Unique double base expansions

Wolfgang Steiner
(joint work with Vilmos Komornik and Yuru Zou)

IRIF, CNRS, Université de Paris

One World Numeration Seminar, February 15, 2022

Alphabet-base systems

Zou–Komornik–Lu:
alphabet-base systems $\{(d_0, \beta_0), (d_1, \beta_1), \ldots, (d_M, \beta_M)\}$, $d_i \in \mathbb{R}$, $\beta_i > 1$,
expansions

$$x = \sum_{k=1}^{\infty} \frac{d_{i_k}}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}} \qquad \big(i_1 i_2 \cdots \in \{0, 1, \ldots, M\}^\infty\big)$$

Rényi '57, Parry '60, Pedicini '05, ...:
$\beta_0 = \cdots = \beta_M = \beta$, β-expansions with digit set $\{d_0, d_1, \ldots, d_M\}$

Neunhäuserer '21, Y.-Q. Li '21: $d_i = i$

Avanzi, Berthé, Doche, Imbert, Dimitrov, Krenn, Wagner, ... '06–:
double base expansions of integers $\sum_{(a,b)\in F} \pm p^a q^b$
applications to elliptic curve cryptosystems

Double base expansions

alphabet-base systems $\{(d_0,\beta_0),(d_1,\beta_1)\}$, w.l.o.g., $\{(0,\beta_0),(1,\beta_1)\}$:

$$\sum_{k=1}^{\infty}\frac{d_{i_k}}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}}=\frac{d_0}{\beta_0-1}+\left(d_1-d_0\frac{\beta_1-1}{\beta_0-1}\right)\sum_{k=1}^{\infty}\frac{i_k}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}}$$

Proof: $T_{d_i,\beta_i}\circ\varphi=\varphi\circ T_{i,\beta_i}$, $i\in\{0,1\}$, with

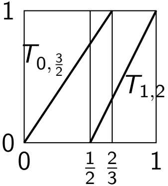
$$T_{d,\beta}:\mathbb{C}\to\mathbb{C},\quad x\mapsto\beta x-d$$

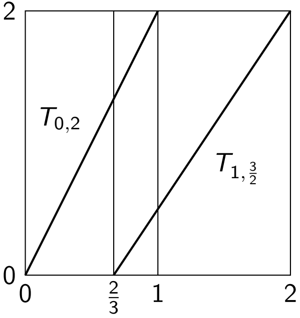
$$\varphi:\mathbb{C}\to\mathbb{C},\quad x\mapsto\tfrac{d_0}{\beta_0-1}+\left(d_1-d_0\tfrac{\beta_1-1}{\beta_0-1}\right)x$$

$$\varphi(x_0)=T_{d_{i_1},\beta_{i_1}}^{-1}\circ\cdots\circ T_{d_{i_n},\beta_{i_n}}^{-1}\circ\varphi\circ T_{i_n,\beta_{i_n}}\circ\cdots\circ T_{i_1,\beta_{i_1}}(x_0)$$

$$=\sum_{k=1}^{n}\frac{d_{i_k}}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}}+\frac{\varphi(x_n)}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}}$$

$$x_n:=\sum_{k=n+1}^{\infty}\frac{i_k}{\beta_{i_{n+1}}\beta_{i_{n+2}}\cdots\beta_{i_k}}$$

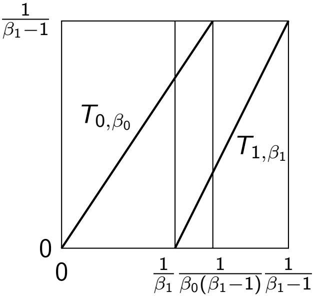




Unique expansions

$$\pi_{\beta_0,\beta_1}(i_1 i_2 \cdots) := \sum_{k=1}^{\infty} \frac{i_k}{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_k}}$$

$$U_{\beta_0,\beta_1} := \left\{\mathbf{u} \in \{0,1\}^\infty : \pi_{\beta_0,\beta_1}(\mathbf{u}) \neq \pi_{\beta_0,\beta_1}(\mathbf{v}) \text{ for all } \mathbf{v} \neq \mathbf{u}\right\}$$
$$= \left\{i_1 i_2 \cdots \in \{0,1\}^\infty : \pi_{\beta_0,\beta_1}(i_n i_{n+1} \cdots) \notin \left[\tfrac{1}{\beta_1}, \tfrac{1}{\beta_0(\beta_1-1)}\right] \forall n \geq 1\right\}$$



$$U_{\beta_0,\beta_1} = \{0,1\}^\infty \iff \frac{1}{\beta_1} > \frac{1}{\beta_0(\beta_1-1)} \iff \frac{1}{\beta_0} + \frac{1}{\beta_1} < 1$$

$$\beta_0 \leq \beta_0',\ \beta_1 \leq \beta_1' \implies U_{\beta_0,\beta_1} \subseteq U_{\beta_0',\beta_1'}$$

$$U_{\beta_0,\beta_1} = \left\{(1-i_1)(1-i_2)\cdots : i_1 i_2 \cdots \in U_{\beta_1,\beta_0}\right\}$$

Critical constants for bases with unique expansions

Generalised golden ratio

$$\mathcal{G}(\beta_0) := \inf\left\{\beta_1 > 1 \,:\, U_{\beta_0,\beta_1} \neq \{00\cdots, 11\cdots\}\right\}$$
$$= \inf\left\{\beta_1 > 1 \,:\, U_{\beta_0,\beta_1} \text{ is infinite}\right\}$$

Erdős–Joó–Komornik '90, Glendinning–Sidorov '01:

$$\inf\left\{\beta > 1 \,:\, U_{\beta,\beta} \neq \{00\cdots, 11\cdots\}\right\} = \tfrac{1+\sqrt{5}}{2} \approx 1.618034$$

Generalised Komornik–Loreti constant

$$\mathcal{L}(\beta_0) := \inf\{\beta_1 > 1 \,:\, U_{\beta_0,\beta_1} \text{ is uncountable}\}$$
$$= \inf\{\beta_1 > 1 \,:\, U_{\beta_0,\beta_1} \text{ has positive topological entropy}\}$$
$$= \inf\{\beta_1 > 1 \,:\, \pi_{\beta_0,\beta_1}(U_{\beta_0,\beta_1}) \text{ has positive Hausdorff dimension}\}$$

Glendinning–Sidorov '01:

$$\min\{\beta > 1 \,:\, U_{\beta,\beta} \text{ is uncountable}\} = \beta_{\mathrm{KL}} \approx 1.78723165$$

$\pi_{\beta_{\mathrm{KL}},\beta_{\mathrm{KL}}}(110100110010110\cdots) = 1$, Thue–Morse word $0110100110010110\cdots$

$\mathcal{G}(\frac{1+\sqrt{5}}{2}) = \frac{1+\sqrt{5}}{2}$, $\mathcal{L}(\beta_{\mathrm{KL}}) = \beta_{\mathrm{KL}}$

Generalised golden ratio and Komornik–Loreti constant

Komornik–St–Zou:

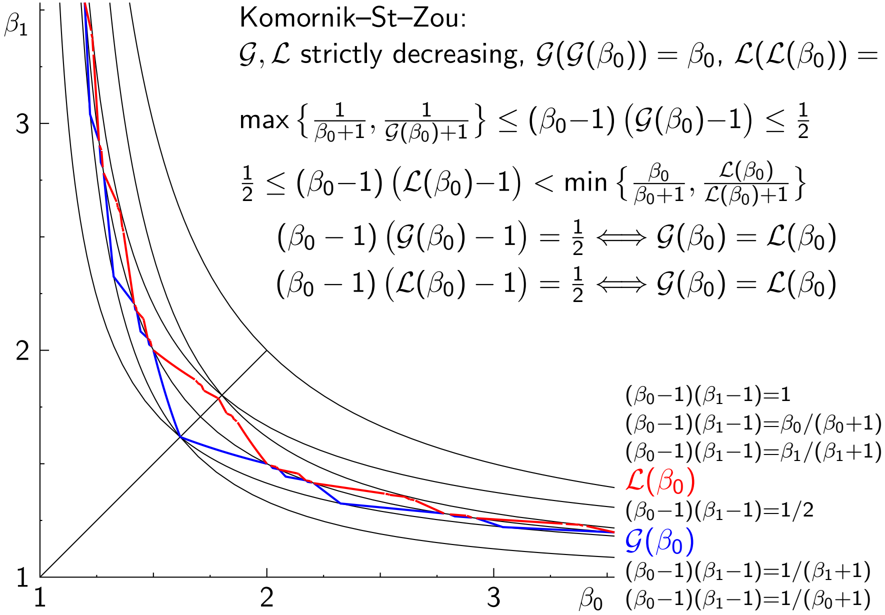
$\mathcal{G}, \mathcal{L}$ strictly decreasing, $\mathcal{G}(\mathcal{G}(\beta_0)) = \beta_0$, $\mathcal{L}(\mathcal{L}(\beta_0)) = \beta_0$

$$\max\left\{\frac{1}{\beta_0+1}, \frac{1}{\mathcal{G}(\beta_0)+1}\right\} \le (\beta_0-1)\left(\mathcal{G}(\beta_0)-1\right) \le \frac{1}{2}$$

$$\frac{1}{2} \le (\beta_0-1)\left(\mathcal{L}(\beta_0)-1\right) < \min\left\{\frac{\beta_0}{\beta_0+1}, \frac{\mathcal{L}(\beta_0)}{\mathcal{L}(\beta_0)+1}\right\}$$

$$(\beta_0-1)\left(\mathcal{G}(\beta_0)-1\right) = \frac{1}{2} \iff \mathcal{G}(\beta_0) = \mathcal{L}(\beta_0)$$

$$(\beta_0-1)\left(\mathcal{L}(\beta_0)-1\right) = \frac{1}{2} \iff \mathcal{G}(\beta_0) = \mathcal{L}(\beta_0)$$



Generalised golden ratio and Komornik–Loreti constant

Komornik–St–Zou:

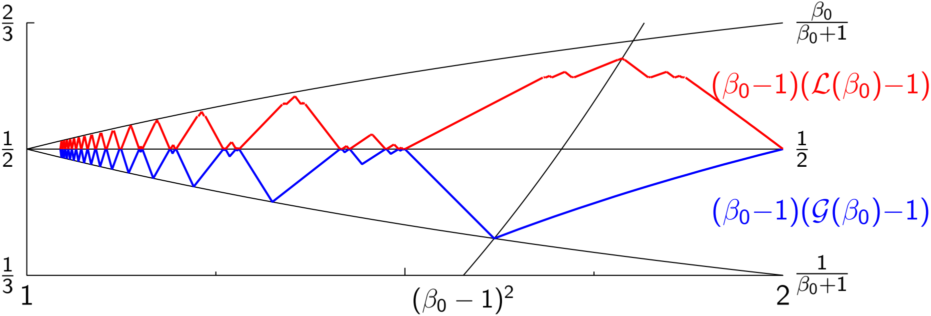
$\mathcal{G}, \mathcal{L}$ strictly decreasing, $\mathcal{G}(\mathcal{G}(\beta_0)) = \beta_0$, $\mathcal{L}(\mathcal{L}(\beta_0)) = \beta_0$

$$\max\left\{\tfrac{1}{\beta_0+1}, \tfrac{1}{\mathcal{G}(\beta_0)+1}\right\} \le (\beta_0 - 1)\big(\mathcal{G}(\beta_0) - 1\big) \le \tfrac{1}{2}$$

$$\tfrac{1}{2} \le (\beta_0 - 1)\big(\mathcal{L}(\beta_0) - 1\big) < \min\left\{\tfrac{\beta_0}{\beta_0+1}, \tfrac{\mathcal{L}(\beta_0)}{\mathcal{L}(\beta_0)+1}\right\}$$

$$(\beta_0 - 1)\big(\mathcal{G}(\beta_0) - 1\big) = \tfrac{1}{2} \iff \mathcal{G}(\beta_0) = \mathcal{L}(\beta_0)$$

$$(\beta_0 - 1)\big(\mathcal{L}(\beta_0) - 1\big) = \tfrac{1}{2} \iff \mathcal{G}(\beta_0) = \mathcal{L}(\beta_0)$$

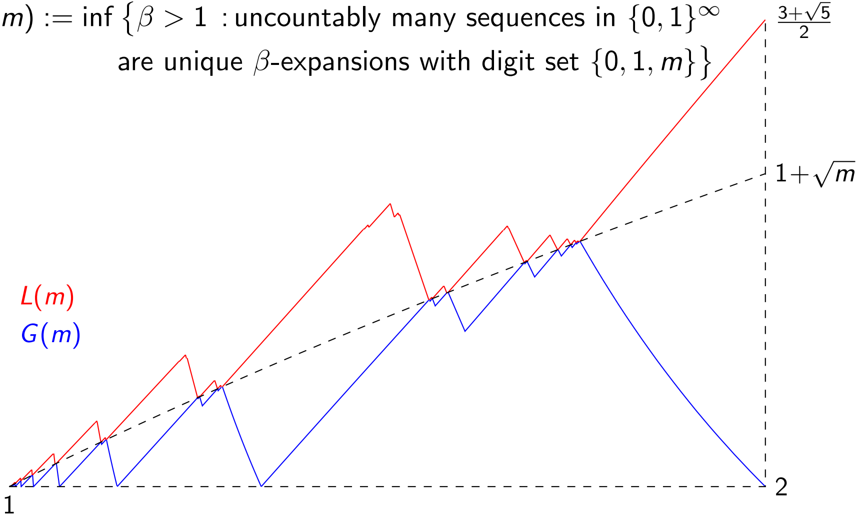


Unique β-expansions with 3 digits

Komornik–Lai–Pedicini '11, Baker–St '17, Komornik–Pedicini '17, St '20:

$$G(m) := \inf\Big\{\beta > 1 : \text{set of unique } \beta\text{-expansions with digit set } \{0,1,m\} \text{ is non-trivial}\Big\}$$

$$L(m) := \inf\Big\{\beta > 1 : \text{uncountably many sequences in } \{0,1\}^\infty \text{ are unique } \beta\text{-expansions with digit set } \{0,1,m\}\Big\}$$



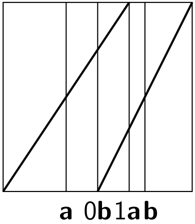
Unique double base expansions and lexicographic world

assume $\frac{1}{\beta_0} + \frac{1}{\beta_1} \geq 1$, i.e., $(\beta_0 - 1)(\beta_1 - 1) \leq 1$

$$\begin{aligned}
U_{\beta_0,\beta_1} &= \left\{\mathbf{u} \in \{0,1\}^\infty \,:\, \pi_{\beta_0,\beta_1}(\mathbf{u}) \neq \pi_{\beta_0,\beta_1}(\mathbf{v}) \text{ for all } \mathbf{v} \neq \mathbf{u}\right\} \\
&= \left\{i_1 i_2 \cdots \in \{0,1\}^\infty \,:\, \pi_{\beta_0,\beta_1}(i_n i_{n+1} \cdots) \notin \left[\tfrac{1}{\beta_1}, \tfrac{1}{\beta_0(\beta_1 - 1)}\right] \forall n \geq 1\right\} \\
&= \left\{i_1 i_2 \cdots \in \{0,1\}^\infty \,:\, i_n i_{n+1} \cdots < 0\mathbf{b} \text{ or } 1\mathbf{a} < i_n i_{n+1} \cdots \ \forall n \geq 1\right\}
\end{aligned}$$

$0\mathbf{b}$ is the quasi-greedy (β_0, β_1)-expansion of $\frac{1}{\beta_1} = \pi_{\beta_0,\beta_1}(1\overline{0})$, $\overline{0} := 00\cdots$

$1\mathbf{a}$ is the quasi-lazy (β_0, β_1)-expansion of $\frac{1}{\beta_0(\beta_1 - 1)} = \pi_{\beta_0,\beta_1}(0\overline{1})$, $\overline{1} := 11\cdots$



$$\begin{aligned}
U'_{\beta_0,\beta_1} &:= \left\{i_1 i_2 \cdots \in \{0,1\}^\infty \,:\, \mathbf{a} < i_n i_{n+1} \cdots < \mathbf{b} \ \forall n \geq 1\right\} \\
&= \left\{\mathbf{u} \in \{0,1\}^\infty \,:\, 0\mathbf{u} \in U_{\beta_0,\beta_1} \text{ and } 1\mathbf{u} \in U_{\beta_0,\beta_1}\right\}
\end{aligned}$$

$$U_{\beta_0,\beta_1} = 0^* U'_{\beta_0,\beta_1} \cup 1^* U'_{\beta_0,\beta_1} \cup \{\overline{0}, \overline{1}\}$$

$$\begin{aligned}
U''_{\beta_0,\beta_1} &:= \left\{i_1 i_2 \cdots \in \{0,1\}^\infty \,:\, \mathbf{a} \leq i_n i_{n+1} \cdots \leq \mathbf{b} \ \forall n \geq 1\right\} \\
&= \bigcap_{\beta_0' > \beta_0} U'_{\beta_0',\beta_1} = \bigcap_{\beta_1' > \beta_1} U'_{\beta_0,\beta_1'}
\end{aligned}$$

$$\mathcal{G}(\beta_0) \leq \beta_1 \ \Leftrightarrow\ U_{\beta_0,\beta_1'} \neq \{\overline{0}, \overline{1}\}\ \forall \beta_1' > \beta_1 \ \Leftrightarrow\ U'_{\beta_0,\beta_1'} \neq \emptyset\ \forall \beta_1' > \beta_1 \ \Leftrightarrow\ U''_{\beta_0,\beta_1} \neq \emptyset$$

$$\mathcal{L}(\beta_0) \leq \beta_1 \ \Leftrightarrow\ U_{\beta_0,\beta_1'}, U'_{\beta_0,\beta_1'}, U''_{\beta_0,\beta_1'} \text{ uncountable } \forall \beta_1' > \beta_1$$

Lexicographic world (Labarca–Moreira '06, ...)

$$\Omega_{\mathbf{a},\mathbf{b}} := \left\{ i_1 i_2 \cdots \in \{0,1\}^\infty \;:\; i_n i_{n+1} \cdots \in [\mathbf{a},\mathbf{b}] \;\forall n \geq 1 \right\}$$

$$\mathbf{a} \leq \overline{01},\ \mathbf{b} \geq \overline{10} \iff \{\overline{01}, \overline{10}\} \subseteq \Omega_{\mathbf{a},\mathbf{b}}$$

$$\mathbf{a} > \overline{01},\ \mathbf{b} \leq 1\overline{10} \implies \Omega_{\mathbf{a},\mathbf{b}} = \emptyset$$

$$\mathbf{a} \geq 0\overline{01},\ \mathbf{b} < \overline{10} \implies \Omega_{\mathbf{a},\mathbf{b}} = \emptyset$$

$$\mathbf{a} \geq \overline{01} \implies \Omega_{\mathbf{a},\mathbf{b}} \subseteq R(\{0,1\}^\infty)$$

$$\mathbf{b} \leq \overline{10} \implies \Omega_{\mathbf{a},\mathbf{b}} \subseteq L(\{0,1\}^\infty)$$

$$L : 0 \mapsto 0 \qquad M : 0 \mapsto 01 \qquad R : 0 \mapsto 01$$

$$\quad 1 \mapsto 10 \qquad\quad 1 \mapsto 10 \qquad\quad 1 \mapsto 1$$

$$\mathbf{a} < 0\overline{01},\ \mathbf{b} \geq \overline{10} \implies \exists k \geq 0 : \{0(01)^k, 0(01)^{k+1}\}^\infty \subseteq \Omega_{\mathbf{a},\mathbf{b}}$$

$$\mathbf{a} \leq \overline{01},\ \mathbf{b} > 1\overline{10} \implies \exists k \geq 0 : \{1(10)^k, 1(10)^{k+1}\}^\infty \subseteq \Omega_{\mathbf{a},\mathbf{b}}$$

$$\mathbf{a} \geq 0\overline{01},\ \mathbf{b} \leq 1\overline{10} \implies \Omega_{\mathbf{a},\mathbf{b}} \subseteq M(\{0,1\}^\infty) \cup \{0,1\}M(\{0,1\}^\infty)$$

Thue–Morse–Sturmian words, partitions of $\{0,1\}^\infty$

$$L: 0 \mapsto 0 \qquad M: 0 \mapsto 01 \qquad R: 0 \mapsto 01$$
$$\quad 1 \mapsto 10 \qquad\quad 1 \mapsto 10 \qquad\quad 1 \mapsto 1$$

$$\boldsymbol{\sigma}(\mathbf{u}) := \lim_{n\to\infty} \sigma_1\sigma_2\cdots\sigma_n(\mathbf{u}) \quad (\boldsymbol{\sigma} = (\sigma_n)_{n\ge 1} \in \{L,M,R\}^\infty,\ \mathbf{u} \in \{0,1\}^\infty)$$

$$0\{0,1\}^\infty = \dot{\bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty}} \tilde{I}_{\boldsymbol{\sigma}} = \dot{\bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty}} \tilde{J}_{\boldsymbol{\sigma}}, \quad 1\{0,1\}^\infty = \dot{\bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty}} I_{\boldsymbol{\sigma}} = \dot{\bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty}} J_{\boldsymbol{\sigma}}$$

$$\tilde{I}_{\boldsymbol{\sigma}} := \begin{cases} [\inf(\boldsymbol{\sigma}(1\overline{0})), \inf(\boldsymbol{\sigma}(\overline{0}))] & \text{if } \boldsymbol{\sigma} \in \{L,R\}^* M\overline{L} \cup \{L,R\}^\infty \setminus \{L,R\}^*\{L\overline{R}, R\overline{L}\} \\ \emptyset & \text{otherwise} \end{cases}$$

$$I_{\boldsymbol{\sigma}} := \begin{cases} [\sup(\boldsymbol{\sigma}(\overline{1})), \sup(\boldsymbol{\sigma}(0\overline{1}))] & \text{if } \boldsymbol{\sigma} \in \{L,R\}^* M\overline{R} \cup \{L,R\}^\infty \setminus \{L,R\}^*\{L\overline{R}, R\overline{L}\} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\tilde{J}_{\boldsymbol{\sigma}} := \begin{cases} \emptyset & \text{if } \boldsymbol{\sigma} \in \{L,M,R\}^* \{L\overline{R}, R\overline{L}\} \\ [\inf(\boldsymbol{\sigma}(1\overline{0})), \inf(\boldsymbol{\sigma}(0\overline{1}))] & \text{otherwise} \end{cases}$$

$$J_{\boldsymbol{\sigma}} := \begin{cases} \emptyset & \text{if } \boldsymbol{\sigma} \in \{L,M,R\}^* \{L\overline{R}, R\overline{L}\} \\ [\sup(\boldsymbol{\sigma}(1\overline{0})), \sup(\boldsymbol{\sigma}(0\overline{1}))] & \text{otherwise} \end{cases}$$

$$\inf(i_1 i_2\cdots) := \inf(\{i_n i_{n+1}\cdots : n \ge 1\}),\ \sup(i_1 i_2\cdots) := \sup(\{i_n i_{n+1}\cdots : n \ge 1\})$$

$$
\begin{array}{lll}
\tilde{I}_{\overline{L}} & \overline{0} & \tilde{J}_{\overline{L}} \\[1ex]
\tilde{I}_{LM\overline{L}} & \left[\begin{array}{l} 0\overline{001} \quad \tilde{J}_{LM\overline{L}} \\ \left.\begin{array}{c} 0001\overline{010} \\ \overline{001} \end{array}\right] \tilde{J}_{LM\overline{R}} \end{array}\right. & \\[1ex]
\tilde{I}_{M\overline{L}} & \left[\begin{array}{l} 0\overline{01} \quad \tilde{J}_{M\overline{L}} \\ 001\overline{0110} \quad \tilde{J}_{MM\overline{L}} \\ \left.\begin{array}{c} 00101101\overline{1001} \\ \overline{0011} \end{array}\right] \tilde{J}_{MM\overline{R}} \\ \left.\begin{array}{c} 001\overline{10} \\ \overline{01} \end{array}\right] \tilde{J}_{M\overline{R}} \end{array}\right. & \\[1ex]
\tilde{I}_{RM\overline{L}} & \left[\begin{array}{l} 0\overline{101} \quad \tilde{J}_{RM\overline{L}} \\ \left.\begin{array}{c} 01011\overline{101} \\ \overline{011} \end{array}\right] \tilde{J}_{RM\overline{R}} \end{array}\right. & \\[1ex]
\tilde{I}_{\overline{R}} & 0\overline{1} & \tilde{J}_{\overline{R}}
\end{array}
$$

$$
\begin{array}{lll}
I_{\overline{L}} & 1\overline{0} & J_{\overline{L}} \\[1ex]
I_{LM\overline{R}} & \left[\begin{array}{l} \left.\begin{array}{c} \overline{100} \\ 10100\overline{010} \end{array}\right] J_{LM\overline{L}} \\ 1\overline{010} \quad J_{LM\overline{R}} \end{array}\right. & \\[1ex]
I_{M\overline{R}} & \left[\begin{array}{l} \left.\begin{array}{c} \overline{10} \\ 110\overline{01} \end{array}\right] J_{M\overline{L}} \\ \left.\begin{array}{c} \overline{1100} \\ 1101001\overline{0110} \end{array}\right] J_{MM\overline{L}} \\ 110\overline{1001} \quad J_{MM\overline{R}} \\ 1\overline{10} \quad J_{M\overline{R}} \end{array}\right. & \\[1ex]
I_{RM\overline{R}} & \left[\begin{array}{l} \left.\begin{array}{c} \overline{110} \\ 1110\overline{101} \end{array}\right] J_{RM\overline{L}} \\ 1\overline{110} \quad J_{RM\overline{R}} \end{array}\right. & \\[1ex]
I_{\overline{R}} & \overline{1} & J_{\overline{R}}
\end{array}
$$

$$
\begin{array}{rl}
L: & 0 \mapsto 0 \\
 & 1 \mapsto 10 \\[1ex]
M: & 0 \mapsto 01 \\
 & 1 \mapsto 10 \\[1ex]
R: & 0 \mapsto 01 \\
 & 1 \mapsto 1
\end{array}
$$

$$
\begin{aligned}
M\overline{L}(\overline{0}) &= \overline{01} \\
M\overline{L}(0\overline{1}) &= 0110\overline{01} \\
M\overline{L}(1\overline{0}) &= 10\overline{01} \\
M\overline{L}(\overline{1}) &= 10\overline{01}
\end{aligned}
$$

$$
\begin{aligned}
M\overline{R}(\overline{0}) &= 01\overline{10} \\
M\overline{R}(0\overline{1}) &= 01\overline{10} \\
M\overline{R}(1\overline{0}) &= 1001\overline{10} \\
M\overline{R}(\overline{1}) &= \overline{10}
\end{aligned}
$$

Lexicographic world and unique expansions

$$\Omega_{\mathbf{a},\mathbf{b}} = \left\{ i_1 i_2 \cdots \in \{0,1\}^\infty \ : \ i_n i_{n+1} \cdots \in [\mathbf{a},\mathbf{b}] \ \forall n \geq 1 \right\}$$

$$r : \{0,1\}^\infty \to \{L,M,R\}^\infty, \quad \mathbf{u} \mapsto \boldsymbol{\sigma} \quad \text{when} \quad \mathbf{u} \in \tilde{I}_{\boldsymbol{\sigma}} \cup I_{\boldsymbol{\sigma}}$$

$$s : \{0,1\}^\infty \to \{L,M,R\}^\infty, \quad \mathbf{u} \mapsto \boldsymbol{\sigma} \quad \text{when} \quad \mathbf{u} \in \tilde{J}_{\boldsymbol{\sigma}} \cup J_{\boldsymbol{\sigma}}$$

Labarca–Moreira '06, St '20, Komornik–St–Zou:

$$\Omega_{\mathbf{a},\mathbf{b}} \neq \emptyset \iff r(\mathbf{a}) \leq r(\mathbf{b})$$

$$\Omega_{\mathbf{a},\mathbf{b}} \text{ has positive entropy} \iff s(\mathbf{a}) < s(\mathbf{b})$$

$$\Omega_{\mathbf{a},\mathbf{b}} \text{ uncountable with zero entropy} \iff s(\mathbf{a}) = s(\mathbf{b}) \notin \{L,M,R\}^*\{\overline{L},\overline{R}\}$$

($\Omega_{\mathbf{a},\mathbf{b}}$ is the S-adic shift given by the primitive sequence $s(\mathbf{a}) = s(\mathbf{b})$)

$0\mathbf{b}$ quasi-greedy (β_0,β_1)-exp. of $\frac{1}{\beta_1}$, $1\mathbf{a}$ quasi-lazy (β_0,β_1)-exp. of $\frac{1}{\beta_0(\beta_1-1)}$:

$$\mathcal{G}(\beta_0) \leq \beta_1 \iff \Omega_{\mathbf{a},\mathbf{b}} = U''_{\beta_0,\beta_1} \neq \emptyset \iff r(\mathbf{a}) \leq r(\mathbf{b})$$

$$\mathcal{L}(\beta_0) \leq \beta_1 \iff s(\mathbf{a}) \leq s(\mathbf{b})$$

Critical constants

0**b** quasi-greedy (β_0, β_1)-exp. of $\frac{1}{\beta_1}$, 1**a** quasi-lazy (β_0, β_1)-exp. of $\frac{1}{\beta_0(\beta_1-1)}$

$$\mathcal{G}(\beta_0) \le \beta_1 \iff r(\mathbf{a}) \le r(\mathbf{b})$$

$$\iff \exists\, \boldsymbol{\sigma} \in \{L,R\}^* M\bar{L} \cup \{L,R\}^\infty \backslash \{L,R\}^* \{\bar{L},\bar{R}\} : \mathbf{a} \le \inf(\boldsymbol{\sigma}(\bar{0})), \mathbf{b} \ge \sup(\boldsymbol{\sigma}(\bar{0}))$$

$$\big(\sup(\sigma M\bar{L}(\bar{0})) = \sup(\sigma(\overline{01})) = \sup(\sigma(\overline{10})) = \sup(\sigma M\bar{R}(\bar{1}))\big)$$

$$\iff \exists\, \boldsymbol{\sigma} \in \{L,R\}^* M\bar{L} \cup \{L,R\}^\infty \backslash \{L,R\}^* \{\bar{L},\bar{R}\} : \beta_1 \ge \max\big(\tilde{g}_{\boldsymbol{\sigma}(\bar{0})}(\beta_0), g_{\boldsymbol{\sigma}(\bar{0})}(\beta_0)\big)$$

$g_{\mathbf{u}}(\beta_0), \tilde{g}_{\mathbf{u}}(\beta_0) > 1$ are defined by

$$\pi_{\beta_0, g_{\mathbf{u}}(\beta_0)}(0\sup(\mathbf{u})) = \frac{1}{g_{\mathbf{u}}(\beta_0)}, \quad \pi_{\beta_0, \tilde{g}_{\mathbf{u}}(\beta_0)}(1\inf(\mathbf{u})) = \frac{1}{\beta_0(\tilde{g}_{\mathbf{u}}(\beta_0) - 1)}$$

$$\mathcal{G}(\beta_0) = \min_{\boldsymbol{\sigma} \in \{L,R\}^* M\bar{L} \cup \{L,R\}^\infty \backslash \{L,R\}^* \{\bar{L},\bar{R}\}} \max\big(\tilde{g}_{\boldsymbol{\sigma}(\bar{0})}(\beta_0), g_{\boldsymbol{\sigma}(\bar{0})}(\beta_0)\big)$$

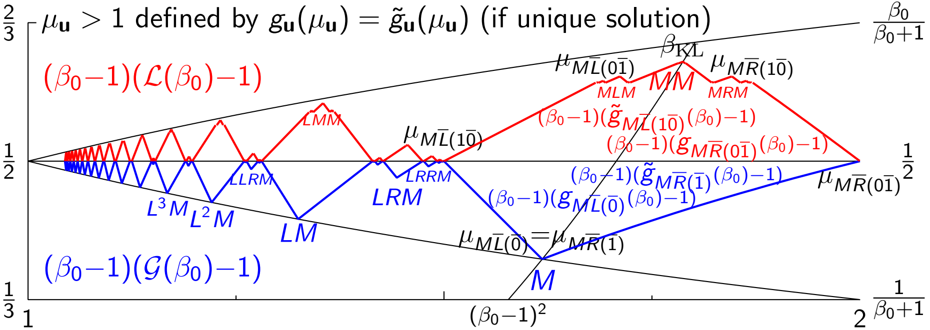
$$\mathcal{L}(\beta_0) = \min_{\boldsymbol{\sigma} \in \{L,M,R\}^* M\{\bar{L},\bar{R}\} \cup \{L,M,R\}^\infty \backslash \{L,M,R\}^* \{\bar{L},\bar{R}\}} \max\big(\tilde{g}_{\boldsymbol{\sigma}(1\bar{0})}(\beta_0), g_{\boldsymbol{\sigma}(0\bar{1})}(\beta_0)\big)$$

Main theorem (Komornik–St–Zou)

$$\mathcal{G}(\beta_0) = \begin{cases} g_{\boldsymbol\sigma(\bar 0)}(\beta_0) & \text{if } \beta_0 \in [\mu_{\boldsymbol\sigma(1\bar 0)}, \mu_{\boldsymbol\sigma(\bar 0)}], \\ & \quad \boldsymbol\sigma \in \{L,R\}^* M\bar L \cup \{L,R\}^\infty \setminus \{L,R\}^*\{\bar L, \bar R\} \\ \tilde g_{\boldsymbol\sigma(\bar 1)}(\beta_0) & \text{if } \beta_0 \in [\mu_{\boldsymbol\sigma(\bar 1)}, \mu_{\boldsymbol\sigma(0\bar 1)}], \\ & \quad \boldsymbol\sigma \in \{L,R\}^* M\bar R \cup \{L,R\}^\infty \setminus \{L,R\}^*\{\bar L, \bar R\} \end{cases}$$

$$\mathcal{L}(\beta_0) = \begin{cases} \tilde g_{\boldsymbol\sigma(1\bar 0)}(\beta_0) & \text{if } \beta_0 \in [\mu_{\boldsymbol\sigma(1\bar 0)}, \mu_{\boldsymbol\sigma(0\bar 1)}], \\ & \boldsymbol\sigma \in \{L,M,R\}^* M\bar L \cup \{L,M,R\}^\infty \setminus \{L,M,R\}^*\{\bar L, \bar R\} \\ g_{\boldsymbol\sigma(0\bar 1)}(\beta_0) & \text{if } \beta_0 \in [\mu_{\boldsymbol\sigma(1\bar 0)}, \mu_{\boldsymbol\sigma(0\bar 1)}], \\ & \boldsymbol\sigma \in \{L,M,R\}^* M\bar R \cup \{L,M,R\}^\infty \setminus \{L,M,R\}^*\{\bar L, \bar R\} \end{cases}$$

$\mu_{\mathbf u} > 1$ defined by $g_{\mathbf u}(\mu_{\mathbf u}) = \tilde g_{\mathbf u}(\mu_{\mathbf u})$ (if unique solution)



Example: $\sigma = L^k M,\ k \geq 0$

$$g_{\sigma(\bar{0})}(\beta_0) = \frac{1}{\beta_0^k(\beta_0 - 1)}, \qquad 2\,\mu^{k+1}_{\sigma(1\bar{0})} = \mu^{k}_{\sigma(1\bar{0})} + 2$$

$$\tilde{g}_{\sigma(\bar{1})}(\beta_0) = \frac{\beta_0^{k+2} - 1}{\beta_0^{k+1}(\beta_0 - 1)}, \qquad \mu^{k+2}_{\sigma(\bar{0})} = \mu_{\sigma(\bar{0})} + 1,\ \ \mu_{\sigma(\bar{1})} = \mu_{\sigma(\bar{0})}$$

$$g_{\sigma(0\bar{1})}(\beta_0) = \frac{2\beta_0 - 1}{\beta_0^{k+1}(\beta_0 - 1)}, \qquad \mu^{k+1}_{\sigma(0\bar{1})} = 2$$

$$\tilde{g}_{\sigma(1\bar{0})}(\beta_0) = \tfrac{1}{2\beta_0^{k+1}}\left(\tfrac{\beta_0^{k+2}-1}{\beta_0-1} + 1 + \sqrt{\left(\tfrac{\beta_0^{k+3}-1}{\beta_0-1} - \beta_0 + 3\right)\tfrac{\beta_0^{k+1}-1}{\beta_0-1}}\right)$$

$$\mu^{2k+3}_{\sigma(01\bar{0})} = \mu^{2k+2}_{\sigma(0\bar{1})} + 2\,\mu^{k+2}_{\sigma(01\bar{0})} - 3\,\mu^{k+1}_{\sigma(0\bar{1})} - \mu_{\sigma(01\bar{0})} + 3$$

$$3\,\mu^{k+3}_{\sigma(10\bar{1})} = 2\,\mu^{k+2}_{\sigma(10\bar{1})} + 6\,\mu^{2}_{\sigma(10\bar{1})} - 5\,\mu_{\sigma(10\bar{1})} + 1$$

$$k = 0: \quad \mathcal{G}(\beta_0) = \begin{cases} \frac{1}{\beta_0-1} & \text{if } \beta_0 \in \left[\frac{3}{2}, \frac{1+\sqrt{5}}{2}\right] \\ \frac{\beta_0+1}{\beta_0} & \text{if } \beta_0 \in \left[\frac{1+\sqrt{5}}{2}, 2\right] \end{cases}$$

$$\mathcal{L}(\beta_0) = \begin{cases} \frac{\beta_0+2+\sqrt{\beta_0^2+4}}{2\beta_0} & \text{if } \beta_0 \in [\frac{3}{2}, 1.6823278] \\ \frac{2\beta_0-1}{\beta_0(\beta_0-1)} & \text{if } \beta_0 \in [1.8711568, 2] \end{cases}$$