Exactness and Ergodicity of Certain Markovian Multidimensional Fraction Algorithms

David Siukaev
scientific adviser:: PhD, Alexandra Skripchenko

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Introduction

A multidimensional continued fraction algorithm is a multidimensional generalization of well-known continued fraction algorithms of small dimensions: Gauss and Euclidean.

Plan of our work:

1. define the MCF algorithms, their corresponding Rauzy graphs and partitions into subcones.
2. check that the proof of the exactness of the Euclidean algorithm given by T.Miernowski and A.Nogueira (see [1]) generalizes to the multidimensional case.
3. proof the exactness of non-homogeneous MCF Selmer, Brun and Jacobi-Perron algorithms by applying Miernowski and Nogueira's proof with solving the technical difficulties of generalizing to the multidimensional case.

Definition

A ***multidimensional continued fraction algorithm*** *is specified by two piecewiese continuous maps:*

$$f : [0,1]^n \to [0,1]^n \qquad \textit{and} \qquad A : [0,1]^n \to GL(n, \mathbb{Z})$$

- $[0,1]^n$ - *parameter space*
- $A^{(k)}(x) = A\big(f^{k-1}(x)\big)$ - *matrix k-th step for the vector* $x \in [0,1]^n$
- *We can define a cycle* $C^{(k)}(x) = A^{(k)} \cdot \ldots \cdot A^{(1)}(x)$

We will consider **non-homogeneous** algorithms. In this case the map f is determined by the formula $f(x) = A^{-1}(x) \cdot x$.

Selmer and Brun algorithms in dimension 3

In space $V = \Delta^3$ map of the **Selmer algorithm** $\mathcal{F}(x_1, x_2, x_3) = (x_1', x_2', x_3')$ is defined as:

$$(x'_{\pi(1)}, x'_{\pi(2)}, x'_{\pi(3)}) = (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)} - x_{\pi(1)}),$$

where $\pi \in S_n$, s.t. $x_{\pi(1)} < x_{\pi(2)} < x_{\pi(3)}$.

Map of the **Brun algorithm** is defined as:

$$(x'_{\pi(1)}, x'_{\pi(2)}, x'_{\pi(3)}) = (x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)} - x_{\pi(2)}).$$

Matrix definition. $M(x) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where

$$M(x) = Id + E_{\pi^{-1}(3),\pi^{-1}(1)} \text{ for Selmer case,}$$

$$M(x) = Id + E_{\pi^{-1}(3),\pi^{-1}(2)} \text{ for Brun's case.}$$

Jacobi-Perron algorithm in dimension 3

Map of the Jacobi-Perron algorithm is defined as:

$$(x'_1, x'_2, x'_3) = (x_{\pi(2)} - b_2 \cdot x_{\pi(1)}, x_{\pi(3)} - b_3 \cdot x_{\pi(1)}, x_{\pi(1)}), \quad \text{where } b_i = \left[\frac{x_{\pi(i)}}{x_{\pi(1)}}\right].$$

To prove the exactness, it is also necessary to define a **tagged** version of the Jacobi-Perron algorithm.
Let $min(x) = \{i : \ \ x_i \leqslant x_j \ \ \forall\, j\}$ and $nz(x) = min\{i : x_i > x_{min}\}$. Let's define number $m(x)$ First we assume $m(x) = min(x)$. Then

$$m(x') = \begin{cases} min(x') & \text{if } x_j \leqslant x_m \ \forall j \neq nz(x),\ x_{nz} < 2x_m, \\ m(x) & \text{else.} \end{cases}$$

Then map $\mathcal{S}(x)$ of the tagged version of the Jacobi-Perron algorithm is defined as

$$\mathcal{S}(x) = M(x)^{-1}x, \quad \text{where } M(x) = Id + E_{\pi^{-1}(m),\pi^{-1}(nz)}.$$

At the same time, we note that $F(x) = \mathcal{S}^{b_3} \circ \mathcal{S}^{b_2} \circ \mathcal{S}^{b_1}(x)$.

Permutations and Partition into subcones

Define the *Rauzy graph for the MCF algorithm of dimension n* to be the directed graph $\mathcal{G}(n)$ whose vertices are the permutations in S_n and the arrows go from π^i to π^{i+1}.

Lemma. For all π_1, π_2 in S_n there is a path in $\mathcal{G}(n)$ that starts in π_1 and ends in π_2. Every π in S_n has exactly $\gamma(n)$ incoming and exactly $\gamma(n)$ outgoing arrows in $\mathcal{G}(n)$. For Selmer and Brun algorithms $\gamma(n) = n$, for tagged Jacobi-Perron algorithm $\gamma(n) = \frac{n^2-n+2}{2}$.

Using the vector x, we construct a sequence of permutations $(\pi_x) = \pi_x^1, \ldots, \pi_x^k, \ldots$ and a sequence of matrices $M_\pi^1, \ldots, M_\pi^k, \ldots$. Finite sequence $\pi^1, \ldots, \pi^k \in \mathcal{G}(n)$ defines the set

$$C^{\pi^1, \ldots, \pi^k} = \{x \in V : \pi_x^i = \pi^i, 1 \leqslant i \leqslant k\}$$

For $k \geqslant 1$ denote by $\mathcal{P}^{(k)}$ the set of cones $C^{\pi^1, \ldots, \pi^k}$ where $\pi^1, \ldots, \pi^k$ goes through all possible paths of length k in $\mathcal{G}(n)$. We call it a *partition* of the space V into $\gamma^k(n)$ cones (by previous lemma) and $\mathcal{P}^{(k+1)}$ is a refinement of this partition.

Ergodic properties

Let (X, Σ, μ) be a measure space and let $T : X \longrightarrow X$ be a measurable map.

Definition

- *T is said to be ergodic with respect to μ if for every $\Omega \in \Sigma$ such that $T^{-1}(\Omega) = \Omega$, $\mu(\Omega) = 0$ or $\mu(X \setminus \Omega) = 0$*
- *T is said to be nonsingular if for $\Omega \in \Sigma$ $\mu(T^{-1}(\Omega)) = 0$ if and only if $\mu(\Omega) = 0$*
- *T is said to be exact if $\Omega \in \bigcap_{m \geqslant 1} T^{-m}(\Sigma)$ implies $\mu(\Omega) = 0$ or $\mu(X \setminus \Omega) = 0$*
- *T is said to be bi-measurable if for every $\Omega \in \Sigma$ we have $T(\Omega) \in \Sigma$*

The bi-measurable map T *satisfies the intersection property* with respect to the measure μ if

$$\forall\, \Omega \in \Sigma, \text{ s.t. } \mu(\Omega) > 0,\ \exists k \geqslant 1, \text{ s.t. } \mu\big(T^k(\Omega) \cap T^{k+1}(\Omega)\big) > 0.$$

Idea of our work

In the article ([1] T.Miernowski, A.Nogueira) the following has been proved:

Lemma

Let T be bi-measurable, nonsingular and ergodic. Then T is exact if and only if it satisfies the intersection property.

After that, it was proved that Euclid algorithm and the non-homogeneous Rauzy induction satisfy the intersection property and, as a consequence, are exact. At the end of the article it is stated that other non-homogeneous multidimensional continued fraction algorithms - in particular, Selmer, Brun and Jacobi-Perron algorithms - also satisfy the intersection property and they are also exact. However, there is no proof of this. In our work, we only check that this statement is true. So, the idea of the proof belongs to T.Miernowski and A.Nogueira (see [1]), while we only prove the mentioned statement, which was given in their article without proof. This requires changing the proof of the auxiliary lemma and changing (constructing new sets unique for each algorithm) at the end of the proof of the main statement.

Lemmas

Lemma 1. ([7], Section 4). There is a positive constant C such that for almost every $x \in V$ there are infinitely many integers $k \geqslant 1$, such that $M^k(x)$ is a C-balanced matrix.

Lemma 2. There exists a partition of V whose elements are subcones $C^{(k)}(x)$ which satisfy Lemma 1.

Lemma 3. For almost every x, $\bigcap_{k \geqslant 1} C^{(k)}(x) = \{\alpha x : \alpha \geqslant 0\}$.

Auxiliary lemma. For every $N \geqslant 1$ there exists a partition $\mathcal{P}_N$ of V which satisfies the following properties:
1. Its elements are subcones of type $C^{(k)}(x)$,
2. If F is a map of the *Selmer* algorithm, then $\frac{\|I_i^{(k)}(x)\|_1}{\|I_j^{(k)}(x)\|_1} \geqslant N$ for some j and all $i \neq j$.

If F is a map of the *Brun* algorithm, then $\frac{\|I_i^{(k)}(x)\|_1}{\|I_j^{(k)}(x)\|_1} \geqslant N$ and $\frac{\|I_j^{(k)}(x)\|_1}{\|I_p^{(k)}(x)\|_1} \geqslant N$for some j, p and all $i \neq j$

If F is a map of the *Jacobi-Perron* algorithm, then $\frac{\|I_j^{(k)}(x)\|_1}{\|I_i^{(k)}(x)\|_1} \geqslant N$for some j and all $i \neq j$.

Exactness of algorithms

We now use the proof of T.Miernowski, A.Nogueira. The difference lies in the construction of new sets P_+ for each of the three algorithms. This is the plan of the proof (see [1]) and then we will construct the indicated sets P_+.

1. Lemma 1.1 implies that it is sufficient to prove that for any subset of $X \subset V$ of positive measure there exists $k \geqslant 1$ such that $\mu(F^{k+1}(X) \cap F^k(X)) > 0$.
2. Let $x^0 \in X$ be a density point satisfying the lemma 3 and $(n-1)$-dimensional ball $D_\rho = \{x^0 + x : \epsilon \in X,\ \epsilon_1 x_1^0 + \ldots + \epsilon_n x_n^0 = 0, \|\epsilon\|_1 < \rho\}$ of radius ρ with the center at x^0 is entirely contained in X.
3. $\Sigma(x^0, \rho) = \{tx : x \in D_\rho, \delta \leqslant t \leqslant 1\}$ $(\delta = 1 - \dfrac{\rho}{\|x^0\|_1})$ - cylindrical cone. By the density point theorem, for a given $\epsilon > 0$ and a sufficiently small ρ the inequality
$$\mu(X \cap \Sigma(x^0, \rho)) > (1-\epsilon)\mu(\Sigma(x^0, \rho))$$
is true.

Exactness of the algorithms

4. Let $N \geqslant 1$ and $\mathcal{P}_N$ is a C-balanced partition of V. The sets $\Sigma(x^0, \rho) \cap C_N$, where $C_N \in \mathcal{P}_N$, divide the cone $\Sigma(x^0, \rho)$. There exists $C_N \in \mathcal{P}_N$ such that
$$\mu(X \cap \Sigma(x^0, \rho) \cap C_N) > (1 - 2\epsilon)\mu(\Sigma(x^0, \rho) \cap C_N).$$
5. Equality $C_N = C^{(k)}(x)$ implies that for some $k \geqslant 1$ the equality $F^k(C_N) = X$ is true.
6. Using the properties of $\Sigma(x^0, \rho) \cap C_N$, for each of the three algorithms (Selmer, Brun and Jacobi-Perron) we construct set P_+ such that
$$\mu\big(F^k(X) \cap P_+ \cap F(P_+)\big) > \frac{1}{2}\mu\big(P_+ \cap \mathcal{F}(P_+)\big),$$
$$\mu\big(F^{k+1}(X) \cap P_+ \cap F(P_+)\big) > \frac{1}{2}\mu\big(P_+ \cap F(P_+)\big),$$
then
$$\mu\big(F^k(X) \cap F^{k+1}(X)\big) > 0.$$
We get the intersection property. ∎

Set P_+ for Selmer algorithm

Now let's build the sets P_+. Consider the Selmer algorithm.
1. Let $P_+ = \{x \in P : x_{min(k)}$ - the largest coordinate of the vector $x\}$.
Vertices of the set P_+ ($2n$ vertices):

$$(0, 0, \dots, 0, \overbrace{\alpha_{min(k)}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0),$$

$$(0, 0, \dots, 0, \overbrace{\beta_{min(k)}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0),$$

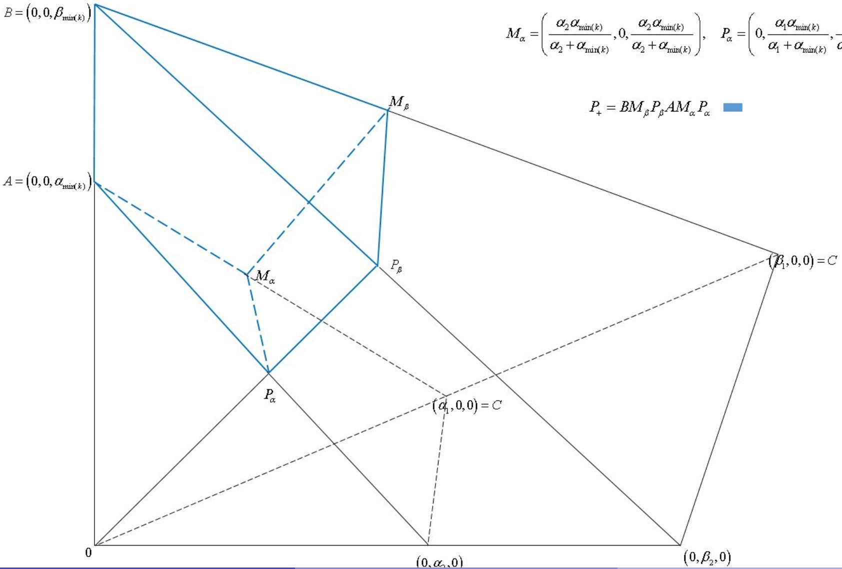
$$(0, 0, \dots, 0, \overbrace{\frac{\alpha_i \alpha_{min(k)}}{\alpha_i + \alpha_{min(k)}}}^{i\textbf{-th coordinate}}, 0, \dots, 0, \overbrace{\frac{\alpha_i \alpha_{min(k)}}{\alpha_i + \alpha_{min(k)}}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0) \textbf{ if } i \neq min(k),$$

$$(0, 0, \dots, 0, \overbrace{\frac{\beta_i \beta_{min(k)}}{\beta_i + \beta_{min(k)}}}^{i\textbf{-th coordinate}}, 0, \dots, 0, \overbrace{\frac{\beta_i \beta_{min(k)}}{\beta_i + \beta_{min(k)}}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0) \textbf{ if } i \neq min(k).$$

Set P_+ for Selmer algorithm (case $n = 3$)

$$M_\beta = \left(\frac{\beta_2 \beta_{\min(k)}}{\beta_2 + \beta_{\min(k)}}, 0, \frac{\beta_2 \beta_{\min(k)}}{\beta_2 + \beta_{\min(k)}} \right), \quad P_\beta = \left(0, \frac{\beta_1 \beta_{\min(k)}}{\beta_1 + \beta_{\min(k)}}, \frac{\beta_1 \beta_{\min(k)}}{\beta_1 + \beta_{\min(k)}} \right)$$

$$M_\alpha = \left(\frac{\alpha_2 \alpha_{\min(k)}}{\alpha_2 + \alpha_{\min(k)}}, 0, \frac{\alpha_2 \alpha_{\min(k)}}{\alpha_2 + \alpha_{\min(k)}} \right), \quad P_\alpha = \left(0, \frac{\alpha_1 \alpha_{\min(k)}}{\alpha_1 + \alpha_{\min(k)}}, \frac{\alpha_1 \alpha_{\min(k)}}{\alpha_1 + \alpha_{\min(k)}} \right)$$



Set P_+ for Selmer algorithm

Now we can estimate the measure of the set P_+:

$$\frac{\mu(P_+)}{\mu(P)} = \left(\frac{\beta^n_{min(k)} \cdot \prod\limits_{i \neq min(k)} \beta_i}{\prod\limits_{i \neq min(k)} (\beta_i + \beta_{min(k)})} - \frac{\alpha^n_{min(k)} \cdot \prod\limits_{i \neq min(k)} \alpha_i}{\prod\limits_{i \neq min(k)} (\alpha_i + \alpha_{min(k)})} \right) \cdot \frac{1}{\beta_1 \dots \beta_n - \alpha_1 \dots \alpha_n} =$$

$$= (1 - \delta^{-n}) \cdot \left(\frac{\beta^n_{min(k)} \cdot \prod\limits_{i \neq min(k)} \beta_i}{\prod\limits_{i \neq min(k)} (\beta_i + \beta_{min(k)})} \right) \cdot \frac{1}{(1 - \delta^{-n})\beta_1 \cdot \dots \cdot \beta_n} =$$

$$= \left(\prod_{i \neq min(k)} (\beta_i / \beta_{min(k)} + 1) \right)^{-1} \geqslant \left(\frac{2}{N} + 1 \right)^{1-n} = \left(\frac{N}{N+2} \right)^{n-1}.$$

So, the following equality holds:

$$\mu(P_+) \geqslant \left(\frac{N}{N+2} \right)^{n-1} \cdot \mu(P).$$

Therefore, for sufficiently large N, the following inequality holds:

$$\mu(F^k(X) \cap P_+) \geqslant (1 - 3\varepsilon)\mu(P_+). \qquad (1)$$

2. The only coordinate P_+ that changes when we apply F is $x_{min(k)}$. Let x_1 be the smallest coordinate of the vector x. Selmer algorithm acts on a pair of $(x_1, x_{min(k)})$ as Euclidean algorithm.

Set P_+ for Selmer algorithm

3. Now we want prove that for sufficiently large N the inequality

$$\mu(P_+ \cap F(P_+)) > \frac{1}{2}\mu(P_+) = \frac{1}{2}\mu(F(P_+)). \qquad (2)$$

holds. Vertices of the set $F(P_+)$ ($2n$ vertices):

$$(0, 0, \dots, 0, \overbrace{\alpha_{min(k)}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0),$$

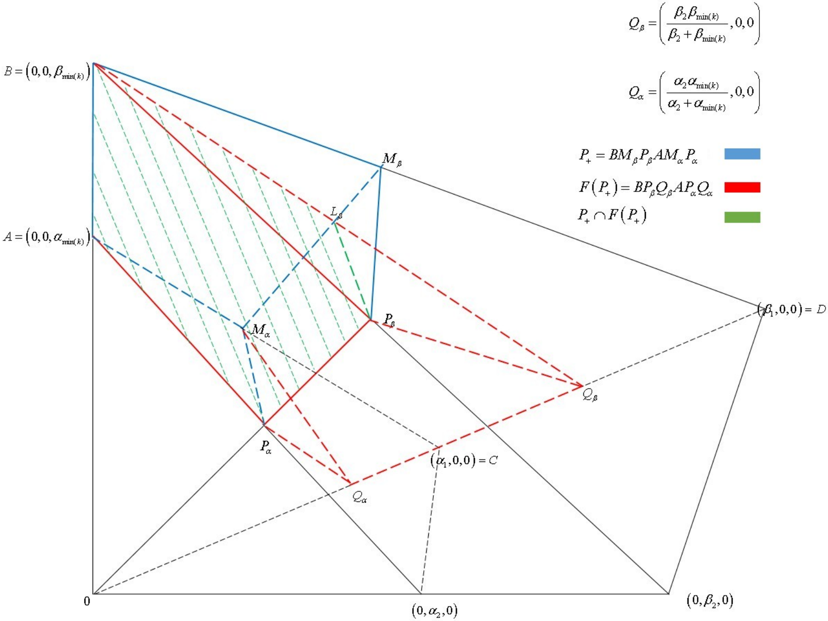
$$(0, 0, \dots, 0, \overbrace{\beta_{min(k)}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0),$$

$$(\overbrace{\frac{\alpha_1\alpha_{min(k)}}{\alpha_1 + \alpha_{min(k)}}}^{\textbf{first coordinate}}, 0, \dots, 0, 0), \qquad (\overbrace{\frac{\beta_1\beta_{min(k)}}{\beta_1 + \beta_{min(k)}}}^{\textbf{first coordinate}}, 0, \dots, 0, 0),$$

$$(0, 0, \dots, 0, \overbrace{\frac{\alpha_i\alpha_{min(k)}}{\alpha_i + \alpha_{min(k)}}}^{i\textbf{-th coordinate}}, 0, \dots, 0, \overbrace{\frac{\alpha_i\alpha_{min(k)}}{\alpha_i + \alpha_{min(k)}}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0) \text{ if } i \neq 1, min(k),$$

$$(0, 0, \dots, 0, \overbrace{\frac{\beta_i\beta_{min(k)}}{\beta_i + \beta_{min(k)}}}^{i\textbf{-th coordinate}}, 0, \dots, 0, \overbrace{\frac{\beta_i\beta_{min(k)}}{\beta_i + \beta_{min(k)}}}^{min(k)\textbf{-th coordinate}}, 0, \dots, 0, 0) \text{ if } i \neq 1, min(k).$$

Sets P_+ and $F(P_+)$ for Selmer algorithm (case $n = 3$)



Set P_+ for Selmer algorithm

We want to prove that for a sufficiently large N the inequality

$$\frac{\beta_1 \beta_{min(k)}}{\beta_1 + \beta_{min(k)}} > \frac{\beta_1 + \alpha_1}{2}. \qquad (3)$$

holds and the inequality (2) holds as a consequence.
4. For a sufficiently large N and small ρ, from the inequalities (1) and (2) we get that

$$\mu(F^k(X) \cap P_+ \cap F(P_+)) \overset{(1),(2)}{\geqslant} (\frac{1}{2} + t - 3\varepsilon) \cdot \mu(P_+) > \frac{1}{2}\mu(P_+) \geqslant$$
$$\geqslant \frac{1}{2}\mu(P_+ \cap F(P_+)). \qquad (4)$$

The map F preserves the Lebesgue measure, so from the inequality (1) follows the inequality

$$\mu(F^{k+1}(X) \cap F(P_+)) \geqslant (1 - 3\varepsilon)\mu(F(P_+)), \qquad (5)$$

so, as a consequence,

$$\mu(F^{k+1}(X) \cap P_+ \cap F(P_+)) \overset{(5),(2)}{\geqslant} (\frac{1}{2} + t - 3\varepsilon) \cdot \mu(F(P_+)) > \frac{1}{2}\mu(F(P_+)) \geqslant$$
$$\geqslant \frac{1}{2}\mu(P_+ \cap F(P_+)). \qquad (6)$$

It follows from the inequalities (4) and (6) that

$$\mu(F^k(X) \cap F^{k+1}(X)) > 0. \qquad \blacksquare$$

Set P_+ for Brun algorithm

Consider the Brun algorithm.
1. Let $P_+ = \{x \in P : x_1$ and x_2 - the largest and second largest coordinates of the vector $x\}$.
Vertices of the set P_+ ($2n$ vertices):

$$(\alpha_1, 0, \dots, 0, 0),$$

$$(\beta_1, 0, \dots, 0, 0),$$

$$(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, 0 \dots, 0, 0),$$

$$(\frac{\beta_1\beta_2}{\beta_1+\beta_2}, \frac{\beta_1\beta_2}{\beta_1+\beta_2}, 0 \dots, 0, 0),$$

$$(\frac{2\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_1\alpha_2}, \frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_1\alpha_2}, 0, \dots, 0, \overbrace{\frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_1\alpha_2}}^{i\textbf{-th coordinate}}, 0, \dots, 0) \text{ if } i > 2,$$

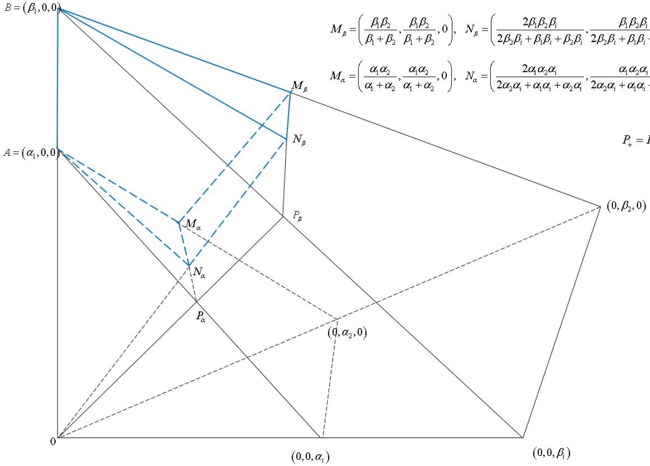
$$(\frac{2\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_1\beta_2}, \frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_1\beta_2}, 0, \dots, 0, \overbrace{\frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_1\beta_2}}^{i\textbf{-th coordinate}}, 0, \dots, 0) \text{ if } i > 2.$$

Set P_+ for Brun algorithm (case $n = 3$)

$$M_\beta = \left(\frac{\beta_1\beta_2}{\beta_1+\beta_2}, \frac{\beta_1\beta_2}{\beta_1+\beta_2}, 0\right), \quad N_\beta = \left(\frac{2\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}, \frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}, \frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}\right), \quad P_\beta = \left(\frac{\beta_1\beta_i}{\beta_1+\beta_i}, 0, \frac{\beta_1\beta_i}{\beta_1+\beta_i}\right)$$

$$M_\alpha = \left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, 0\right), \quad N_\alpha = \left(\frac{2\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}, \frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}, \frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}\right), \quad P_\alpha = \left(\frac{\alpha_1\alpha_i}{\alpha_1+\alpha_i}, 0, \frac{\alpha_1\alpha_i}{\alpha_1+\alpha_i}\right)$$

$P_+ = BM_\beta N_\beta AM_\alpha N_\alpha$



Set P_+ for Brun algorithm

Now we can estimate the measure of the set P_+:

$$\frac{\mu(P_+)}{\mu(P)} = \left(\frac{N}{N+2}\right) \cdot \left(\frac{N^2}{N^2+2N+8}\right)^{n-2} \to 1 \text{ for } N \to \infty.$$

Therefore, for sufficiently large N, the following inequality holds:

$$\mu(F^k(X) \cap P_+) \geqslant (1-3\varepsilon)\mu(P_+). \qquad (7)$$

2. The only coordinate P_+ that changes when we apply F is x_1. Brun algorithm acts on a pair of (x_2, x_1) as Euclidean algorithm.
3. Now we want prove that for sufficiently large N the inequality

$$\mu(P_+ \cap F(P_+)) > \frac{1}{2}\mu(P_+) = \frac{1}{2}\mu(F(P_+)) \qquad (8)$$

holds. It follows from the inequalities

$$\frac{\beta_1\beta_2}{\beta_1+\beta_2} > \frac{1}{\sqrt[n-1]{2}}\beta_2 + \left(1 - \frac{1}{\sqrt[n-1]{2}}\right)\alpha_2, \qquad (9)$$

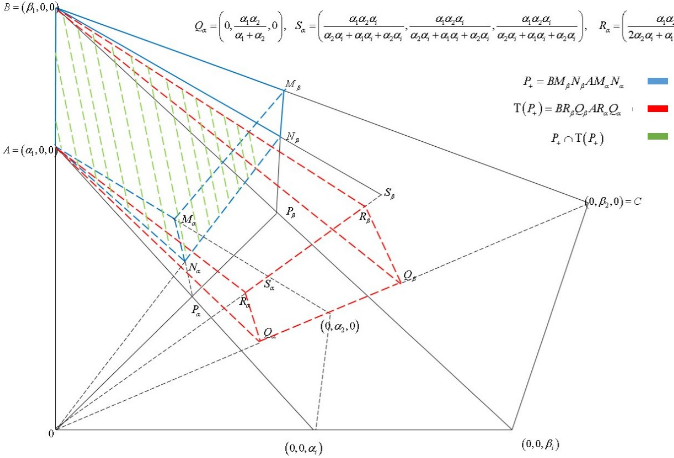
$$\frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_1\beta_2} > \frac{1}{\sqrt[n-1]{2}} \cdot \frac{\beta_1\beta_2\beta_i}{\beta_2\beta_i+\beta_1\beta_i+\beta_1\beta_2} + \left(1-\frac{1}{\sqrt[n-1]{2}}\right) \cdot \frac{\alpha_1\alpha_2\alpha_i}{\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_1\alpha_2} \quad (i>2). \qquad (10)$$

The end of the proof is the same as in the case of Selmer. ∎

Sets P_+ and $F(P_+)$ for Brun algorithm (case $n = 3$)

$$Q_\beta = \left(0, \frac{\beta_1\beta_2}{\beta_1+\beta_2}, 0\right), \quad S_\beta = \left(\frac{\beta_1\beta_2\beta_i}{\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}, \frac{\beta_1\beta_2\beta_i}{\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}, \frac{\beta_1\beta_2\beta_i}{\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}\right), \quad R_\beta = \left(\frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}, \frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}, \frac{\beta_1\beta_2\beta_i}{2\beta_2\beta_i+\beta_1\beta_i+\beta_2\beta_i}\right)$$

$$Q_\alpha = \left(0, \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}, 0\right), \quad S_\alpha = \left(\frac{\alpha_1\alpha_2\alpha_i}{\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}, \frac{\alpha_1\alpha_2\alpha_i}{\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}, \frac{\alpha_1\alpha_2\alpha_i}{\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}\right), \quad R_\alpha = \left(\frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}, \frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}, \frac{\alpha_1\alpha_2\alpha_i}{2\alpha_2\alpha_i+\alpha_1\alpha_i+\alpha_2\alpha_i}\right)$$



Set P_+ for Jacobi-Perron algorithm

Consider the Jacobi-Perron algorithm (tagged version).
1. Let $P_+ = \{x \in P : x_1$ - the smallest coordinate of the vector $x\}$.
Vertices of the set P_+ (n vertices):

$$(0, \dots, 0, \alpha_i, 0, \dots, 0) \quad \text{if } i > 1,$$

$$(0, \dots, 0, \beta_i, 0, \dots, 0) \quad \text{if } i > 1,$$

$$\left(\frac{1}{\frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_n}}, \dots, \frac{1}{\frac{1}{\alpha_1} + \dots + \frac{1}{\alpha_n}} \right),$$

$$\left(\frac{1}{\frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}}, \dots, \frac{1}{\frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}} \right).$$

Now we can estimate the measure of the set P_+:

$$\frac{\mu(P_+)}{\mu(P)} \geqslant \frac{N}{N + 2 \cdot (n-1)} \to 1 \text{ for } N \to \infty. \quad (\delta = \tfrac{m+1}{m})$$

Therefore, for sufficiently large N, the following inequality holds:

$$\mu(\mathcal{S}^k(X) \cap P_+) \geqslant (1 - 3\varepsilon)\mu(P_+). \qquad (11)$$

Set P_+ for Jacobi-Perron algorithm

2. The only coordinate P_+ that changes when we apply $\mathcal{S}$ is x_2-the smallest coordinate of the vector x after x_1. Tagged Jacobi-Perron algorithm acts on a pair of (x_1, x_2) as Euclidean algorithm.
3. Now we want prove that for sufficiently large N the inequality

$$\mu(P_+ \cap \mathcal{S}(P_+)) > \frac{1}{2}\mu(P_+) = \frac{1}{2}\mu(\mathcal{S}(P_+)). \qquad (12)$$

holds. it follows from the inequalities

$$\frac{1}{\frac{1}{\beta_1} + \ldots + \frac{1}{\beta_n}} > \frac{1}{n - \sqrt[1]{2}} \cdot \frac{1}{\frac{1}{\beta_1} + \frac{1}{\beta_3} + \ldots + \frac{1}{\beta_n}} + \left(1 - \frac{1}{n - \sqrt[1]{2}}\right) \cdot \frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_3} + \ldots + \frac{1}{\alpha_n}}$$

4. As in Selmer's and Brun's cases, from the inequalities (11) and (12) it follows that

$$\mu\big(\mathcal{S}^k(X) \cap \mathcal{S}^{k+1}(X)\big) > 0.$$

This means that tagged Jacobi-Perron algorithm satisfies the intersection property, so it is exact with respect to the Lebesgue measure. ■

Set P_+ for Jacobi-Perron algorithm

Thus, we get that the proof of T.Miernowski and A.Nogueira generalizes to the multidimensional case. to do this, we needed to re-prove the auxiliary lemma, as well as to carry out technically complex constructions of sets P_+ for the Selmer, Brun and Jacobi-Perron algorithms. Therefore, the statement about the possibility of generalizing the proof from [1] is true. Also, in order to be able to consider the algorithms as ergodic and use their infinite-partial balance, we conceptually used the paper by J. Chaika and A. Nogueira (see [7]), which deals with non-homogeneous MCF algorithms.

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