Distribution of cycles for one-dimensional random dynamical systems

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the One World Numeration Seminar
December 13, 2022
preprint available at https://arxiv.org/abs/2108.05522

Periodic points of dynamical systems

X: space
$T\colon X \to X$ map

$$T^n(x) := \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times composition}}(x) \quad (n = 0, 1, 2, \dots)$$

PROBLEM: Describe the structure of the orbit $\{T^n(x)\}_{n=0}^{\infty}$ for a majority of initial conditions x.

$$\mathrm{Fix}(T^n) := \{x \in X\colon \ T^n(x) = x\} \quad (n = 1, 2, \dots).$$

For $x \in \mathrm{Fix}(T^n)$, the set $\{x, T(x), \dots, T^{n-1}(x)\}$ is called a periodic orbit. The point x is called a *periodic point of period n*.

$$\mathrm{Fix}(T^n) \ni x \to T(x) \to T^2(x) \to \cdots \to T^n(x) = x.$$

- approximations of dynamical objects by periodic orbits (invariant set, invariant measure, pressure, ...)
- dynamical zeta function, Livschitz's theorem, ...

Distribution of periodic points

X: space
$T\colon X \to X$: map
$\varphi\colon X \to \mathbb{R}$ potential (weight function) $S_n\varphi := \sum_{k=0}^{n-1} \varphi \circ T^k$.

$$\nu_{n,\varphi} := \frac{1}{Z_n(\varphi)} \sum_{x \in \mathrm{Fix}(T^n)} \exp\left(S_n\varphi(x)\right) \delta_x$$

where

$$Z_n(\varphi) := \sum_{x \in \mathrm{Fix}(T^n)} \exp\left(S_n\varphi(x)\right).$$

Theorem 1 (Bowen 1975)

Let X be a topologically mixing subshift of finite type and $T\colon X \to X$ the left shift. For any Hölder continuous function $\varphi\colon X \to \mathbb{R}$, the sequence $\{\nu_{n,\varphi}\}_{n=1}^{\infty}$ converges to the equilibrium state for the potential φ.

How Theorem 1 can be extended to random dynamical systems?

Independently Identically Distributed random dynamical systems

$2 \leq N < \infty$ integer, $T_i \colon X \to X$ $(1 \leq i \leq N)$ maps
$p = (p_1, \ldots, p_N)$: probability vector with $\prod_{i=1}^{N} p_i \neq 0$.
We consider an i.i.d. random dynamical system in which T_i is chosen with probability p_i at each step.
$\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
$\Omega := \{1, 2, \ldots, N\}^{\mathbb{N}}$ sample space
For a *sample path* $\omega = (\omega_n)_{n \in \mathbb{N}} \in \Omega$ and $n \in \mathbb{N}$, consider a random composition

$$T_\omega^n := T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}.$$

For convenience, define T_ω^0 to be the identity map on X.
$\{T_\omega^n(x)\}_{n \in \mathbb{N}_0}$ is called a *random orbit with initial condition* x.

“Periodic points” of random dynamical systems

A *random cycle* is an element of the set

$$\bigcup_{n\in\mathbb{N}}\bigcup_{\omega\in\Omega}\mathrm{Fix}(T_\omega^n),$$

where

$$\mathrm{Fix}(T_\omega^n) := \{x\in X\colon\ T_\omega^n(x)=x\}.$$

$x\in\mathrm{Fix}(T^n)$ implies that the orbit $\{T^n(x)\}_{n\in\mathbb{N}_0}$ is finite as a set, whereas $x\in\mathrm{Fix}(T_\omega^n)$ does not imply the finiteness of the random orbit $\{T_\omega^n(x)\}_{n\in\mathbb{N}_0}$ as a set. Indeed, we have

$$\begin{aligned} T_\omega^{n+1}(x) &= T_{\omega_{n+1}}\circ T_{\omega_n}\circ\cdots\circ T_{\omega_2}\circ T_{\omega_1}(x)\\ &= T_{\omega_{n+1}}(x), \end{aligned}$$

which may not be contained in the set
$\{x, T_{\omega_1}(x), T_{\omega_2}\circ T_{\omega_1}(x),\ldots, T_{\omega_{n-1}}\circ\cdots\circ T_{\omega_1}(x)\}$.

Samplewise//Sample-averaged schemes

$$\bigcup_{n\in\mathbb{N}}\bigcup_{\omega\in\Omega}\mathrm{Fix}(T_\omega^n),$$

- *Samplewise (Quenched)*: Fix $\omega\in\Omega$, and ask behaviors of

$$\mathrm{Fix}(T_\omega^n)$$

  as $n\to\infty$.

- *Sample-averaged (Annealed)*: Ask behaviors of

$$\bigcup_{\omega\in\Omega}\mathrm{Fix}(T_\omega^n)$$

  as $n\to\infty$.

Some results/facts on random cycles

- Dynamical zeta functions defined by random cycles were considered by Ruelle (1990), Buzzi (2002).
- A dynamical zeta function defined by random cycles of certain random matrices cannot be extended holomorphically beyond its disk of holomorphy, almost surely. (Buzzi (2002))
- Distribution of $\bigcup_{\omega\in\Omega}\mathrm{Fix}(T_\omega^n)$ as $n\to\infty$ for Ruelle expanding maps. (Carvalho/Rodrigues/Varandas (2017))
- Growth of $\#\mathrm{Fix}(T_\omega^n)$ as $n\to\infty$ (Asaoka/Shinohara/Turaev (2017)) for random interval maps systems with expansion/ contraction

Fully branched uniformly expanding maps

X: compact interval

A *fully branched map* on X is a map $T\colon \bigcup_{a\in\mathcal{A}} J_a \to X$ where $\mathcal{A}\subset\mathbb{N}$ with $2\le\#\mathcal{A}<\infty$, and $(J_a)_{a\in\mathcal{A}}$ is a collection of pairwise disjoint subintervals of X such that:

- $X=\bigcup_{a\in\mathcal{A}} J_a$;
- for each $a\in\mathcal{A}$, the restriction of T to J_a extends to a C^2 diffeomorphism on $\mathrm{cl}(J_a)$;
- for each $a\in\mathcal{A}$, $\mathrm{cl}(T(J_a))=X$.

A fully branched map T on X is *uniformly expanding* if there exists a constant $\gamma>1$ such that $\inf_{x\in J_a}|(T|_{J_a})'x|\ge\gamma$ for any $a\in\mathcal{A}$.

I.i.d. random dynamical system

$T_1, \ldots, T_N$, $1 \leq N < \infty$ fully branched uniformly expanding maps on X (do not assume a common Markov partition).
$\Omega := \{1, 2, \ldots, N\}^{\mathbb{N}}$, $p = (p_1, \ldots, p_N)$ probability vector with $\prod_{i=1}^{N} p_i > 0$, m_p: Bernoulli measure on Ω determined by p.
For a sample $\omega = (\omega_n)_{n \in \mathbb{N}} \in \Omega$ and $n \geq 1$,

$$T_\omega^n(x) := T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1}(x), \; T_\omega^0(x) = x.$$

By Pelikan's theorem (1984), $\exists!$ a Borel probability measure λ_p on X s.t. $\lambda_p \ll \mathrm{Leb}$ and $\lambda_p = \sum_{i=1}^{N} p_i \lambda_p \circ T_i^{-1}$. From the random ergodic theorem, For m_p-a.e. $\omega \in \Omega$ and any $\phi \colon X \to \mathbb{R}$ continuous,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T_\omega^k(x)) = \int \phi d\lambda_p \quad \text{for } \lambda_p\text{-a.e. } x \in X,$$

namely, for λ_p-a.e. $x \in X$,

$$\delta_x^{\omega,n} := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T_\omega^k(x)} \to \lambda_p \quad \text{in the weak* topology as } n \to \infty.$$

Random cycle measures

For $\omega \in \Omega$, define a *samplewise random cycle measure* ξ_n^ω *on* X by

$$\xi_n^\omega = \frac{1}{Z_{\omega,n}} \sum_{x \in \mathrm{Fix}(T_\omega^n)} |(T_\omega^n)' x|^{-1} \delta_x^{\omega,n} \quad (n = 1, 2, \ldots),$$

where $(T_\omega^n)' x := \prod_{i=1}^n (T_{\omega_i})'(T_\omega^{i-1}(x))$ and $Z_{\omega,n} := \sum_{x \in \mathrm{Fix}(T_\omega^n)} |(T_\omega^n)' x|^{-1}$ is the normalizing constant.

(Distribution of random cycles). Does the sequence $\{\xi_n^\omega\}_{n=1}^\infty$ *converge? If so, what is the limit measure?*

$\mathcal{M}(X)$: the space of Borel probability measures on X.
For $\omega \in \Omega$, define a *samplewise random cycle measure* $\tilde{\xi}_n^\omega$ *on* $\mathcal{M}(X)$ by

$$\tilde{\xi}_n^\omega = \frac{1}{Z_{\omega,n}} \sum_{x \in \mathrm{Fix}(T_\omega^n)} |(T_\omega^n)' x|^{-1} \delta_{\delta_x^{\omega,n}} \quad (n = 1, 2, \ldots),$$

where $\delta_{\delta_x^{\omega,n}}$ is the unit point mass at $\delta_x^{\omega,n}$.

Distribution of random cycles: samplewise result

Theorem A

Let $2 \leq N < \infty$, and let $T_1, \ldots, T_N$ be fully branched uniformly expanding maps on X. Let $p = (p_1, \ldots, p_N)$ be a probability vector with $\prod_{i=1}^N p_i > 0$. For m_p-almost every $\omega \in \Omega$, the sequence $\{\tilde{\xi}_n^\omega\}_{n=1}^\infty$ of samplewise random cycle measures on $\mathcal{M}(X)$ converges to the unit point mass at λ_p in the weak topology.*

i.e., for any continuous function $\tilde{\varphi} \colon \mathcal{M}(X) \to \mathbb{R}$, $\int \tilde{\varphi} d\tilde{\xi}_n^\omega \to \tilde{\varphi}(\lambda_p)$.

Corollary 1

For m_p-almost every $\omega \in \Omega$, the sequence $(\xi_n^\omega)_{n=1}^\infty$ converges in the weak topology to λ_p as $n \to \infty$.*

i.e., for any continuous function $\varphi \colon X \to \mathbb{R}$, $\int \varphi d\xi_n^\omega \to \int \varphi d\lambda_p$.

Proof of Corollary 1.

Given $\varphi \colon X \to \mathbb{R}$, apply Theorem A to the continuous function $\nu \in \mathcal{M}(X) \mapsto \int \varphi d\nu \in \mathbb{R}$. $\square$

Distribution of random cycles: samplewise result

Corollary 2 (Inspired by Olsen (2003))

Let $T_1, \ldots, T_N$ and $p = (p_1, \ldots, p_N)$ be as in Theorem A.

(a) *If $\varphi, \psi \colon X \to \mathbb{R}$ are continuous, then for m_p-almost every $\omega \in \Omega$,*

$$\lim_{n\to\infty} \frac{1}{Z_{\omega,n}} \sum_{x \in \mathrm{Fix}(T_\omega^n)} |(T_\omega^n)'(x)|^{-1} \frac{1}{n^2} \sum_{k=0}^{n-1} \varphi(T_\omega^k(x)) \sum_{k=0}^{n-1} \psi(T_\omega^k(x)) = \int \varphi d\lambda_p \int \psi d\lambda_p.$$

(b) *If $\varphi \colon X \to \mathbb{R}$, $\psi \colon X \to \mathbb{R}$ are continuous with $\inf \psi > 0$, then for m_p-almost every $\omega \in \Omega$,*

$$\lim_{n\to\infty} \frac{1}{Z_{\omega,n}} \sum_{x \in \mathrm{Fix}(T_\omega^n)} |(T_\omega^n)'(x)|^{-1} \frac{\sum_{k=0}^{n-1} \varphi(T_\omega^k(x))}{\sum_{k=0}^{n-1} \psi(T_\omega^k(x))} = \frac{\int \varphi d\lambda_p}{\int \psi d\lambda_p}.$$

Corollary 2 (Continued)

(c) *If* $\pi_1, \pi_2 \colon X \to \mathbb{R}$ *are continuous and* $g \colon \mathbb{R} \to \mathbb{R}$ *is bounded continuous, then for* m_p*-almost every* $\omega \in \Omega$ *we have*

$$\lim_{n\to\infty} \frac{1}{Z_{\omega,n}} \sum_{x\in \mathrm{Fix}(T^n_\omega)} |(T^n_\omega)'(x)|^{-1} \frac{1}{n^2} \times$$
$$\sum_{k_1,k_2=0}^{n-1} g(\pi_1(T^{k_1}_\omega(x)) + \pi_2(T^{k_2}_\omega(x)))$$
$$= \int g d(\lambda_p \circ \pi_1^{-1} \otimes \lambda_p \circ \pi_2^{-1}),$$

where $\otimes$ *denotes the convolution.*

Proof of Corollary 2.

Apply Theorem A to the continuous functions
$\nu \in \mathcal{M}(X) \mapsto \int \varphi d\nu \int \psi d\nu$, $\nu \in \mathcal{M}(X) \mapsto \int \varphi d\nu / \int \psi d\nu$,
$\nu \in \mathcal{M}(X) \mapsto \int g d(\nu \circ \pi_1^{-1} \otimes \nu \circ \pi_2^{-1})$ respectively. $\square$

Distribution of random cycles: sample-averaged result

By Riesz's representation theorem, for each $p = (p_1, \ldots, p_N)$ and $n \in \mathbb{N}$ $\exists!$ a Borel probability measure $\tilde{\eta}_{p,n}$ on $\mathcal{M}(X)$ s.t.

$$\int \tilde{\varphi} d\tilde{\eta}_{p,n} = \int dm_p(\omega) \int \tilde{\varphi} d\tilde{\xi}_n^\omega \quad \text{for any continuous } \tilde{\varphi} \colon \mathcal{M}(X) \to \mathbb{R}.$$

Also, $\exists!$ a Borel probability measure $\eta_{p,n}$ on X s.t.

$$\int \varphi d\eta_{p,n} = \int dm_p(\omega) \int \varphi d\xi_n^\omega \quad \text{for any continuous } \varphi \colon X \to \mathbb{R}.$$

Corollary 3

Let $T_1, \ldots, T_N$ and $p = (p_1, \ldots, p_N)$ be as in Theorem A. Then $(\tilde{\eta}_{p,n})_{n=1}^\infty$ converges to δ_{λ_p} in the weak topology as $n \to \infty$ and $(\eta_{p,n})_{n=1}^\infty$ converges to λ_p in the weak* topology as $n \to \infty$.*

Distribution of random cycles: sample-averaged result

For $\omega \in \Omega$ and $n \in \mathbb{N}$, write $T_{\omega_1 \cdots \omega_n} = T^n_\omega$ and $\delta_x^{\omega_1 \cdots \omega_n} = \delta_x^{\omega, n}$.
For $p = (p_1, \ldots, p_N)$, $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{1, \ldots, N\}^n$, put

$$Q_p(\omega_1 \cdots \omega_n) := \prod_{i=1}^{N} p_i^{\#\{1 \le k \le n :\ \omega_k = i\}}.$$

Define an *averaged random cycle measure* on X by

$$\kappa_{p,n} := \frac{\sum_{\omega_1 \cdots \omega_n \in \{1,\ldots,N\}^n}}{\text{normalize}} \left(Q_p(\omega_1 \cdots \omega_n) \sum_{x \in \mathrm{Fix}(T_{\omega_1 \cdots \omega_n})} |(T_{\omega_1 \cdots \omega_n})' x|^{-1} \delta_x^{\omega_1 \cdots \omega_n} \right),$$

and define an *averaged random cycle measure* on $\mathcal{M}(X)$ by

$$\tilde{\kappa}_{p,n} := \frac{\sum_{\omega_1 \cdots \omega_n \in \{1,\ldots,N\}^n}}{\text{normalize}} \left(Q_p(\omega_1 \cdots \omega_n) \sum_{x \in \mathrm{Fix}(T_{\omega_1 \cdots \omega_n})} |(T_{\omega_1 \cdots \omega_n})' x|^{-1} \delta_x^{\omega_1 \cdots \omega_n} \right).$$

Distribution of random cycles: sample-averaged result

Theorem B

Let $2 \leq N < \infty$, and let $T_1, \ldots, T_N$ be fully branched uniformly expanding maps on a compact interval X. Let $p = (p_1, \ldots, p_N)$ be a probability vector with $\prod_{i=1}^N p_i > 0$. The sequence $\{\tilde{\kappa}_{p,n}\}_{n=1}^\infty$ of sample-averaged random cycle measures converges to the unit point mass at λ_p in the weak topology.*

i.e., for any continuous function $\tilde{\varphi}\colon \mathcal{M}(X) \to \mathbb{R}$,
$\int \tilde{\varphi} d\tilde{\kappa}_{p,n} \to \tilde{\varphi}(\lambda_p)$.

Corollary 4

Let $T_1, \ldots, T_N$ and $p = (p_1, \ldots, p_N)$ be as in Theorem B. The sequence $\{\kappa_{p,n}\}_{n=1}^\infty$ of sample-averaged random cycle measures converges to λ_p in the weak topology.*

i.e., for any continuous function $\varphi\colon X \to \mathbb{R}$, $\int \varphi d\kappa_{p,n} \to \int \varphi d\lambda_p$.

Strategy for a proof of Theorem A

Consider a skew product map

$$R\colon (\omega, x) \in \Omega \times X \mapsto (\theta\omega,\, T_{\omega_1} x) \in \Omega \times X,$$

where $\theta\colon \Omega \to \Omega$ denotes the left shift $(\theta\omega)_k = \omega_{k+1}$.

$$\begin{array}{ccc} \Omega \times X & \xrightarrow{R} & \Omega \times X \\ {\scriptstyle \text{projection}} \Big\downarrow & & \Big\downarrow {\scriptstyle \text{projection}} \\ X & \xrightarrow{T_{\omega_1}} & X. \end{array}$$

Key observation: $x \in \mathrm{Fix}(T^n_\omega) \Longrightarrow (\omega', x) \in \mathrm{Fix}(R^n)$, where ω' is the repetition of $\omega_1\omega_2\cdots\omega_n$ and
$\mathrm{Fix}(R^n) = \{(\omega, x) \in \Omega \times X\colon R^n(\omega, x) = (\omega, x)\}$.

1. (Level-2) large deviation principle on periodic points of R (Kifer (1994))
2. Conversion to samplewise large deviations (adapt Aimino/Nicol/Vaienti (2015))
3. Project to the original space X.

Sample-averaged Level-2 large deviations

$\mathcal{M}(\Omega \times X)$: the space of Borel probability measures on $\Omega \times X$.
For $(\omega, x) \in \Omega \times X$ and $n \geq 1$, let $\delta^n_{(\omega,x)} = (1/n)\sum_{k=0}^{n-1} \delta_{R^k(\omega,x)}$.
Define a Borel probability measure $\tilde{\mu}_n$ on $\mathcal{M}(\Omega \times X)$ by

$$\tilde{\mu}_n := \frac{1}{\mathrm{normalize}} \sum_{(\omega,x)\in \mathrm{Fix}(R^n)} Q_p(\omega_1 \cdots \omega_n)|(T^n_\omega)'x|^{-1}\delta_{\delta^n_{(\omega,x)}},$$

where $\delta_{\delta^n_{(\omega,x)}}$ is the unit point mass at $\delta^n_{(\omega,x)}$.

Proposition 1 (Kifer (1994) Large Deviation Principle)

There exists a lower semicontinuous function $I\colon \mathcal{M}(\Omega \times X) \to [0,\infty]$ such that: (a) $I(\mu) = 0$ iff $\mu = m_p \times \lambda_p$; (b) for any Borel set $\mathcal{B} \subset \mathcal{M}(\Omega \times X)$,

$$-\inf_{\mathrm{int}\mathcal{B}} I \leq \liminf_{n\to\infty} \frac{1}{n} \log \tilde{\mu}_n(\mathrm{int}\mathcal{B}) \leq \limsup_{n\to\infty} \frac{1}{n} \log \tilde{\mu}_n(\mathrm{cl}\mathcal{B}) \leq -\inf_{\mathrm{cl}\mathcal{B}} I.$$

Conversion to samplewise level-2 large deviations

For each $\omega \in \Omega$ and $n \geq 1$, define a Borel probability measure $\tilde{\mu}_n^\omega$ on $\mathcal{M}(\Omega \times X)$ by

$$\tilde{\mu}_n^\omega := \frac{1}{Z_{\omega,n}} \sum_{x \in \mathrm{Fix}(T_\omega^n)} |(T_\omega^n)' x|^{-1} \delta_{\delta^n_{(\omega,x)}}.$$

Proposition 2 (Samplewise large deviations upper bound)

For m_p-almost every $\omega \in \Omega$ and any closed subset $\mathcal{C}$ of $\mathcal{M}(\Omega \times X)$, we have

$$\limsup_{n \to \infty} \frac{1}{n} \log \tilde{\mu}_n^\omega(\mathcal{C}) \leq -\inf_{\mathcal{C}} I.$$

For our purpose, there is no need for a lower bound.
Idea of proof of Proposition 2: Adapt the trick of conversion (sample-averaged $\Longrightarrow$ samplewise) by Aimino/Nicol/Vaienti (2015) to periodic points (random cycles).

Conversion to samplewise level-2 large deviations

Since $\mathcal{M}(\Omega \times X)$ is metrizable, it is separable. So, enough to show that for each closed set $\mathcal{C}$, $\exists$ a Borel set $\Omega_{\mathcal{C}} \subset \Omega$ s.t. for m_p-a.e. $\omega \in \Omega_{\mathcal{C}}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \tilde{\mu}_n^{\omega}(\mathcal{C}) \leq -\inf_{\mathcal{C}} I.$$

We may assume $0 < \inf_{\mathcal{C}} I < \infty$. There is a uniform constant $K > 0$ such that

$$\begin{aligned}
\tilde{\mu}_n(\mathcal{C}) &= \frac{1}{\text{normalize}} \sum_{\substack{(\omega,x)\in \mathrm{Fix}(R^n) \\ \delta^n_{(\omega,x)} \in \mathcal{C}}} Q_p(\omega_1 \cdots \omega_n) |(T^n_\omega)' x|^{-1} \\
&= \int \tilde{\mu}_n^{\omega}(\mathcal{C}) \left(Z_{\omega,n} \Big/ \int Z_{\omega',n} \mathrm{d}m_p(\omega') \right) \mathrm{d}m_p(\omega) \\
&\geq K \int \tilde{\mu}_n^{\omega}(\mathcal{C}) \, \mathrm{d}m_p(\omega).
\end{aligned}$$

Key: $Z_{\omega,n}$ is bounded away from 0 and $+\infty$ uniformly on ω and n.

Conversion to samplewise level-2 large deviations

For $\epsilon \in (0,1)$ and $n \geq 1$, set

$$\Omega_{\epsilon,n} = \left\{ \omega \in \Omega \colon \tilde{\mu}_n^{\omega}(\mathcal{C}) \geq \exp\left(-n(1-\epsilon) \inf_{\mathcal{C}} I \right) \right\}.$$

By Markov's inequality,

$$\begin{aligned} m_p(\Omega_{\epsilon,n}) &\leq \exp\left(n(1-\epsilon) \inf_{\mathcal{C}} I \right) \int \tilde{\mu}_n^{\omega}(\mathcal{C}) \mathrm{d} m_p(\omega) \\ &\leq K^{-1} \exp\left(n(1-\epsilon) \inf_{\mathcal{C}} I \right) \tilde{\mu}_n(\mathcal{C}). \end{aligned}$$

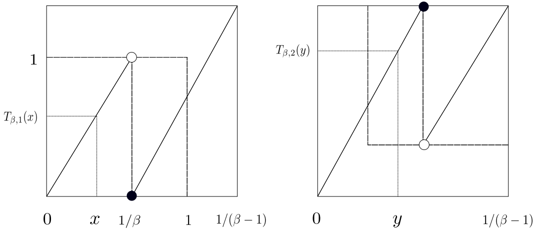
By Proposition, $\tilde{\mu}_n(\mathcal{C})$ decays exponentially as $n \to \infty$, so $m_p(\Omega_{\epsilon,n})$ decays exponentially as $n \to \infty$. By Borel-Cantelli's lemma,

$$\#\{ n \in \mathbb{N} \colon \tilde{\mu}_n^{\omega}(\mathcal{C}) \geq \exp(-n(1-\epsilon) \inf_{\mathcal{C}} I) \} < \infty$$

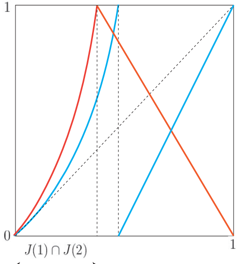
for m_p-almost every $\omega \in \Omega$. Since ϵ is arbitrary, we obtain the desired upper bound for m_p-almost every $\omega \in \Omega$. $\square$

Some possible extensions of the main results

- maps with non-full branches (Dajani/de Vries (2005))



- maps with neutral fixed points (Liverani/Saussol/Vaienti (1999) etc.)



Some possible extensions of the main results

- maps with infinitely many branches (Kalle/Kempton/ Verbitskiy (2017) etc.)

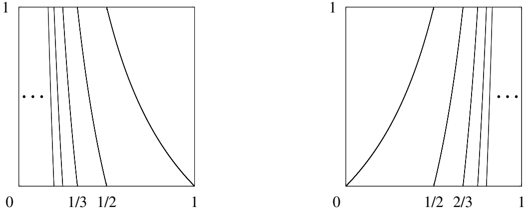


Figure: graphs of the Gauss and Rényi transformations

Thank you for your attention.