algebraic points are mapped to algebraic vectors via φ

Inducing a Height

Given $\boldsymbol{d} = (d_1, \ldots, d_{n-1}) \in \mathbb{N}_{\geq 2}^{n-1}$, $\boldsymbol{H} = (H_1, \ldots, H_{n-1}) \in \mathbb{R}_{\geq 0}^{n-1}$, let

$$P_{\boldsymbol{d}}(\varphi, \boldsymbol{H}) := \Bigg\{ \varphi(L) : L \in \mathbb{P}_{n-1}^{+}(\mathbb{A}),\ \deg(\varphi(L)) = \boldsymbol{d}, \\ \operatorname{height}(\varphi(L)) \in \prod_{j=0}^{n-1} [0, H_j] \Bigg\},$$

where for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{A}^{n-1}$

$$\deg(\boldsymbol{\alpha}) := (\deg(\alpha_1), \ldots \deg(\alpha_{n-1})),$$
$$\operatorname{height}(\boldsymbol{\alpha}) := (\operatorname{height}(\alpha_1), \ldots \operatorname{height}(\alpha_{n-1})).$$

This determines our set of "points of bounded height".

The Subspace Theorem from the Probabilistic Perspective

We want to count how many $(L_1, \ldots, L_n) \in \mathbb{P}_{n-1}(\mathbb{A})^n$ admit solutions to $(\star)$ that lie in subspaces of $\mathbb{Q}^n$ of large height. Let $G_k(\mathbb{Q}^n)$ denote the set of k-dimensional subspaces of $\mathbb{Q}^n$, and let $\psi : \mathbb{P}_{n-1}(\mathbb{Q}) \to \mathbb{R}_{>0}$.

ψ-Solution Subspace

Call $\Pi \in G_k(\mathbb{Q}^n)$ a ψ-solution subspace with respect to a given $(L_1, \ldots, L_n) \in \mathbb{P}_{n-1}(\mathbb{R})^n$ if:

- There exists infinitely many $L \in \mathbb{P}_{n-1}(\mathbb{Q}) \cap \Pi$ satisfying $(\star)$
- there exists k linearly independent $L \in \mathbb{P}_{n-1}(\mathbb{Q}) \cap \Pi$ satisfying $(\star)$.

Let

$$J_\psi(k; Q_1, Q_2)$$

denote the set of all $(L_1, \ldots, L_n) \in \mathbb{P}_{n-1}(\mathbb{R})$ for which there exists a ψ-solution subspace $\Pi \in G_k(\mathbb{Q}^n)$ such that $H(\Pi) \geq Q_1$ and such that there exists k linearly independent $L \in \mathbb{P}_{n-1}(\mathbb{Q}) \cap \Pi$ satisfying $(\star)$ and obeying $H(L) \leq Q_2$.

The Subspace Theorem from the Probabilistic Perspective

Theorem (Adiceam, Shirandami)

Suppose $\psi : \mathbb{P}_{n-1}(\mathbb{Q}) \to \mathbb{R}_{>0}$ *is of the form* $\psi(L) = \tilde{\psi}(H(L))$ *and such that*

$$\sum_{Q \geq 1} \frac{\tilde{\psi}(Q)}{Q} \cdot (\log Q)^{n-1} < \infty.$$

Then,

$$\frac{\#\left(P_{\boldsymbol{d}}(\varphi, \boldsymbol{H})^n \cap J_{\psi}(k; Q_1, Q_1)\right)}{\left(H_1^{d_1+1} \dots H_{d_{n-1}}^{d_{n-1}+1}\right)^n} \ll_{\boldsymbol{d},k} \sum_{Q \geq Q_1^{1/k}} \frac{\tilde{\psi}(Q)}{Q} \cdot (\log Q)^{n-1} + Q_2^n \cdot \max_{1 \leq i \leq n} \frac{(\log H_i)^{\delta_{d_i,2}}}{H_i}.$$

The main building block for the proof of this theorem is the earlier stated...

The Subspace Theorem from the Probabilistic Perspective

Theorem (Adiceam, Shirandami)

Let $Z : \Omega \to \mathcal{P}(\mathbb{R}^n)$ be a semi-algebraic family. Then, uniformly in $\boldsymbol{t} \in \Omega$,

$$\left| \frac{\#(\mathbb{A}_{\boldsymbol{d}}(\boldsymbol{H}) \cap Z(\boldsymbol{t}))}{H_1^{d_1+1} \dots H_n^{d_n+1}} - \int_{Z(\boldsymbol{t})} dx_1 \dots dx_n \, \Phi_{d_1}(x_1) \dots \Phi_{d_n}(x_n) \right|$$

$$\ll_d \max_{1 \le i \le n} \frac{(\log H_i)^{\delta_{d_i,2}}}{H_i}.$$

We pass on to the proof of this theorem.

Proofs

Fix $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}_{\geq 2}^n$.
Given $\boldsymbol{H} \in \mathbb{R}_{\geq 0}^n$, $\nu \in \mathbb{N}$, $A \subset \mathbb{R}^n$, let $\Gamma_{\boldsymbol{d}}(\boldsymbol{H}, \nu; A)$ denote the set of all real tuples

$$\left(a_0^{(1)}, \ldots, a_{d_1}^{(1)}; \ldots; a_0^{(n)}, \ldots, a_{d_n}^{(n)}\right) \in \mathbb{R}^D \qquad \left(D := \sum_{i=1}^n (d_i + 1)\right)$$

such that

$$\forall 1 \leq i \leq n : \max_{0 \leq j \leq d_i} |a_j^{(i)}| \leq H_i,$$

and

$$\#\left\{(x_1, \ldots, x_n) \in A : \forall 1 \leq i \leq n,\ \sum_{j=0}^{d_i} a_j^{(i)} x_i^j = 0\right\} = \nu.$$

Further set $\Delta_{\boldsymbol{d}}(\boldsymbol{H}, \nu; A)$ to be those *integer* tuples in $\Gamma_{\boldsymbol{d}}(\boldsymbol{H}, \nu; A)$ such that for each $1 \leq i \leq n$ the polynomial $\sum_{j=0}^{d_i} a_j^{(i)} X^j$ is irreducible over $\mathbb{Z}$.

Proofs

Then,

$$\#\big(\mathbb{A}_{\boldsymbol{d}}(\boldsymbol{H})\cap A\big)=\sum_{1\le\nu\le d_1\ldots d_n}\nu\cdot\#\Delta_{\boldsymbol{d}}(\boldsymbol{H},\nu;A).$$

The irreducibility criterion may be replaced by a weaker coprimality condition on the coefficients while picking up an error term. In particular, denoting by $\mathcal{Z}_s$ the primitive points in $\mathbb{Z}^s$,

$$\Big|\#\Delta_{\boldsymbol{d}}(\boldsymbol{H},\nu;A)-\#\big(\Gamma_{\boldsymbol{d}}(\boldsymbol{H},\nu;A)\cap\big(\mathcal{Z}_{d_1+1}\times\cdots\times\mathcal{Z}_{d_n+1}\big)\big)\Big|$$
$$\ll_{\boldsymbol{d}}\max_{1\le i\le n}\frac{(\log H)^{\delta_{d_i,2}}}{H_i}.$$

The problem reduces to estimating the number of primitive lattice points falling within a certain set.

Proofs

To estimate

$$\#\big(\Gamma_{\boldsymbol{d}}(\boldsymbol{H},\nu;A)\cap(\mathcal{Z}_{d_1+1}\times\cdots\times\mathcal{Z}_{d_n+1})\big)$$

we first estimate

$$\#\big(\Gamma_{\boldsymbol{d}}(\boldsymbol{H},\nu;A)\cap(\mathbb{Z}_{d_1+1}\times\cdots\times\mathbb{Z}_{d_n+1})\big).$$

From now on the set A is taken as a fibre of a semi-algebraic family $Z:\Omega\to\mathcal{P}(\mathbb{R}^n)$. Let

$$\begin{aligned} W_{\boldsymbol{d},\nu}:\Omega\times\mathbb{R}^n_{\geq 0} &\to \mathcal{P}(\mathbb{R}^n)^D \\ (\boldsymbol{t},\boldsymbol{H}) &\mapsto \Gamma_{\boldsymbol{d}}(\boldsymbol{H},\nu;Z(\boldsymbol{t})). \end{aligned}$$

Useful Fact

$W_{\boldsymbol{d},\nu}$ is a semi-algebraic family.

This is a consequence of the Tarski-Seidenberg principle.

Proofs

Now we apply a result of Barroero & Widmer on the enumeration of lattice points in the fibres of a semi-algebraic family.

Theorem (Barroero & Widmer 2014)

LEt $Y : \Sigma \to \mathbb{R}^N$ be a semi-algebraic family. Then, *uniformly* in $\boldsymbol{s} \in \Sigma$,

$$\left| \#\left(Y(\boldsymbol{s}) \cap \mathbb{Z}^N\right) - \mathrm{Vol}_N\left(Y(\boldsymbol{s})\right) \right| \ll_Y 1 + \sum_{j=1}^{N-1} V_j\left(Y(\boldsymbol{s})\right),$$

where $V_j\left(Y(\boldsymbol{s})\right)$ denotes the sum of the j-dimensional Lebesgue measures of all the j-dimensional coordinate projections of $Y(\boldsymbol{s})$.

Proofs

After some simple calculations, one obtains

$$\left|\frac{\#\left(W_{\boldsymbol{d},\nu}(\boldsymbol{t},\boldsymbol{H})\cap(\mathbb{Z}^{d_1+1}\times\cdots\times\mathbb{Z}^{d_n+1})\right)}{H_1^{d_1+1}\dots H_n^{d_n+1}}-\mathrm{Vol}_D\left(W_{\boldsymbol{d},\nu}(\boldsymbol{t},\boldsymbol{1})\right)\right|$$

$$\ll_{\boldsymbol{d},Z}\frac{1}{\min_{1\leq i\leq n}H_i}\cdot\qquad\text{(where }\boldsymbol{1}:=(1,\dots,1))$$

To get back to counting primitive points one applies some very standard arguments involving the Möbius function, yielding

$$\left|\frac{\#\left(W_{\boldsymbol{d},\nu}(\boldsymbol{t},\boldsymbol{H})\cap(\mathcal{Z}_{d_1+1}\times\cdots\times\mathcal{Z}_{d_n+1})\right)}{H_1^{d_1+1}\dots H_n^{d_n+1}}-\frac{\mathrm{Vol}_D\left(W_{\boldsymbol{d},\nu}(\boldsymbol{t},\boldsymbol{1})\right)}{\zeta(d_1+1)\dots\zeta(d_n+1)}\right|$$

$$\ll_{\boldsymbol{d},Z}\frac{1}{\min_{1\leq i\leq n}H_i}.$$

Proofs

Collecting everything together,

$$\left|\frac{\#\big(\mathbb{A}_{\boldsymbol{d}}(\boldsymbol{H})\cap Z(\boldsymbol{t})\big)}{H_1^{d_1+1}\dots H_n^{d_n+1}} - f_{\boldsymbol{d}}(\boldsymbol{t})\right| \ll_{\boldsymbol{d},Z} \max_{1\leq i\leq n} \frac{(\log H_i)^{\delta_{d_i,2}}}{H_i},$$

with

$$f_{\boldsymbol{d}(\boldsymbol{H})} := \sum_{1\leq\nu\leq d_1\dots d_n} \nu\cdot\frac{\mathrm{Vol}_D\big(W_{\boldsymbol{d},\nu}(\boldsymbol{t},\boldsymbol{1})\big)}{\zeta(d_1+1)\dots\zeta(d_n+1)}.$$

Now use the fact that $Z(\boldsymbol{t})$ is Jordan measurable and appeal to Koleda's Theorem to determine exactly the formula for $f_{\boldsymbol{d}}(\boldsymbol{H})$.