Escape of Mass of the Thue Morse Sequence

(joint with Erez Nesharim)

Noy Soffer Aranov

Technion

One World Numeration

Continued Fractions

Given a quadratic irrational $\Theta \in \mathbb{R}$, we write

$$\Theta = b_0 + \cfrac{1}{b_1 + \cfrac{1}{\ddots + \cfrac{1}{b_k + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_\ell + \cfrac{1}{a_1 + \cfrac{1}{\ddots}}}}}}}} := [b_0; b_1, \ldots, b_k, \overline{a_1, \ldots, a_\ell}].$$

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Given p prime, how does the continued fraction expansion of $p^k\Theta$ change as $k \to \infty$?

Some Known Results

Given a prime p, write the c.f.e. of $p^k\Theta$ as

$$p^k\Theta = [b_{k,0}; b_{k,1}, \ldots, b_{k,m_k}, \overline{a_{k,1}, \ldots, a_{k,\ell_k}}].$$

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2. For every Θ, we have $\lim_{k\to\infty} \max_{i=1,\ldots\ell_k} |a_{k,i}| = \infty$.
3. For every Θ, there is no escape of mass of $p^k\Theta$, that is

$$\lim_{k\to\infty} \frac{\max_{i=1,\ldots,\ell_k} \log |a_{k,i}|}{\sum_{i=1}^{\ell_k} \log |a_{k,i}|} = 0.$$

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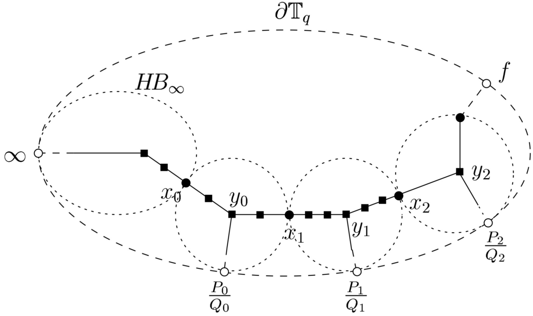
2. Then, connect between the upper half plane model and lattices in $\mathrm{PGL}_2(\mathbb{R})/\,\mathrm{PGL}_2(\mathbb{Z})$.
3. Then, escape of mass of the c.f.e. corresponds to escape of mass of Haar measures supported on a sequence of compact, nested diagonal orbits.



Hecke Friends

Setting

$$G_\infty = \mathrm{PGL}_2(\mathbb{R}), \Gamma_\infty = \mathrm{PGL}_2(\mathbb{Z}), X_\infty = G_\infty / \Gamma_\infty,$$
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Definition (Hecke Friends)

Let p be a prime. We say that two lattices $\Lambda_1, \Lambda_2 \subseteq X_\infty$ are Hecke neighbors if $\Lambda_2 \subseteq \Lambda_1$ and $\Lambda_1/\Lambda_2 \cong \mathbb{Z}/p\mathbb{Z}$.

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If Λ_0 is a lattice then, the set of Hecke friends $\Lambda \subseteq \Lambda_0$ forms a $p+1$ regular tree $T_p(\Lambda_0)$.

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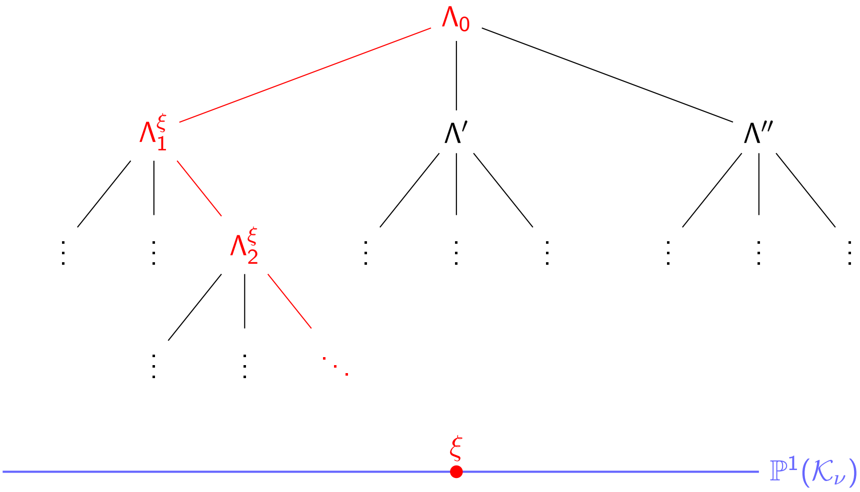
3. $A\Lambda_0$ supports an A invariant probability measure $\mu_{A\Lambda_0}$.
4. Thus, every node in the Hecke tree Λ supports an A invariant probability measure $\mu_{A\Lambda}$.

Therefore, we can study the evolution of nested number fields or c.f.e's by studying the distribution of these measures $\mu_{A\Lambda}$ when Λ is taken along a branch in the tree $T_p(\Lambda_0)$.

The Hecke Tree $T_p(\Lambda_0)$



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We say that probability measures $\{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X_\infty)$ exhibit c escape of mass if every accumulation point $\mu = \lim_{k\to\infty} \mu_{n_k}$ satisfies $\mu(X_\infty) \leq 1 - c$.

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Theorem (Aka-Shapira 2018)

All rational branches in the Hecke tree $T_p(\Lambda_0)$ do not exhibit escape of mass. Furthermore, the measures μ_n^ξ taken along rational branches equidistribute.

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Question

What happens when we replace $\mathbb{R}$ with $\mathbb{F}_q((t^{-1}))$?

Function Field Setting

Let

$$\mathbb{F}_q[t] = \left\{ \sum_{n=0}^{N} a_n t^n : a_n \in \mathbb{F}_q, N \in \mathbb{N} \right\},$$

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Lemma

$\Theta \in$ QI if and only if the c.f.e. of Θ is periodic.

Question

How does the c.f.e. of $t^k\Theta$ distribute?

Dynamics of CFE's in Function Fields

Let the c.f.e. of $t^k\Theta$ be

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2. $\ell_k = O_\Theta(k)$ (Kemarsky-Paulin-Shapira ,Paulin-Shapira 2018)
3. (Paulin-Shapira 2018) For every $\Theta \in \mathrm{QI}$, there exists $c = c(\Theta) > 0$ such that for every $N > 0$

$$\liminf_{n\to\infty} \frac{\max_{i=1,\dots,\ell_k} \max\{\deg(a_{k,i}) - N, 0\}}{\sum_{i=1}^{\ell_k} \deg(a_{k,i})} \geq c.$$

This is called c-**escape of mass**.

Idea of Proof

Method

- Again use Hecke trees $\mathbb{T}_t(\Lambda_\Theta)$ in $\mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))/\,\mathrm{PGL}_2(\mathbb{F}_q[t])$ and study the escape of mass of measures taken along the branches.

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- To study this escape of mass study the evolution of the length of the shortest vector in $a_t^n\Lambda_\Theta$, for $a_t = \begin{pmatrix} t & \\ & 1 \end{pmatrix}$.

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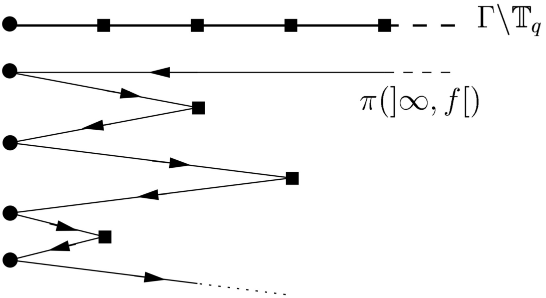
- Use Paulin's results to connect this to the c.f.e.

Escape of Mass in Positive Characteristic



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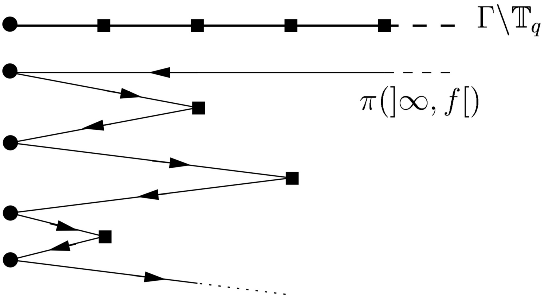

Conjecture (Kemarsky-Paulin-Shapira/Paulin-Shapira 2018)

For every $\Theta \in \mathbf{QI}$, Θ exhibits full escape of mass, that is we can take $c = 1$.

We Disprove This Conjecture!

Thue-Morse Sequence

Define the substitution $0 \mapsto 01$ and $1 \mapsto 10$.

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Properties of Θ

1. Θ_n is the number of 1's in the binary expansion of $n-1$ modulo 2.

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2. Write $\Theta = \sum_{n=1}^{\infty} \Theta_n t^{-n} \in \mathbb{F}_2((t^{-1}))$. Then,

$$(t^2+1)(t+1)\Theta^2 + t(t^2+1)\Theta + t^2 = 0.$$

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In particular Θ is quadratic.

Main Results

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Let $\Theta = \sum_{n=1}^{\infty} \Theta_n t^{-n} \in \mathbb{F}_2((t^{-1}))$. Then,

1. Θ has $\frac{2}{3}$-escape of mass exactly, i.e. for every $N > 0$, we have

$$\liminf_{k\to\infty} \frac{\sum_{i=1}^{\ell_k} \max\{\deg(a_{k,i}) - N, 0\}}{\sum_{i=1}^{\ell_k} \deg(a_{k,i})} = \frac{2}{3}.$$

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2. Moreover, for every $N > 0$, we have

$$\limsup_{k\to\infty} \frac{\sum_{i=1}^{\ell_k} \max\{\deg(a_{k,i}) - N, 0\}}{\sum_{i=1}^{\ell_k} \deg(a_{k,i})} = \frac{2}{3} + \frac{2}{9}$$

Hankel Matrices and Flats

Given $\{\Theta_n\}_{n\in\mathbb{N}} \subseteq \mathbb{F}_q$, we define

$$H_\Theta(k;m) = \begin{pmatrix} \Theta_k & \Theta_{k+1} & \cdots & \Theta_{k+m} \\ \Theta_{k+1} & \Theta_{k+2} & \cdots & \Theta_{k+m+1} \\ \vdots & \ddots & \cdots & \vdots \\ \Theta_{k+m} & \Theta_{k+m+1} & \cdots & \Theta_{k+2m} \end{pmatrix}.$$

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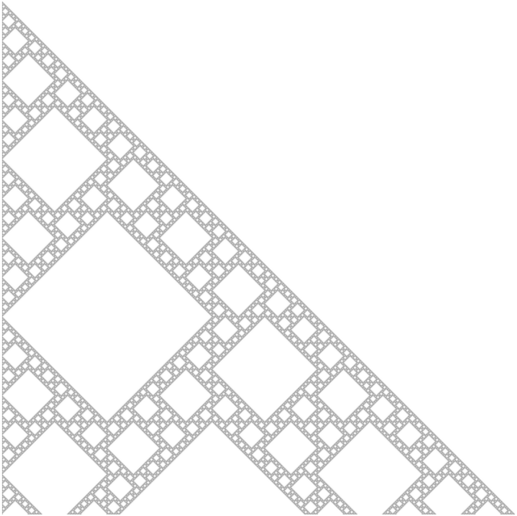
Define the flat of Θ as

$$\mathfrak{F}_{m,n}(\Theta) = \begin{cases} 0 & n > m+1 \\ 1 & n = m+1 \\ 0 & m = n \mod 2 \\ \det H_\Theta\left(\frac{m-n}{2}, \frac{m+n}{2}\right) & m > n \text{ and } m \neq n \mod 2 \end{cases}.$$

Flat of the Thue Morse Sequence



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Corollary

Let $\Theta = \sum_{n=1}^{\infty} \Theta_n t^{-n} \in \mathbf{QI}$ and let $t^k \Theta = [0; \overline{a_{k,1}, \dots, a_{k,\ell_k}}]$. Let $h_{k,j}$ be the j-th non-zero coordinate in the k-th column of $\mathfrak{F}$. Then, Θ exhibits exactly c-escape of mass if and only if for every $N > 0$, we have

$$\liminf_{k \to \infty} \frac{\max_{j=1,\dots,\ell_k - 1} \max\{h_{k,j+1} - h_{k,j} - N, 0\}}{\sum_{j=1}^{\ell_k - 1} (h_{k,j+1} - h_{k,j})} = c.$$

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In other words, escape of mass of the c.f.e. corresponds to the proportion that the larges distances between non-zero coordinates in the column takes up in the period of the column.

Zero Blocks in Flats - Known Results

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Remark

Zero blocks in flats were originally studied to understand the (false) t-adic Littlewood conjecture:

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1. If $\Theta \in \mathbf{QI}$, then, $\mathfrak{F}(\Theta)$ contains zero windows of unbounded size (de Mathan-Teullie).This corresponds to escape of mass for $\{t^k\Theta\}_{k\geq 0}$.
2. There exist flats with zero windows of bounded size (Adiceam-Nesharim-Lunnon 2020, Robertson 2023, Garrett-Robertson 2024)

Remark

Zero blocks in flats were originally studied to understand the (false) t-adic Littlewood conjecture: $\Theta \in \mathbb{F}_q((t^{-1}))$ satisfies this if

$$\inf_{0\neq N\in\mathbb{F}_q[t]} |N|\cdot|N|_t|\langle Nt^k\Theta\rangle| = \inf_{0\neq N\in\mathbb{F}_q[t],k\geq 0} |N|\cdot|\langle Nt^k\Theta\rangle| = 0.$$

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Adiceam-Nesharim-Lunnon proved that Θ satisfies t-adic Littlewood if and only if $\mathfrak{F}(\Theta)$ has zero windows of unbounded size.

Method of Proof

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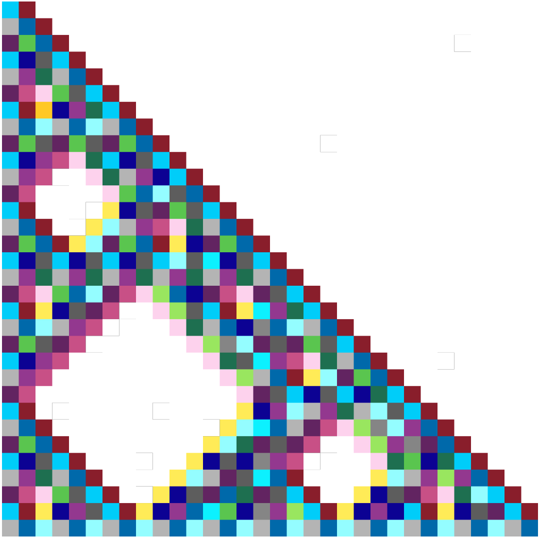
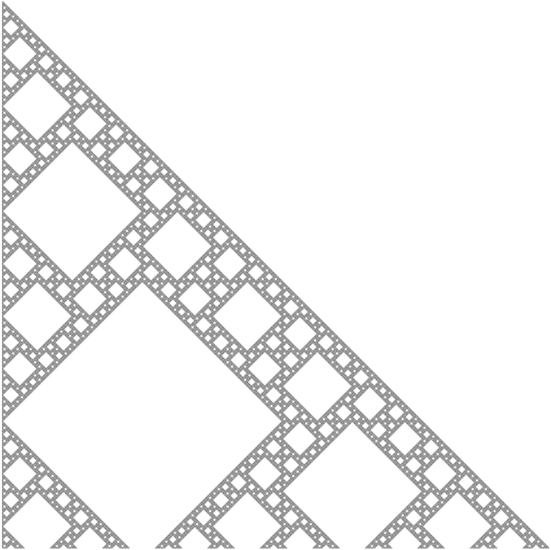
4. Use symmetries and "self similar structure" of the tiling and the coding to prove that that F exhibits $\frac{2}{3}$ escape of mass and at most $\frac{8}{9}$ escape of mass

The Tiling



The Coding



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4. What are the accumulation points of measures taken along a branch in the Hecke tree?
5. What happens in the building of PGL_3?

Thank you very much