On singular substitution $\mathbb{Z}$-actions

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Plan of the talk

- Background: substitution dynamical systems, spectral theory
- Recent results on singularity (in brief)
- Spectral cocycle and statement of result (joint with A. Bufetov)
- About the proof I: reduction, via equidistribution result of B. Host
- About the proof II (if time permits): singularity via local dimension estimates for spectral measures.

Substitutions

- $\mathcal{A}$ is a finite alphabet; $\mathcal{A}^+ = \bigcup_{n=1}^{\infty} \mathcal{A}^n$ (set of nonempty words).
- **Substitution:** $\zeta : \mathcal{A} \to \mathcal{A}^+$, extended to $\mathcal{A}^+$.

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- **Substitution:** $\zeta : \mathcal{A} \to \mathcal{A}^+$, extended to $\mathcal{A}^+$. **A few examples:**
- **Thue-Morse:** $\zeta(0) = 01,\ \zeta(1) = 10.$ Iterate (by concatenation) :

$$0 \to 01 \to 0110 \to 01101001 \to 0110100110010110 \to \ldots$$

- **Fibonacci:** $\zeta(0) = 01, \quad \zeta(1) = 0$; Iterate:

$$0 \to 01 \to 010 \to 01001 \to 01001010 \to 0100101001001 \to \ldots$$

- **Tribonacci:** $\zeta(0) = 012,\ \zeta(1) = 0,\ \zeta(2) = 1.$ Iterate:

$$0 \to 012 \to 01201 \to 012010120 \to 0120101200120101 2 \to \ldots$$

Substitution dynamical system

- **Substitution space:**

$$X_\zeta = \left\{x \in \mathcal{A}^{\mathbb{Z}} : \text{every factor of } x \text{ appears in } \zeta^k(j) \text{ for some } j \in \mathcal{A},\ k \geq 1\right\}$$

- **Shift transformation:** $(Tx)_n = x_{n+1}$, $n \in \mathbb{Z}$.
- $X_\zeta \subset \mathcal{A}^{\mathbb{Z}}$ is closed (in the product topology) and shift-invariant.
- (X_ζ, T_ζ), where $T_\zeta = T|_{X_\zeta}$, is the (topological) substitution dynamical system.

Substitution matrix, primitivity, aperiodicity

- **Substitution matrix:**

$$M = M_\zeta = (m_{ij})_{i,j\in\mathcal{A}}, \quad \text{where} \;\; m_{ij} = \text{number of } i\text{'s in } \zeta(j)$$

  Note that we *ignore* the order of symbols in $\zeta(j)$.

- The substitution is **primitive** if M_ζ^k has all entries > 0 for some $k \geq 1$.

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- We assume that the substitution is **aperiodic** , i.e., there are no periodic sequences in X_ζ. In the primitive case this is equivalent to $\mathrm{card}(X_\zeta) = \infty$. *Primitive aperiodic will be our standing assumptions.*

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- **Basic references:** [M. Queffélec, *Substitution Dynamical Systems — Spectral Analysis*, LNM 1294, 1987, 2010 (2nd edition)]; [Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, ed. by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel, LNM 1794, 2002]

Unique ergodicity

Theorem (P. Michel)

If ζ is a primitive substitution, then (X_ζ, T_ζ) is uniquely ergodic, i.e., it has is a unique invariant Borel probability measure μ.

For a cylinder set $[w]$ ($=$ the set of sequences $x \in X_\zeta$ containing a given word w at a given position) the measure is

$$\mu([w]) = \text{frequency of } w \text{ in } x \in X_\zeta$$

Perron-Frobenius theory shows that such frequencies exist uniformly.

Remark. $(\mu([j]))_{j \in \mathcal{A}}$ is the Perron-Frobenius eigenvector for M_ζ.

Spectral Analysis

Koopman operator:

$$(X_\zeta, T_\zeta, \mu) \ \leadsto \ Uf(x) = f(T_\zeta x) \text{ on } L^2(X_\zeta, \mu)$$

This is a unitary operator on a Hilbert space.

Questions:

- What is the spectral type of U?

Spectral Analysis

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- What is the spectral type of U?
- When does U have nontrivial (nonconstant) eigenfunctions? Describe the discrete component of U (that is, the closed linear span of eigenfunctions). When is the spectrum pure discrete?

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Questions:

- What is the spectral type of U?
- When does U have nontrivial (nonconstant) eigenfunctions? Describe the discrete component of U (that is, the closed linear span of eigenfunctions). When is the spectrum pure discrete?
- Describe the continuous component of the spectrum. When is it singular? Recall that a finite measure on the line decomposes: $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sing}}, \quad \nu_{\mathrm{sing}} = \nu_{\mathrm{sing.cont.}} + \nu_{\mathrm{disc.}}.$

Examples

1. **"Thue-Morse":** $\zeta(0) = 01,\ \zeta(1) = 10$, primitive, constant length 2.

$$M_\zeta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Discrete spectrum on "even functions" w.r.t. $0 \leftrightarrow 1$; singular continuous spectrum on "odd" functions [Kakutani].

2. **"Tribonacci":** $\zeta(0) = 012,\ \zeta(1) = 0,\ \zeta(2) = 1$, primitive, non-constant length.

$$M_\zeta = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Pure discrete spectrum, isomorphic to a translation on $\mathbb{T}^2$ [G. Rauzy].

Examples (cont.)

- **"Rudin-Shapiro":** $\mathcal{A} = \{0, 1, \overline{0}, \overline{1}\}$.

$$\zeta: \; 0 \mapsto 0\overline{1}, \quad 1 \mapsto \overline{0}\overline{1}, \quad \overline{0} \mapsto \overline{0}1, \quad \overline{1} \mapsto 01$$

Discrete spectrum on "even functions" w.r.t. the "bar" operation; Lebesgue spectrum on "odd" functions [Kamae], [Queffélec].

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- There are generalizations, with a Lebesgue component, due to [Frank] and [Chan-Grimm], but all are constant length or isomorphic to constant length.

- [Dekking '78] criterion for pure discrete spectrum of **constant length** substitutions. However, it is an open question, in general, when there is a Lebesgue component.

Nonconstant length substitutions

- A substitution ζ is of Pisot type if its matrix M_ζ has all its eigenvalues, except the largest one, of modulus less than 1.

- **Conjecture:** *every Pisot type substitution system is pure discrete*

  Many special cases are known: in particular, true for $m = 2$.

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- **Known:** Pisot type $\Longrightarrow$ the substitution system has a discrete component isomorphic to an irrational translation on the torus $\mathbb{T}^{m-1}$.

- **Question:** what about non-Pisot substitutions? When do they have pure singular spectrum? (Singular component is always present.)

Recent work on self-similar substitution flows

- Along with the substitution $\mathbb{Z}$-action (X_ζ, T_ζ) one can consider the **substitution flow** – $\mathbb{R}$-action on the tiling space.
- The tilings are on the line $\mathbb{R}$, with the **tiles = line segments** of fixed length $s_j > 0$ and a label j for each $j \in \mathcal{A}$. To each $x \in X_\zeta$ we associate a tiling $\mathcal{T}_x$ – replace every symbol x_i by the tile with the same label; the tile corresponding to x_0 has 0 as its left endpoint.
- The space is the **"hull"** – the closure of $\{\mathcal{T}_x - t : \ t \in \mathbb{R}\}$ in the natural topology. The flow acts by translation.

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- The space is the **"hull"** – the closure of $\{\mathcal{T}_x - t : \ t \in \mathbb{R}\}$ in the natural topology. The flow acts by translation.
- **Self-similar flow:** choose $(s_j)_{j\in\mathcal{A}} =$ PF eigenvector for M_ζ^t.
- Pure singular spectrum has been recently confirmed for a class of self-similar flows: $\boxed{0 \mapsto 0111, \ 1 \mapsto 0}$ by [Baake, Frank, Grimm, and Robinson (2019)], extensions by Baake, Gähler, Grimm, and Mañibo.
- However, pure singular spectrum for $\mathbb{Z}$-actions doesn't follow easily in the **non-constant length** case.

Spectral cocycle (joint work with A.I. Bufetov)

Matrix-function associated with a substitution

- Let ζ be a substitution on $\mathcal{A} = \{1, \ldots, m\}$.
- For a word $w = w_1 \ldots w_k$ in alphabet $\mathcal{A}$ let $\mathcal{Z}(w) = z_{w_1} \cdots z_{w_k}$, where $z_1, \ldots, z_m$ are free commuting variables.
- Define $m \times m$ matrix-function $\widetilde{\mathcal{M}}_\zeta$ with polynomial entries:

$$\widetilde{\mathcal{M}}_\zeta(z_1, \ldots, z_m)_{(b,c)} := \sum_{j:\ \zeta(b)_j = c} \mathcal{Z}\big(\mathrm{Pref}_j \zeta(b)\big),$$

where $\mathrm{Pref}_j \zeta(b)$ is the prefix of $\zeta(b)$, that is, $\zeta(b)[1, j-1]$ (empty if $j = 1$). Now let

$$\mathcal{M}_\zeta(\xi) = \mathcal{M}_\zeta(\xi_1, \ldots, \xi_m) = \widetilde{\mathcal{M}}_\zeta(z_1, \ldots, z_m) \ \text{ where } z_j = e^{-2\pi i \xi_j}.$$

Matrix-function (cont.)

Example: $\mathcal{A} = \{1,2,3\}$, $\zeta:\ 1 \mapsto 1213,\ 2 \mapsto 232,\ 3 \mapsto 1332.$

$$\widetilde{\mathcal{M}}_\zeta(z_1,z_2,z_3) = \begin{bmatrix} 1+z_1z_2 & z_1 & z_1z_2z_1 \\ 0 & 1+z_2z_3 & z_2 \\ 1 & z_1z_3^2 & z_1+z_1z_3 \end{bmatrix}$$

Matrix-function (cont.)

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Properties:

1. $\mathcal{M}_\zeta(\xi) = \widetilde{\mathcal{M}}_\zeta(e^{-2\pi i\xi_1},\dots,e^{-2\pi i\xi_m})$ is $\mathbb{Z}^m$-periodic, consider it on the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$.
2. $\mathcal{M}_\zeta(\mathbf{0}) = M_\zeta^t$.
3. all the coefficients are 0's and 1's; ζ is determined by $\mathcal{M}_\zeta$.
4. **Cocycle property:** if ζ_1 and ζ_2 are two substitutions on $\mathcal{A}$, then

$$\mathcal{M}_{\zeta_1\circ\zeta_2}(\xi) = \mathcal{M}_{\zeta_2}(M_{\zeta_1}^t\xi)\cdot\mathcal{M}_{\zeta_1}(\xi).$$

Cocycle over the toral endomorphism

- Assume that $\det M_\zeta \neq 0$ and consider the toral endomorphism:

$$E_\zeta : \xi \mapsto M_\zeta^t \xi \pmod{\mathbb{Z}^m}.$$

Then

$$\mathcal{M}_\zeta(\xi, n) := \mathcal{M}_\zeta\big(E_\zeta^{n-1}\xi\big) \cdot \ldots \cdot \mathcal{M}_\zeta(\xi) = \mathcal{M}_{\zeta^n}(\xi)$$

is a complex matrix cocycle over $(\mathbb{T}^m, E_\zeta)$.

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- A related matrix product (cocycle over $\omega \mapsto \theta\omega$ on $\mathbb{R}$) was considered by [Baake et al.]:

$$\mathcal{C}_\zeta(\omega, n) = \mathcal{M}_\zeta(\omega\vec{s}, n), \quad \omega \in \mathbb{R}, \quad \text{where } M_\zeta^t \vec{s} = \theta\vec{s},$$

which corresponds to the self-similar substitution tiling flow. Here θ = PF eigenvalue of M_ζ.

Lyapunov exponents

- If E_ζ is **ergodic**, i.e. M_ζ has no eigenvalues roots unity, then by [Furstenberg-Kesten], there is **Lyapunov exponent**

$$\chi_\zeta := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{M}_{\zeta^n}(\xi)\|,$$

where the limit exists for Lebesgue (Haar) a.e. $\xi \in \mathbb{T}^m$.

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- One can consider the **local upper Lyapunov exponent** at any $\xi \in \mathbb{T}^m$:

$$\chi_\zeta^+(\xi) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{M}_{\zeta^n}(\xi)\|,$$

or $\chi_\zeta(\xi)$ if the limit exists. Note $\chi_\zeta(\mathbf{0}) = \log\theta$.

Statement of result

Theorem (A. Bufetov, B. S.)

Suppose that ζ is a primitive aperiodic substitution, and the substitution matrix M_ζ has irreducible over $\mathbb{Q}$ characteristic polynomial. Let θ be the PF eigenvalue of M_ζ. If

$$\chi_\zeta < \frac{1}{2}\log\theta, \tag{1}$$

then the substitution $\mathbb{Z}$-action (X_ζ, T_ζ, μ) has pure singular spectrum.

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Remarks: 1. We do not know if this condition is necessary.

2. One can show that $\chi_\zeta \le \frac{1}{2}\log\theta$. A nice direct proof following [Baake, Gähler, Mañibo (2019)].

How to check; examples

Lemma

We have

$$\chi_\zeta = \lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{T}^m} \log \|\mathcal{M}_{\zeta^k}(\xi)\| \, d\xi = \inf_{k\ge 1} \frac{1}{k} \int_{\mathbb{T}^m} \log \|\mathcal{M}_{\zeta^k}(\xi)\| \, d\xi$$

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Examples: We obtain singular spectrum for all examples of [Baake et al.], using their calculations, and for many others, by hand, e.g.:

$$\boxed{0 \mapsto 0^m 1,\ \ 1 \mapsto 012,\ \ 2 \mapsto 1},\ m \ge 12.$$

References

[1] Bufetov, Alexander I.; Solomyak, Boris. On singular substitution $\mathbb{Z}$-actions. *Preprint*: arXiv:2003.11287

Builds on earlier work:

[2] Bufetov, Alexander I.; Solomyak, Boris. On the modulus of continuity for spectral measures in substitution dynamics. *Adv. Math.* **260** (2014), 84–129.

[3] Bufetov, Alexander I.; Solomyak, Boris. A spectral cocycle for substitution systems and translation flows. *J. d'Analyse*, to appear; *Preprint* arXiv: 1802.04783.

About the proof

Theorem ([2,3])

Let $\vec{1} = (1, \ldots, 1)^t$. *If for Lebesgue-a.e.* $\omega \in \mathbb{R}$,

$$\chi_\zeta^+(\omega\vec{1}) = \limsup_{n\to\infty} \frac{1}{n} \log \|\mathcal{M}_\zeta(\omega\vec{1}, n)\| < \frac{\log\theta}{2}, \tag{2}$$

then the substitution $\mathbb{Z}$*-action* (X_ζ, T_ζ, μ) *has pure singular spectrum.*

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Condition (2) looks very similar to (1): $\chi_\zeta < \frac{1}{2}\log\theta$.

Problem: the Lyapunov exponent χ_ζ exists a.e. on the torus $\mathbb{T}^m$, but the diagonal $\{\omega\vec{1} : \ \omega \in \mathbb{R}\}$ has zero Haar measure!

About the proof (cont.)

Equidistribution results of [Meiri '98], [Host '00] come to the rescue!

Theorem (Host)

- $A = (m \times m)$ *integer matrix, such that for every* $r > 0$, *the characteristic polynomial of* A^r *is irreducible over* $\mathbb{Q}$.
- $B = (m \times m)$ *integer matrix, such that all eigenvalues of* B *have modulus* > 1, *and* $q = |\det(B)|$ *is relatively prime with* $\det(A)$.
- $\nu =$ *probability measure on* $\mathbb{T}^m$ *is invariant, ergodic and of positive entropy for* $\mathbf{t} \mapsto B\mathbf{t}$ *(mod* $\mathbb{Z}^m$*).*

Then the sequence $\{A^n\mathbf{t}\}_{n\geq 0}$ *is equidistributed on* $\mathbb{T}^m$ *for* ν*-a.e.* $\mathbf{t} \in \mathbb{T}^m$.

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Recall: a sequence $\{\mathbf{t}_n\}_{n\ge 1}$ is **equidistributed** on $\mathbb{T}^m$ if for any $f \in \mathcal{C}(\mathbb{T}^m)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{t}_n) = \int_{\mathbb{T}^m} f(\xi)\, d\xi.$$

About the proof (cont.)

- In our application, $A = M_\zeta^t$, which defines the endomorphism E_ζ. The characteristic polynomial of M_ζ is irreducible, in our case this implies irreducibility for all powers.
- $\nu =$ Lebesgue measure on the diagonal $(x, \dots, x)$; it is invariant and ergodic, with positive entropy, for $B = pI$ (I is the identity matrix), with $p \geq 2$, which is just the time-p map on the diagonal.
- It is trivial to satisfy the condition $GCD(\det A, \det B) = 1$ since $p \geq 2$ can be taken arbitrary.
- The conclusion is that for Lebesgue-a.e. ω, the sequence $\{E_\zeta^n(\omega\vec{1})\}$ is uniformly distributed on the torus $\mathbb{T}^m$.
- This makes it possible to apply the earlier theorem in singularity. $\square$

Spectral measures

- Given a measure-preserving system (Y, T, μ) and $f \in L^2(Y, \mu)$, the **spectral measure** is defined by

$$\widehat{\sigma}_f(-k) = \int_0^1 e^{2\pi i k\omega}\, d\sigma_f(\omega) = \langle f \circ T^k, f\rangle, \quad k \in \mathbb{Z}.$$

- If f is an eigenfunction, $f \circ T = e^{2\pi i \omega} f$ in L^2, then $\sigma_f = \|f\|_2^2 \cdot \delta_\omega$.
- (Y, T, μ) has pure singular spectrum if $\sigma_f \perp \mathcal{L}$ for all $f \in L^2$.

Spectral measures II

For (X_ζ, T_ζ, μ) the most basic case is when f depends only on the 0-th symbol: $f(x) = \phi(x_0)$. For a word $v = v_0 v_1 \ldots \in \mathcal{A}^+$ let

$$\Phi_f(v, \omega) = \sum_{n=0}^{|v|-1} \phi(v_n) e^{-2\pi i \omega n}.$$

Then one can show (see e.g. [Queffélec, 4.9]) that for any $x \in X_\zeta$ and $a \in \mathcal{A}$,

$$\begin{aligned} \sigma_f &= \text{weak*-} \lim_{N\to\infty} \frac{1}{N} |\Phi_f(x[0, N-1]|^2)\, d\omega \\ &= \text{weak*-} \lim_{n\to\infty} \frac{1}{|\zeta^n(a)|} |\Phi_f(\zeta^n(a), \omega)|^2\, d\omega \end{aligned}$$

(the limit exists and is independent of $x \in X_\zeta$ and a).

Spectral cocycle (a.k.a generalized matrix Riesz product)

- For *constant length* substitutions, M. Queffélec expressed the spectral measures using *generalized Riesz products* (sometimes, scalar; usually, matrix). In the TM case, with $f(x) = \phi(x_0),\ \phi(0) = 1, \phi(1) = -1$:

$$\sigma_f = \text{weak*-}\lim_{n\to\infty} 2^{-n} \left| \prod_{k=0}^{n-1} (1 - e^{-2\pi i \omega 2^k}) \right|^2 d\omega.$$

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Theorem (A. Bufetov, B.S. 2014)

For a primitive aperiodic substitution holds

$$\Sigma_\zeta = (\sigma_{a,b})_{a,b\in\mathcal{A}} = \text{weak*-}\lim_{n\to\infty} C\cdot\theta^{-n}\cdot\overline{\mathcal{M}^*_\zeta(\omega\vec{1},n)\cdot\mathcal{M}_\zeta(\omega\vec{1},n)}\,d\omega,$$

where Σ_ζ is the matrix of spectral (correlation) measures.

Key Lemma: local spectral estimates

Lemma

Let $(u_n)_{n=1}^{\infty}$ *be a substitution sequence,* $\omega \in [0,1]$*. Suppose that*

$$\left|\sum_{n=1}^{N} \phi(u_n) e^{-2\pi i \omega n}\right| \le C N^{1-\delta}, \; N \in \mathbb{N}. \tag{3}$$

Then

$$\sigma_f(\omega - r, \omega + r) \le C' r^{2\delta}, \; r > 0, \quad \text{where } f(x) = \phi(x_0).$$

Key Lemma: local spectral estimates

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Rough idea:

- "Best" $\delta = 0$ at a given ω corresponds to an eigenvalue;
- $\delta = 1/2$ for a.e. ω corresponds to Lebesgue measure;
- $\delta > 1/2$ for a.e. ω corresponds to a singular measure.

Remarks

- Estimates of type (3) have been studied earlier in special cases, with the focus on **all** ω.
- for Rudin-Shapiro, get (3) with $\delta = 1/2$, **for all** ω.
- (3) is impossible with $\delta > 1/2$ **for all** ω, but can happen **for a.e.** ω.
- for Thue-Morse get (3) with $\delta = \frac{1}{4}\log_2 \frac{27}{16}$, **for all** ω [Queffélec 8.1].

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- assuming $\int f\, d\mu = 0$, the inequality (3) holds at $\omega = 0$ with

$$1 - \delta = \frac{\log|\theta_2|}{\log \theta},$$

  where θ_2 is the 2nd eigenvalue of M_ζ in absolute value [Dumont-Thomas].

Thank You!