Primes as sums of Fibonacci numbers

Lukas Spiegelhofer (MU Leoben, Austria)
Joint work with Michael Drmota and Clemens Müllner (TU Wien, Austria)

October 11, 2022
OWNS

The Zeckendorf expansion and the Fibonacci word

The greedy expansion of a nonnegative integer into a sum of Fibonacci numbers yields the Zeckendorf expansion. No adjacent 1 appear in such an expansion, and it is the unique representation as a sum of distinct Fibonacci numbers with this property.

0	0	0	8	10000	1	16	100100	2
1	1	1	9	10001	2	17	100101	3
2	10	1	10	10010	2	18	101000	2
3	100	1	11	10100	2	19	101001	3
4	101	2	12	10101	3	20	101010	3
5	1000	1	13	100000	1	21	1000000	1
6	1001	2	14	100001	2	22	1000001	2
7	1010	2	15	100010	2	23	1000010	2

The least significant digit is given by the Fibonacci word $0 \mapsto 01$, $1 \mapsto 0$.
The Fibonacci substitution is compatible with the "Zeckendorf shift"!

The Zeckendorf–Thue–Morse sequence

$$\text{Fibonacci: } \left\{ \begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 0 \end{array} \right\}, \qquad \text{Thue–Morse: } \left\{ \begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 10 \end{array} \right\}.$$

The *Zeckendorf–Thue–Morse sequence* ztm:

$$\sigma : \left\{ \begin{array}{ccc} \mathtt{a} & \mapsto & \mathtt{ad} \\ \mathtt{b} & \mapsto & \mathtt{a} \\ \mathtt{c} & \mapsto & \mathtt{cb} \\ \mathtt{d} & \mapsto & \mathtt{c} \end{array} \right\}, \qquad \pi : \left\{ \begin{array}{ccc} \mathtt{a} & \mapsto & 0 \\ \mathtt{b} & \mapsto & 0 \\ \mathtt{c} & \mapsto & 1 \\ \mathtt{d} & \mapsto & 1 \end{array} \right\}.$$

What does this sequence do? It counts the number of Zeckendorf digits equal to 1, modulo 2.

In our paper, we proved in particular a prime number theorem for ztm.

Theorem 1 (Drmota, Müllner, S., Mem. Amer. Math. Soc. 2022+)

$$\#\{p \le x : \mathrm{ztm}(p) = 0\} \sim \frac{\pi(x)}{2}.$$

The sequence of prime numbers along Sturmian sequences (the Fibonacci word, for example) is quite well understood: the proportion of primes with value 0 is as expected.

Keywords *Nilsequences, Pure discrete spectrum*

The word ztm is morphic/substitutive, not automatic, and not Sturmian. Our result appears to be the first prime number theorem for this class of sequences.
Prime number theorems, and Möbius orthogonality for automatic sequences to were treated by Müllner (Duke Math. J. 2017), but the general case of morphic sequences is wide open.

We want to sketch the proof of the following theorem.

Proposition 1

There exist constants $c > 0$ and C such that

$$\sum_{p \le x} \mathrm{e}\bigl(\vartheta \mathrm{z}(p)\bigr) \le C(\log x)^4 x^{1-c\|\vartheta\|^2}$$

for all real ϑ and $x \ge 2$.

- p is a prime,
- $\mathrm{e}(x) = \exp(2\pi i\, x)$,
- $\mathrm{z}(n)$ is the minimal number of Fibonacci numbers needed to write n as their sum,
- $\|\vartheta\|$ is the distance of ϑ to $\mathbb{Z}$.

The sum over primes can be rewritten, using summation by parts:

$$\sum_{p\le N} \mathrm{e}\bigl(\vartheta \mathrm{z}(p)\bigr) \le \frac{2}{\log N} \max_{t\le N} \left| \sum_{n\le t} \mathrm{e}\bigl(\vartheta \mathrm{z}(n)\bigr)\Lambda(n) \right| + O\bigl(\sqrt{N}\bigr).$$

Von Mangoldt function:

$$\Lambda(n) = \begin{cases} \log p, & n = p^k \text{ for some } k \ge 1 \text{ and some prime } p, \\ 0, & \text{otherwise.} \end{cases}$$

That is, we have to estimate twisted partial sums of the von Mandoldt function. (Note: the prime number theorem asserts that $\sum_{n<x} \Lambda(n) \sim x$.)

Vaughan's identity leads to the starting point of our method.

Lemma (cf. Davenport, Multiplicative number theory)

Let $f:\mathbb{N}\to\mathbb{C}$ such that $|f(n)|\le 1$ for all $n\ge 1$. For all $N,U,V\ge 2$ such that $UV\le N$ we have

$$\sum_{n\le N} f(n)\Lambda(n) \ll U + (\log N)\boxed{\sum_{t\le UV}\max_{w}\left|\sum_{w\le r\le N/t} f(rt)\right|}$$
$$+\sqrt{N}(\log N)^3\left(\boxed{\max_{\substack{U\le M\le N/V\\ V\le q\le N/M}}\sum_{V<p\le N/M}\left|\sum_{\substack{M<m\le 2M\\ m\le\min(N/p,N/q)}} f(mp)\overline{f(mq)}\right|}\right)^{1/2},$$

with an absolute implied constant.

Let us write

$$S_{\mathrm{I}}(N,U,V) := \sum_{t\le UV} \max_{w} \left| \sum_{w\le r\le N/t} f(rt) \right|,$$

$$S_{\mathrm{II}}(N,U,V) := \max_{\substack{U\le M\le N/V\\ V\le q\le N/M}} \sum_{V<p\le N/M} \left| \sum_{\substack{M<m\le 2M\\ m\le \min(N/p,N/q)}} f(mp)\overline{f(mq)} \right|.$$

Good control over S_{I} will simplify the treatment of S_{II}.

Choose $U = N^{3/4}$ (and V small).

- S_{I}, inner sum: the difference t is usually *large* ($\gg N^{3/4-\varepsilon}$) compared to the length of the sum ($\ll N^{1/4+\varepsilon}$).
- S_{II}, inner sum: the differences p and q will be *small* ($\ll N^{1/4}$) compared to the length of the sum ($\gg N^{3/4}$).

We establish a strong estimate for type-I sums —

"the Zeckendorf sum-of-digits function has level of distribution 1".

Theorem (DMS 2022+)

Let $\varepsilon > 0$. There exist $c_1 = c_1(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$ such that for all $\vartheta \in \mathbb{R}$ and all real $x \geq 2$,

$$\sum_{1\le d\le D} \max_{\substack{y,z\ge 0\\ z-y\le x}} \max_{0\le a<d} \left| \sum_{\substack{y\le n<z\\ n\equiv a \bmod d}} \mathrm{e}\bigl(\vartheta \mathrm{z}(n)\bigr) \right| \le C\,(\log x)^{11/4}\, x^{1-c_1\|\vartheta\|^2},$$

where $D = x^{1-\varepsilon}$.

This is a statement on z along *very sparse finite arithmetic progressions*

$$A + \bigl(0, d, 2d, 3d, \ldots, (N-1)d\bigr),$$

for average moduli d.

- The proof of this statement is based on the corresponding paper for the Thue–Morse sequence $\mathtt{t}(n) = (-1)^{\mathsf{s}_2(n)}$ (S., Compos. Math 2020):

  Sketch of proof of S2020.
- Apply the van der Corput inequality repeatedly in order to obtain *higher order correlations*.
- Estimate

$$\sum_{\substack{0 \le n < 2^\rho \\ 0 \le r_1, \ldots, r_m < 2^\rho}} \mathrm{e}\left(\frac{1}{2} \sum_{\varepsilon \in \{0,1\}^m} \mathsf{s}_2^{(\rho)}(n + \varepsilon \cdot r) \right),$$

  nontrivially, where $\mathsf{s}_2^{(\rho)}(n) = s_2(n \bmod 2^\rho)$.
- Such an estimate of a *Gowers norm* of $\mathtt{t}$ was basically given by Konieczny (2019).

Eliminating digits

Consider a digital expansion along a sparse arithmetic progression:

in each step, *many digits* change!

Each application of van der Corput's inequality "eliminates" digits with indices in a certain interval $[A,B)$.

It is easy to detect base-q digits $\delta_j(n)$ with indices in $[A,B)$: we have

$$\bigl(\delta_A(n),\ldots,\delta_{B-1}(n)\bigr) = (\nu_A,\ldots,\nu_{B-1}) \quad \text{if and only if} \quad \left\{\frac{n}{q^B}\right\} \in J,$$

where

$$J = \left[\frac{m}{q^{B-A}}, \frac{m+1}{q^{B-A}}\right) \quad \text{and} \quad m = \sum_{A\le j<B} \nu_j q^{j-A}.$$

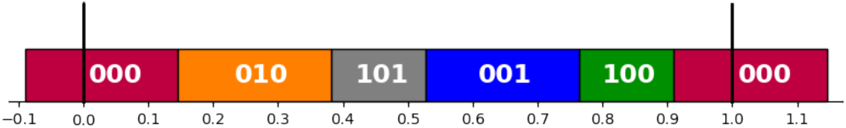
The indicator function 1_J can be approximated by trigonometric polynomials.

The Zeckendorf case: digit detection

Ostrowski expansion, $\varphi = \frac{\sqrt{5}+1}{2}$: The Zeckendorf digits of n with indices below B are equal to prescribed values if and only if

$$n\varphi$$

is contained in a certain interval modulo 1.



The blue and red intervals are **separated** $\rightsquigarrow$ integers having Zeckendorf expansion $\cdots **00*$ cannot be detected by a single interval.

Wrap

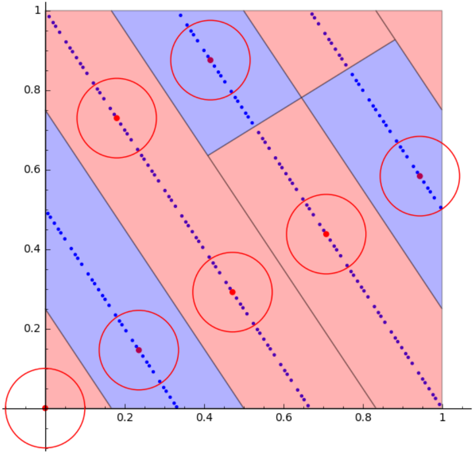
Wrap this around the two-dimensional torus in such a way that adjacent digit combinations (w.r.t. the lexicographical ordering) lie "parallel to each other" (illustration in a moment): set

$$p(n) = \left(\frac{n}{\varphi^B}, \frac{n}{\varphi^{B+1}} \right).$$

The closure of the set of points $p(n)$ mod 1×1 is a union of finitely many line segments, since

$$\frac{F_{B+1}}{\varphi^B} + \frac{F_B}{\varphi^{B+1}} = 1.$$

Detecting the lowest Zeckendorf digit: $B = 3$



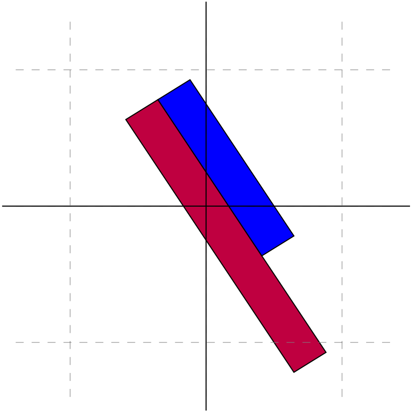
0
1
10
100
101
1000
1001

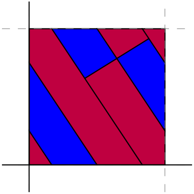
The least significant digit is given by the Fibonacci word

$$F = (0100101001001010010100100101001001\cdots),$$

the fixed point of the Fibonacci substitution $\sigma : 0 \mapsto 01, 1 \mapsto 0$.

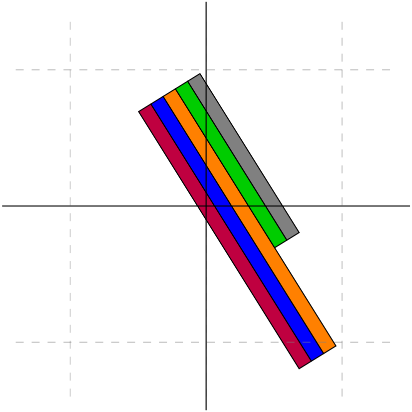
One significant digit: B=3

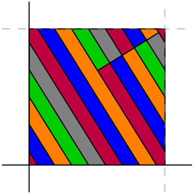




$0, 1$

Three significant digits





$\underbrace{\texttt{000}, \texttt{001}}_{\text{no longer separated!}}, \texttt{010}, \texttt{100}, \texttt{101}$

The Zeckendorf digits of n with indices in $[A, B)$ are equal to prescribed values $\omega_A, \ldots, \omega_{B-1}$ if and only if

$$\left(\frac{n}{\varphi^B}, \frac{n}{\varphi^{B+1}}\right)$$

is contained in $Q + \mathbb{Z}^2$, where Q is a certain parallelogram depending on the values ω_j. $\rightsquigarrow$ trigonometric approximation of Q!

Together with the following lemma, we may eliminate Zeckendorf digits in our sum $\sum_n \mathrm{e}(\vartheta \mathrm{z}(nd + a))$.

Lemma (generalized vdC inequality)

Let I be a finite interval in $\mathbb{Z}$ containing M integers and $x_m \in \mathbb{C}$ for $m \in I$. Assume that $K \subset \mathbb{N}$ is a finite nonempty set. Then

$$\left|\sum_{m \in I} x_m\right|^2 \leq \frac{M + \max K - \min K}{|K|^2} \sum_{(k,k') \in K^2} \sum_{m \in I \cap (I-k+k')} x_m \overline{x_{m+k-k'}}.$$

- Apply this process repeatedly, until only few digits remain.
- In analogy to the Thue–Morse case, it remained to estimate a Gowers (type) norm for the Zeckendorf sum-of-digits function.

$\square$

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773,787,797,809,811,821,823,827,829,839,853,857,859,863,877,881,883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997,

. . . . . .

Theorem 2 (DMS)

There exists a constant ("effective") k_0 with the following property. For each integer $k \geq k_0$, there exists a prime number that is the sum of exactly k different, non-consecutive Fibonacci numbers.

Underlying this result, there is a "local limit theorem" involving a normal distribution of the relevant quantities around a (moving) expected value.

Generalizations

For the (Ostrowski) α-sum-of-digits function s_α, we expect that the method generalizes, at least for the case that α is an algebraic number. Algebraicity is probably needed for multi-dimensional detection.

More interestingly, we are interested in *morphic sequences*: take a fixed point of a substitution over a finite alphabet, and possibly rename the letters afterwards. Examples: the Thue–Morse sequence defined by

$$\sigma : 0 \mapsto 01, \quad 1 \mapsto 10,$$

or the Zeckendorf sum-of-digits function modulo 2,
Automatic sequences were treated by Müllner (Duke Math. J. 2017), but the general case (morphic sequences) is wide open.

Linear recurrent number systems

The *Tribonacci* numeration system is based on the Tribonacci numbers

$$\begin{gathered} a_n = a_{n-1} + a_{n-2} + a_{n-3}, \\ a_0 = a_1 = 0, a_2 = 1, \quad \text{that is,} \\ (a_n)_{n \geq 3} = (1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots). \end{gathered}$$

Every natural number is the unique sum of pairwise different a_n, $n \geq 3$, where taking three consecutive numbers is forbidden.
The lowest two Tribonacci digits exhibit a close connection to the Tribonacci word

$$\sigma : \left\{ \begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 02 \\ 2 & \mapsto & 0 \end{array} \right\}.$$

Tribonacci sum of digits

Meanwhile, the Tribonacci sum-of-digits function modulo 2 appears to be given by

$$\sigma : \left\{ \begin{array}{ccc} \mathtt{a} & \mapsto & \mathtt{ae} \\ \mathtt{b} & \mapsto & \mathtt{af} \\ \mathtt{c} & \mapsto & \mathtt{a} \\ \mathtt{d} & \mapsto & \mathtt{db} \\ \mathtt{e} & \mapsto & \mathtt{dc} \\ \mathtt{f} & \mapsto & \mathtt{d} \end{array} \right\}, \qquad \pi : \left\{ \begin{array}{ccc} \mathtt{a} & \mapsto & 0 \\ \mathtt{b} & \mapsto & 0 \\ \mathtt{c} & \mapsto & 0 \\ \mathtt{d} & \mapsto & 1 \\ \mathtt{e} & \mapsto & 1 \\ \mathtt{f} & \mapsto & 1 \end{array} \right\}.$$

$$\mathtt{tr} = \mathtt{011010010010110010110110100101101101001101001001011} \cdots$$

Detecting a given letter in the Tribonacci word leads to the (classical) two-dimensional *Rauzy fractal*.

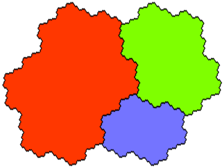


Figure: Jolivet, Loridant, Luo 2014

For detecting Tribonacci digits with indices in $[A, B)$, we will have to consider three-dimensional *cylinders* with the Rauzy fractal as base!

Thank you!