Distribution of Reduced Quadratic Irrationals

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Definition

A real number x is called a quadratic irrational if there exist integers A, B, C with $A \neq 0$, such that $Ax^2 + Bx + C = 0$, with gcd(A, B, C) = 1 and $D = B^2 - 4AC > 0$ not a perfect square.

Definition

A quadratic irrational ω is called (regular) reduced if $\omega > 1$ and its conjugate ω^* lies between -1 and 0.

In this talk, I will be describing a number theoretical approach (Kallies et al, Boca, Ustinov) which gives an effective estimate in the asymptotic formula for the distribution of various types of quadratic irrationals.

The Division Algorithms (Vallée)

By-default division (G): By-excess division (B): Centered division (K): Even division (E): Odd division (O): Subtractive algorithm (T):

$$\begin{array}{ll} v=cu+r, & 0\leq r< u\\ v=cu-r, & 0\leq r< u\\ v=cu+\epsilon r, & \epsilon=\pm 1, \ 0\leq r< u/2\\ v=cu+\epsilon r, & \epsilon=\pm 1, 0\leq r< u, \ c \ \text{ even}\\ v=cu+\epsilon r, & \epsilon=\pm 1, \ 0\leq r< u, \ c \ \text{ odd}\\ \text{replaces by-default division by exactly}\\ c \ \text{subtractions of the form } v=u+r \end{array}$$

• Each of these algorithms is associated to a continued fraction (CF) expansion:

E)— even CF (ECF)

$$O$$
)— odd CE (OCE)

The RCF expansion

• For example:

$$\frac{5}{8} = \frac{5}{1 \cdot 5 + 3} = \frac{1}{1 + \frac{3}{5}} = \frac{1}{1 + \frac{3}{1 \cdot 3 + 2}} = \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}$$
$$= \frac{1}{1 + \frac{1}{1 + \frac{2}{1 \cdot 2 + 1}}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5}}}} = [1, 1, 1, 2]$$

Every x ∈ (0,1] \ Q has a unique regular continued fraction (RCF) expansion

$$x = [a_1, a_2, a_3, \ldots] := rac{1}{a_1 + rac{1}{a_2 + rac{1}{a_3 + \ddots}}}, \quad a_i \ge 1.$$

The RCF expansion

- Another example: the golden mean $G = \frac{\sqrt{5}+1}{2} > 1$ satisfies $G^2 = G + 1$, or $G = 1 + \frac{1}{G} = 1 + \frac{1}{1 + \frac{1}{G}}$. Hence $\frac{1}{G} = g = [\overline{1}]$.
- Truncating the RCF expansion of $x = [a_1, a_2, ...]$ at level *n* yields the *n*th convergent of *x*

$$\frac{p_n}{q_n} = \frac{p_n}{q_n}(x) := [a_1, \dots, a_n] = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}}$$

The sequence $\left(\frac{p_n}{q_n}(x)\right)_n$ converges to *x*:

$$\lim_{n\to\infty}\frac{p_n}{q_n}=x$$

.

The RCF Gauss map

The continuous extension of the standard Euclidean division algorithm is the RCF Gauss map

$$T(x):=rac{1}{x}-\left\lfloorrac{1}{x}
ight
floor,\ T(0):=0.$$

The RCF digits of \boldsymbol{x} are fully recaptured by \mathcal{T} 's iterates as

$$a_1 = \left\lfloor \frac{1}{x} \right\rfloor = \left\lfloor \frac{1}{T^0(x)} \right\rfloor, \quad a_n = \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor.$$

• *T* acts as a shift on the digits:
If
$$x = [a_1, a_2, a_3, ...]$$
, then $T(x) = [a_2, a_3, ...]$.

• The unique Lebesgue absolutely continuous *T*-invariant probability measure is the Gauss measure

$$d\mu(x) = h(x)dx$$
, where $h(x) = \frac{1}{\log 2} \frac{1}{(x+1)}$.





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Qls as periodic points of the Gauss map

A number x ∈ [0, 1] is a *periodic point* of T if there exists an n ≥ 1 such that Tⁿx = x.

The periodic points of *T* are the numbers with purely periodic RCF expansion, $x = [\overline{a_1, a_2, \dots, a_n}]$.

• Lagrange: The reduced quadratic irrationals (RQIs) are precisely the periodic points of *T*.

Remark— (Galois) If
$$\omega = [\overline{a_1, ..., a_n}]$$
, then $-\frac{1}{\omega^*} = [\overline{a_n, ..., a_1}]$.

• Ordering the RQIs ω can be done by the spectral radius of a specific matrix $\widetilde{M}(\omega)$ associated to ω (Pollicott 1986, Faivre 1992), by the trace of that matrix (Kallies et al 2001, Boca 2007, Ustinov 2013), and by the size of the fundamental discriminant $\Delta = \Delta(\omega)$ (Duke 1988).

• Consider the hyperbolic plane $\mathbb{H} = \{x + yi \mid y > 0\}$ with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. Elements of PSL(2, \mathbb{Z}) are isometries of \mathbb{H} , and $\mathbb{H}/\text{PSL}(2, \mathbb{Z})$ is the modular surface.



Distribution of Reduced Quadratic Irrationals

Fix an RQI ω and its conjugate ω*. Let γ_ω be the geodesic in ℍ with endpoints ω, ω*. Then p ∘ γ_ω is a closed primitive geodesic on ℍ/PSL(2, ℤ). We define the length *ρ*(ω) of the RQI ω to be the length of p ∘ γ_ω.



 Let a ≥ 1. Take x = [a₁, a₂, ...] := a₁ + [a₂, ...] > 1. The "building block" g_a := ^a 1 1 0 acts on x as a linear fractional transformation:

$$g_a \circ x = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \circ x = \frac{ax+1}{x+0} = a + \frac{1}{x} = [a, a_1, a_2, \ldots].$$

 Suppose that ω = [a₁, a₂,..., a_n] > 1, with n = per(ω), and consider the matrices

$$M(\omega) = \begin{pmatrix} a_1 & 1\\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1\\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1}\\ q_n & q_{n-1} \end{pmatrix}, \text{ and}$$
$$\widetilde{M}(\omega) = \begin{cases} M(\omega), & \text{if } n \text{ is even}\\ M(\omega)^2, & \text{if } n \text{ is odd.} \end{cases}$$

• It is clear that $\widetilde{M}(\omega) \circ \omega = \omega$.

The matrix *M*(ω) is a generator of {g ∈ PSL(2, Z) | g ∘ ω = ω}. In fact, it is a generator of the fixed-point group {g ∈ SL(2, Z) | g ∘ ω = ω}.

• The length of the closed primitive geodesic $p\circ\gamma_\omega$ is

$$L_{p\circ\gamma_{\omega}} = 2\log \mathcal{R}(\widetilde{M}(\omega)),$$

where $\mathcal{R}(\widetilde{M}(\omega))$ is the spectral radius of a generator $\widetilde{M}(\omega)$ of the fixed-point group $\{g \in \mathsf{PSL}(2,\mathbb{Z}) \mid g \circ \omega = \omega\}.$

 $\bullet\,$ By definition, the length of the RQI ω is

$$\varrho(\omega) = 2\log \mathcal{R}(\widetilde{M}(\omega)).$$

•
$$G = \frac{1+\sqrt{5}}{2} = [\overline{1}]$$
, so per(G) is odd.



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Set

$$\begin{split} G_{\omega} &:= \{g \in \mathsf{GL}(2,\mathbb{Z}) \mid g \circ \omega = \omega\} \\ E_{\Delta} &:= \{u + v\sqrt{\Delta} \mid u, v \in \mathbb{Z}, \ u^2 - \Delta v^2 = \pm 4\}, \end{split}$$

and denote by $\epsilon_0^+(\omega)$ the fundamental solution to the Pell equation $u^2 - \Delta v^2 = 4$, $\Delta = \operatorname{disc}(\omega)$.

• The homomorphism of abelian groups

$$\Lambda_{\omega}: G_{\omega} \to E_{\Delta}, \qquad \Lambda_{\omega} \left(egin{pmatrix} a & b \\ c & d \end{pmatrix}
ight) \mapsto c \omega + d$$

maps each matrix σ to its eigenvalue $\lambda_{\sigma,\omega} = c\omega + d$.

It turns out that Λ_{ω} is an isomorphism, and therefore maps $\widetilde{M}(\omega)$ to $\epsilon_0^+(\omega)$, so that $\mathcal{R}(\widetilde{M}(\omega)) = \epsilon_0^+(\omega)$. Therefore, ordering the RQIs ω by their length is equivalent to ordering them by the size of $\epsilon_0^+(\omega)$.

The periodic points of T have been shown to be uniformly distributed w.r.t. the Gauss measure μ on [0, 1] (identified with $[1, \infty)$ via $z \mapsto \frac{1}{z}$):

Theorem (Pollicott 1986, Faivre 1992)

$$(A) \quad \frac{\sum\limits_{\varrho(M) \le R} 1}{\sum\limits_{\varrho(M) \le R} 1} \sim \int_{\alpha}^{\beta} d\mu, \qquad \text{as } R \to \infty \qquad (0 \le \alpha < \beta \le 1)$$
$$(B) \quad \sum\limits_{\varrho(M) \le R} 1 \sim \frac{3 \log 2}{\pi^2} e^R, \qquad \text{as } R \to \infty.$$

 Consider the function π₁ counting the RQI's of bounded length, and its Laplace-Stieltjes transform F:

$$\pi_1(R) = \sum_{\varrho(\omega) \le R} 1,$$

$$F(s) = \int_0^\infty e^{-st} d\pi_1(t) = \sum_{\omega} e^{-s\varrho(\omega)} = \sum_{n=1}^\infty \sum_{a_1,\dots,a_n} (xTx\dots T^{n-1}x)^{2\alpha_n s}.$$

- Then $\pi_1(R) = O(e^R)$, and F is holomorphic for $\operatorname{Re}(s) > 1$. In fact, F extends meromorphically to the half plane $\{\operatorname{Re}(s) > \frac{1}{2}\}$, with only one pole on the line $\operatorname{Re}(s) = 1$, which is simple, located at s = 1, and with residue equal to $\frac{3 \log 2}{\pi^2}$.
- Applying the Wiener-Ikehara Tauberian theorem on *F* yields the result:

$$\lim_{R\to\infty} e^{-R}\pi_1(R) = \frac{3\log 2}{\pi^2}.$$

• The transfer operator associated with the Gauss map:

$$L(s)(f)(z) = \sum_{n=1}^{\infty} f\left(\frac{1}{z+n}\right) \left(\frac{1}{z+n}\right)^{2s}$$

restricted to the disc algebra $A(\{|z-1| \le \frac{3}{2}\})$ is a nuclear operator.

- Spectral analysis of L shows that 1 is the maximal eigenvalue of L(1), it is simple, and all other eigenvalues have modulus < 1. Similarly, all eigenvalues of L(s), for $\operatorname{Re}(s) \ge 1$, $s \ne 1$ have modulus < 1.
- For the eigenvalues

 $\theta_1(s) \geq \theta_2(s) \geq \ldots$

of L(s), for $s \neq 1$ but close to 1, $\operatorname{Re}(s) \geq 1$, we have that for $i \geq 2$

 $\theta_i(s) < R < \theta_1 < 1,$

where 0 < R < 1 does not depend on *s*.



• Denote by v(s) the Fredholm kernel associated to L(s). The the functions $f_1(z,s) = \det(I - zv(s))$ and $F_1(z,s) = f_1(z,s)f_z(-z,s)$ are holomorphic for $z \in \mathbb{C}$ and $\operatorname{Re}(s) > \frac{1}{2}$, and $F_1(1,s) \neq 0$ for $\operatorname{Re}(s) > 1$. Then F can be expressed as

$$F(s) = -rac{1}{2}rac{\partial F_1}{\partial z}(1,s)rac{1}{F_1(1,s)} + G(s),$$

where G is a holomorphic function for $\operatorname{Re}(s) > \frac{1}{2}$.

- The function F₁(1, s) has a simple zero at s = 1, and the residue of F at s = 1 is -¹/_{2θ'(1)}.
- The number $-\theta_1'(1)$ is shown to be real and equal to the entropy of the RCF Gauss map.
- The following trace formula for $n \ge 1$ is important

$$\mathsf{Tr}(\mathbf{v}^{n}(s)) = \sum_{a_{1},\ldots,a_{n}} \frac{(\prod_{i=1}^{n-1} T^{i}x)^{2s}}{1 - (-1)^{n}(\prod_{i=1}^{n-1} T^{i}x)^{2}},$$

where $x = [0; \overline{a_1, \ldots, a_n}]$.

Ordering by Trace

- Instead of ordering the RQIs ω > 1 by the spectral radius λ_ω > 1 of the matrix M̃(ω), consider the trace of M̃(ω), Tr(M̃(ω)) = λ_ω + λ_ω⁻¹.
- In fact, the isomorphism Λ_{ω} preserves the trace:

$$\operatorname{tr} \circ \Lambda_{\omega} = \operatorname{Tr},$$

where $Tr(\sigma)$ is the trace of the matrix σ and $tr(\eta)$ is the trace of the QI η in the associated quadratic field.

When $\eta \in E_{\Delta}^+$, we have $0 < \operatorname{tr}(\eta) - \eta < \frac{1}{2}$. For $\eta = \epsilon_0^+(\omega)$, we get that the trace and the spectral radius of $\widetilde{M}(\omega)$ "grow together":

$$0 < \operatorname{Tr} \widetilde{M}(\omega) - \mathcal{R}(\widetilde{M}(\omega)) < \frac{1}{2}$$

This suggests that ordering by length and ordering by Trace could give the same asymptotics for the number of RQIs.

Ordering by Trace

- Every purely periodic ω = [a₁, a₂, ..., a_n] gives rise to a word w = a₁...a_n, where n is not necessarily the period of ω. Instead of counting the RQI's ω, we will count the number of words w.
- These words w are in one-to-one correspondence with the matrices $\widetilde{M}(w) = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$, where m is even.
- For the "building blocks" of $\widetilde{M}(w)$, we have

$$\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ell & 1 \\ 1 & 0 \end{pmatrix} = B^k A^\ell,$$

where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A^t.$

• The monoid \mathcal{M} generated by A and B is free. Therefore, with some overcounting, we can order the RQIs by the Trace of the products of even length of these matrices.

Ordering by Trace

• We want to find asymptotic estimates for the cardinality of the set

$$\mathcal{L} := \{ B^{a_1} A^{a_2} \dots B^{a_{2n-1}} A^{a_{2n}} \mid n \ge 1, \ M := [\overline{a_1, \dots, a_{2n}}] > 1, \\ \mathsf{Tr}(B^{a_1} A^{a_2} \dots B^{a_{2n-1}} A^{a_{2n}}) \le N \}.$$

• This set can be re-written as

$$egin{aligned} \mathcal{L} = igg\{egin{aligned} q' & q \ p' & p \end{pmatrix} \mid \ 0 \leq p \leq q, \ 0 \leq p' \leq q', \ q < q', \ p + q' \leq N, \ pq' - p'q = 1 igg\}, \end{aligned}$$

and can be parametrized as

$$\mathcal{S} := \{(q,q') \mid \ q < q' \leq \mathsf{N}, \ \mathsf{gcd}(q,q') = 1, \ q' + \overline{q'} \leq \mathsf{N} \}$$

• Hence its cardinality is given by

$$|\mathcal{L}| = |\mathcal{S}| = \sum_{\substack{q < N \ 0 < x \leq \min\{q, N-y\} \ xy \equiv 1 \mod q}} 1$$

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Ordering by Trace (sketch of proof)

- Counting the number of such matrices with Trace ≤ N amounts to counting the lattice points within certain regions A and on the modular hyperbolas H_q = {(x, y) ∈ Z² | xy ≡ 1 mod q}, q < N.
- Weil bounds on Kloosterman sums are used to estimate the number N_q(M) of such points when A is a rectangle I × J:

$$\mathcal{N}_{\boldsymbol{q}}(\boldsymbol{I} imes \boldsymbol{J}) = rac{\phi(\boldsymbol{q})}{\boldsymbol{q}^2} \; |\boldsymbol{I}| \; |\boldsymbol{J}| + O_{\epsilon}(\boldsymbol{q}^{1/2+\epsilon}).$$

• Standard techniques and Perron's formula are then used to estimate

$$\sum_{n \le N} \frac{\phi(n)}{n^2} = c_1 \log N + c_2 + O\left(\frac{\log N}{N}\right),$$
$$\sum_{\lambda N \le n \le \mu N} \frac{\phi(n)}{n^2} = c_1 \log \frac{\lambda}{\mu} + O\left(\frac{\log N}{N}\right).$$

Ordering by Trace and length

Hence we get an effective version of the Pollicott-Faivre results:

Theorem (Boca, 2007)

$$\sum_{(\omega) \leq R} 1 = \frac{3\log 2}{\pi^2} e^R + O_{\epsilon}(e^{(7/8 + \epsilon)R}), \qquad \text{as } R \to \infty.$$

Theorem (Ustinov, 2013)

For $\alpha, \beta \in [0, 1]$, we have $(R \to \infty)$:

(A)
$$\sum_{\substack{\varrho(\omega) \le R}} 1 = \frac{3\log 2}{\pi^2} e^R + O_{\epsilon}(e^{(3/4+\epsilon)R}), \quad \text{and}$$

(B)
$$\sum_{\substack{\varrho(\omega) \le R\\ 0 \le \omega \le \alpha\\ 0 \le -1/\omega^* \le \beta}} 1 = \frac{e^R}{2\zeta(2)} \iint_{[0,\alpha] \times [0,\beta]} \frac{dx \, dy}{(xy+1)^2} + O_{\varepsilon}(e^{(3/4+\epsilon)R}).$$

Taking $\beta = 1$, this shows that the RQIs are uniformly distributed w.r.t. μ .

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The ECF Gauss map

• (Schweiger) Every irrational number x > 1 has a unique ECF-expansion

$$x = [(a_1, e_1), (a_2, e_2), \ldots] := a_1 + \frac{e_1}{a_2 + \frac{e_2}{a_3 + \cdots}} \ge 1,$$

where $a_i \in 2\mathbb{N}$ and $e_i \in \{\pm 1\}$.

• The corresponding ECF Gauss shift $\mathcal{T}_{\mathcal{E}}$ acts on $[1,\infty)$ by

$$T_E(x) = \begin{pmatrix} 0 & e_1 \\ 1 & -a_1 \end{pmatrix} \circ x = rac{e_1}{x-a_1}$$

where $a_1 = a_1(x) = 2\lfloor \frac{x+1}{2} \rfloor \in 2\mathbb{N}$ and $e_1 = e_1(x) = \operatorname{sgn}(x - a_1(x)) \in \{\pm 1\}.$

• T_E acts as a shift on the digits: If $x = [(a_1, e_1), (a_2, e_2), (a_3, e_3), ...]$, then $T(x) = [(a_2, e_2), (a_3, e_3), ...]$. • The ECF Gauss shift on $[1,\infty)$ and (0,1) respectively:



The ECF Gauss map

- The map T_E has the invariant measure $\mu_E = \frac{2dx}{(x-1)(1+x)}$, which is infinite .
- The periodic points of *T_E* are the *E-reduced* Qls, and their set *R_E* consists exactly of the Qls *ω* > 1 with *ω*^{*} ∈ [-1, 1].

Remark— The Galois formula in this case takes the following form: if

$$\omega = [\overline{(a_1, e_1), \dots, (a_n, e_n)}], \text{ then } -\omega^* = \langle \langle \overline{(a_n, e_n), \dots, (a_1, e_1)} \rangle \rangle = \frac{e_n}{a_n + \frac{e_{n-1}}{a_{n-1} + \cdots}}.$$

• If
$$\omega = [\overline{(a_1, e_1), \dots, (a_n, e_n)}] \in R_E$$
, $n = \operatorname{per}(\omega)$, we have

$$M_E(\omega) := \begin{pmatrix} a_1 & e_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & e_n \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1}e_n \\ q_n & q_{n-1}e_n \end{pmatrix},$$

$$\widetilde{M}_E(\omega) = \begin{cases} M_E(\omega) & \text{if } (-e_1) \cdots (-e_n) = +1 \\ M_E(\omega)^2 & \text{if } (-e_1) \cdots (-e_n) = -1. \end{cases}$$

Remark— In the RCF case, $(-e_1)\cdots(-e_n) = +1 \iff n$ is even.

Distribution of the E-RQIs

• The *length* of $\omega \in R_E$ is

$$\varrho_E(\omega) := 2 \log \mathcal{R}(\widetilde{M}_E(\omega)),$$

where $\mathcal{R}(\widetilde{M}_E(\omega))$ is the spectral radius of $\widetilde{M}_E(\omega)$.

• In fact, $\widetilde{M}_{E}(\omega)$ is the generator of the fixed-point group $\Theta_{\omega} = \{ \sigma \in \Theta \mid \sigma \circ \omega = \omega \}$, where

 $\Theta = \{ \sigma \in \mathsf{SL}(2,\mathbb{Z}) \mid \sigma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mod 2 \}.$

- Set $\mathcal{F}_{\Delta} = \{ u + v\sqrt{\Delta} \mid u, v \in \mathbb{Z}, u^2 \Delta v^2 = \pm 1 \}$, and $\widetilde{\Theta} = \{ \sigma \in \mathsf{GL}(2,\mathbb{Z}) \mid \sigma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2 \}.$
- The corresponding homomorphism Λ^E_ω : Θ̃_ω → F_Δ is not always surjective. In particular, it is onto when Δ = Δ(ω) ≡ 1 mod 4. When Δ = 4Δ₀, Δ₀ not a perfect square, Δ₀ odd, and B ≡ 0 mod 4, where Ax² + Bx + C is the minimal polynomial of ω, we have Λ^E_ω(Θ̃_ω) = F_{Δ₀}.

Theorem (Boca, S, 2020)

For every $\alpha, \beta_1, \beta_2 \geq 1$ with $(\alpha, \beta_1) \neq (1, 1)$,

$$\sum_{\substack{\omega \in R_E \\ \varrho_E(\omega) \le R \\ \omega \ge \alpha}} 1 = C(\alpha, \beta_1, \beta_2) e^R + O_{\alpha, \beta_1, \varepsilon} (e^{(3/4+\varepsilon)R})$$

$$= \frac{1}{\pi^2} \log \left(\frac{\alpha \beta_2 + 1}{\alpha \beta_2} \cdot \frac{\alpha \beta_1}{\alpha \beta_1 - 1} \right)$$

$$= \frac{1}{\pi^2} \iint_{[\alpha, \infty) \times [-\frac{1}{\beta_2}, \frac{1}{\beta_1}]} \frac{du \, dv}{(u-v)^2}.$$

Taking $\beta_1 = \beta_2 = 1$, this shows that the E-reduced QIs are uniformly distributed w.r.t the T_E -invariant measure μ_E .

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where

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• As in the case of the RCF, the Trace and the spectral radius of the matrices \widetilde{M}_E "grow together", therefore counting the *E*-reduced QIs ordered by their length amounts to counting the number of matrices

$$\begin{split} & \mathcal{M} = \begin{pmatrix} p' & pe \\ q' & qe \end{pmatrix} \in \mathsf{SL}(2,\mathbb{Z}), \text{ with } p' > p > q > 0, \ p' > q' > q, \ e = \pm 1, \\ & \mathcal{M} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod 2, \ \mathsf{Tr}\mathcal{M} \leq \mathcal{N}. \end{split}$$

• Passing from the constraints $\omega > 1$, $\omega^* \in [-1, 1]$ on QIs to the constraints on their convergents p/q is allowed because of the approximations

$$\omega pprox rac{p}{q} \quad ext{and} \quad -rac{1}{\omega^*} pprox rac{p'}{pe}.$$

Distribution of the E-RQIs

- Counting these matrices M with Trace $\leq N$ amounts to counting the lattice points within certain regions A and on the modular hyperbolas $\mathcal{H}_{q,h} = \{(x, y) \in \mathbb{Z}^2 \mid xy \equiv h \mod q\}, (q, h) = 1, q < N$, with additional conditions on the parity of x, y and q.
- The number theoretical arguments giving the estimates in this case involve proving formulas such as

$$\sum_{\substack{n \le N \\ n \text{ even}}} \frac{\varphi(2n)}{n^2} = \frac{2(\log N + c_1)}{3\zeta(2)} + O\left(\frac{\log^2 N}{N}\right), \qquad c_1 = \gamma - \frac{4\log 2}{3} - \frac{\zeta'(2)}{\zeta(2)},$$
$$\sum_{\substack{n \le N \\ n \text{ odd}}} \frac{\varphi(n)}{n^2} = \frac{2(\log N + c_2)}{3\zeta(2)} + O\left(\frac{\log^2 N}{N}\right), \qquad c_2 = c_1 + \frac{6\log 2}{3}.$$

Distribution of the B-RQIs

- The BCF Gauss map on $(1, \infty)$ can be obtained from T_E by setting $e_i = -1$, $\forall i \ge 1$, and letting $a_i \ge 2$ not necessarily even. Its (infinite) invariant measure is $\mu_B = \frac{dx}{x(x-1)}$ (Rényi).
- The periodic points of *T_B* are the *B*-reduced QIs, and their set *R_B* consists exactly of the QIs *ω* > 1 with *ω*^{*} ∈ [0, 1].

Theorem (Boca, S, 2020)

For every
$$lpha, eta \geq 1$$
 with $(lpha, eta) \neq (1, 1)$,

$$\sum_{\substack{\omega \in R_B, \varrho_B(\omega) \le R \\ \omega \ge \alpha, 0 < \omega^* \le \frac{1}{\beta}}} 1 = \frac{e^R}{2\zeta(2)} \iint_{[\alpha,\infty) \times [0,1/\beta]} \frac{du \, dv}{(u-v)^2} + O_{\alpha,\beta,\varepsilon}(e^{(3/4+\varepsilon)R}).$$

Taking $\beta = 1$ this shows that the B-reduced QIs are uniformly distributed w.r.t the T_B -invariant measure μ_B .

The distribution of the O-QIs

• (Rieger) Every irrational number x > 1 has a unique OCF-expansion

$$x = [(a_1, e_1), (a_2, e_2), \ldots] := a_1 + \frac{e_1}{a_2 + \frac{e_2}{a_3 + \cdots}} \ge 1,$$

where $a_i \in 2\mathbb{N} - 1$, $e_i \in \{\pm 1\}$ and $a_i + e_i \geq 2$, $\forall i$.

- The OCF Gauss map T_O acts as a shift on the digits, and the T_O -invariant measure μ_o is a probability measure.
- The *length* of $\omega \in R_O$ is

$$\varrho_O(\omega) := 2 \log \mathcal{R}(\widetilde{M}_O(\omega)),$$

where $\mathcal{R}(\widetilde{M}_O(\omega))$ is the spectral radius of $\widetilde{M}_O(\omega)$, and $\widetilde{M}_O(\omega)$ is the generator of the fixed-point group $\Gamma_{\omega} = \{g \in \Gamma \mid g \circ \omega = \omega\}$, where

 $\Gamma = \{ \sigma \in \mathsf{SL}(2,\mathbb{Z}) \mid \sigma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \mod 2 \}.$

The distribution of the O-RQIs

• In this case, we count the number of matrices in the set

 $\mathcal{P}:=\big\{\left.\left(\begin{smallmatrix}a_1&e_1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}a_2&e_2\\1&0\end{smallmatrix}\right)\cdots\left(\begin{smallmatrix}a_n&e_n\\1&0\end{smallmatrix}\right)\right|\ n\geq 1,\ a_i\in 2\mathbb{N}-1,\ e_i=\pm 1,\ a_i+e_i\geq 2\big\},$

• The set of matrices $\mathcal P$ can be parametrized as

$$\mathcal{P} = \mathcal{S}_{+1} \cup \mathcal{S}_{-1}, \quad \text{with}$$

 $\mathcal{S}_{+1} = \left\{ egin{array}{c} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma} \mid \ 0 \leq d \leq b, \ 1 \leq c \leq a, \ a/b > g, \ ad - bc = \pm 1
ight\}, \quad ext{and}$ $\mathcal{S}_{-1} = \left\{ egin{array}{c} a & -b \\ c & -d \end{pmatrix} \in \widetilde{\Gamma} \mid \ 0 \leq d \leq b, \ 1 \leq c \leq a, \ a/b > G + 1, \ ad - bc = \pm 1
ight\}.$

Theorem (S, 2021)

For every $\alpha \geq 1$, $\beta_1 \geq G+1$, and $\beta_2 \geq G-1$ we have that

$$\sum_{\substack{\omega \in \mathcal{R}_O, \ \varrho_o(\omega) \le R \\ \omega \ge \alpha, \ -\frac{1}{\beta_2} \le \omega^* \le \frac{1}{\beta_1}}} 1 = \frac{3 \log G}{4\zeta(2)} \iint_{[\alpha,\infty) \times [-\frac{1}{\beta_1}, \frac{1}{\beta_2}]} d\widetilde{\mu}_o \ + \ O(e^{(\frac{3}{4} + \epsilon)R}).$$

Taking $\beta_1 = G + 1$, $\beta_2 = G - 1$ we get that the O-reduced QIs are equidistributed with respect to the T_O -invariant probability measure μ_o : For every $\alpha \ge 1$ we have

$$\frac{\sum\limits_{\substack{\omega\in\mathcal{R}_{O},\ \omega\geq\alpha\\ \varrho_{o}(\omega)\leq R}}1}{\sum\limits_{\substack{\omega\in\mathcal{R}_{O}\\ \varrho_{o}(\omega)\leq R}}1} \sim \int_{\alpha}^{\infty}d\mu_{o}.$$

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Thank you!

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