Finite β-expansions of natural numbers

Fumichika Takamizo
(Osaka Metropolitan University)

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Contents

Notation

- $\beta > 1$
- $\mathbb{N} = \{1, 2, 3, \cdots\}$ and $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $\lfloor x \rfloor$ is the integer part of x, and $\{x\} := x - \lfloor x \rfloor$.
- $T(x) := \{\beta x\}$ for $x \in [0, 1]$, T is called the β-transformation.
- $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$ where $a, b \in \mathbb{Z}$ and $a < b$
- $\nu(c_1 c_2 \cdots) = \sum_{n \geq 1} c_n \beta^{-n}$ where $c_1 c_2 \cdots \in \mathbb{Z}^{\mathbb{N}}$ satisfies $\sum_{n \geq 1} |c_n| \beta^{-n} < \infty$.

What is β-expansion?

Let $x_n = T(x_{n-1})$ and $c_n = \lfloor \beta x_{n-1} \rfloor$ for $x_0 = x \in [0,1]$. Then $c_n \in [\![0, \lfloor \beta \rfloor]\!]$. Moreover,

$$x_1 = \beta x_0 - c_1 \quad \text{i.e.,} \quad x_0 = c_1\beta^{-1} + x_1\beta^{-1}$$
$$x_2 = \beta x_1 - c_2 \quad \text{i.e.,} \quad x_1 = c_2\beta^{-1} + x_2\beta^{-1}$$
$$\cdots .$$

So we have the expansion of x, that is,

$$x = x_0 = c_1\beta^{-1} + x_1\beta^{-1} = c_1\beta^{-1} + c_2\beta^{-2} + x_2\beta^{-2} = \cdots = \sum_{n=1}^{\infty} c_n\beta^{-n}$$

and write

$$d_\beta(x) = c_1 c_2 \cdots c_n \cdots .$$

The β-*expansion of* x is given by

$$x = c_1\beta^{L(x)-1} + \cdots c_{L(x)-1}\beta + c_{L(x)} + \frac{c_{L(x)+1}}{\beta} + \cdots = \sum_{n=1}^{\infty} c_n\beta^{L(x)-n}$$

where

$$L(x) := \min\{n \in \mathbb{N}_0 \mid x\beta^{-n} < 1\} \quad \& \quad d_\beta(\beta^{-L(x)}x) = c_1c_2\cdots .$$

Symbolically, we write

$$x = c_1 \cdots c_{L(x)}.c_{L(x)+1} \cdots .$$

Example:

(1) $d_\beta(1) = \lfloor\beta\rfloor d_\beta(\{\beta\})$

(2) the β-expansion of $1 = 1.0^\infty$

Main Theorem

Def

x has the finite β-expansion if $T^n(\beta^{-L(x)}x) = 0$ for some $n \in \mathbb{N}$.

Let $\mathrm{Fin}(\beta) = \{x \in [0, \infty) \mid x \text{ has the finite } \beta\text{-expansion}\}$.

Def (Frougny-Solomyak)

(F$_1$) $\mathbb{N}_0 \subset \mathrm{Fin}(\beta)$

(PF) $\mathbb{N}_0[1/\beta] \subset \mathrm{Fin}(\beta)$ where

$$\mathbb{N}_0[1/\beta] := \{\textstyle\sum_{k=0}^{n} c_k \beta^{-k} \mid c_k \in \mathbb{N}_0, n \in \mathbb{N}_0\}.$$

(F) $\mathbb{Z}[1/\beta]_{\geq 0} \subset \mathrm{Fin}(\beta)$ where

$$\mathbb{Z}[1/\beta]_{\geq 0} := \{x = \textstyle\sum_{k=0}^{n} c_k \beta^{-k} \mid x \geq 0, n \in \mathbb{N}_0, c_k \in \mathbb{Z}\}.$$

Def

An algebraic integer β is a Pisot number if

(i) $\beta > 1$ (ii) $\gamma\ (\neq \beta)$ is a conjugetes of $\beta \Rightarrow |\gamma| < 1$

Note: β is a Pisot number with degree $1 \Rightarrow \beta \in \mathbb{Z}_{\geq 2}$ and so β has (F).

Thm (Frougny-Solomyak & Akiyama)

β has $(\mathrm{F}_1) \Rightarrow \beta$ is a Pisot number

Thm (Frougny-Solomyak)

β is a Pisot number with degree $2 \Rightarrow \beta$ has (PF)

Example (Frougny-Solomyak & Hollander): Let β be an algebraic intger with minimal polynomial $x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0$. Suppose that a_j ($j \in [\![0, d-1]\!]$) satisfy one of the following:

(FS) $a_{d-1} \geq \cdots a_1 \geq a_0 \geq 1$. (H) $a_{d-1} > \sum_{j=0}^{d-2} a_j$, $a_j \geq 0$.

Then β has (F).

Example: Let β be a Pisot number with minimal polynomial $x^2 - 3x + 1$. Then β has (PF) without (F). For example,

$$d_\beta(1 - \beta^{-1}) = 1^\infty.$$

Example (Frougny-Solomyak): Let β be a Pisot number with minimal polynomial $x^3 - 3x^2 + 2x - 2$. Then β does not have (F_1). Indeed,

$$\text{the } \beta\text{-expansion of } 3 = 10.111(00012)^\infty.$$

Main Theorem

Let $\beta > 1$ be an algebraic integer with minimal polnomial $x^3 - 2tx^2 + 2tx - t$ ($t \in \mathbb{Z}_{\geq 2}$). Then β has (F_1) without (PF).

By using the following theorem, we can show that above β does not have (PF).

Thm (Akiyama)

Let $\beta > 1$. β has (PF) without (F) if and only if β is a Pisot number whose minimal polynomial is of the form:

$$x^d - (\lfloor \beta \rfloor + 1)x^{d-1} + \textstyle\sum_{j=2}^{d} a_j x^{d-j}$$

with $a_j \geq 0$ and $\sum_{j=2}^{d} a_j < \lfloor \beta \rfloor$.

Key Lemma

Let $x \in [0, \infty)$ and the β-expansion of x be

$$x = c_1 \cdots c_{L(x)} . c_{L(x)+1} \cdots .$$

We define

$$\{x\}_\beta = \sum_{n \geq L(x)+1} c_n \beta^{L(x)-n} = .c_{L(x)+1} c_{L(x)+2} \cdots$$

and call it *β-fractional part of x*.

Key Lemma

${}^\forall x \in [0, \infty)$, ${}^\exists \theta \in \{0, 1\}$, $\omega_n \in \mathbb{N}_0$ $(n \in [\![0, r]\!])$ s.t.

$$\{x+1\}_\beta - \{x\}_\beta = \theta - \sum_{n=0}^{r} \omega_n T^n(1).$$

Now $<_{lex}$ means the lexicographic order and $\leq_{lex}$ means $<_{lex}$ or $=$.

Remark: For $x, y \in [0,1]$, $x < y \Leftrightarrow d_\beta(x) <_{lex} d_\beta(y)$.

Define ρ on $\mathbb{N}_0^{\mathbb{N}}$ by $\rho(c_1 c_2 \cdots, c'_1 c'_2 \cdots) := (\inf\{n \mid c_n \neq c'_n\})^{-1}$. We can define

$$d_\beta^*(x) := \lim_{y \uparrow x} d_\beta(y) \text{ for each } x \in (0,1].$$

Since ν is continuous on $[\![0, \lfloor \beta \rfloor]\!]^{\mathbb{N}}$, we get $x = \nu(d_\beta^*(x))$.

Thm (Parry)

$$c_1 c_2 \cdots \in d_\beta([0,1)) \Leftrightarrow 0^\infty \leq_{lex} c_n c_{n+1} \cdots <_{lex} d_\beta^*(1) \ (\forall n \in \mathbb{N}_0)$$

Moreover we have

$$d_\beta^*(1) = \begin{cases} (d_1 \cdots d_q)^\infty & \text{if } d_\beta(1) = d_1 \cdots d_{q-1}(d_q + 1)0^\infty \\ d_\beta(1) & \text{otherwise} \end{cases}$$

Def: For $\boldsymbol{c} = c_1 c_2 \cdots$, let

$$\boldsymbol{c}[i,j] := \begin{cases} c_i c_{i+1} \cdots c_j & \text{if } i \leq j < \infty \\ c_i c_{i+1} \cdots & \text{if } j = \infty \\ \text{empty word} & \text{if } i > j \end{cases}$$

Note: $\forall n \in \mathbb{N}, \exists p \in \mathbb{N}_0$ s.t. $T^p(1) = \nu(d^*_\beta(1)[n,\infty])$

For $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{N}_0^{\mathbb{N}}$, define

$$\boldsymbol{c} \overset{\nu}{=} \boldsymbol{c}' \overset{def}{\iff} \nu(\boldsymbol{c}) = \nu(\boldsymbol{c}')$$

$$\boldsymbol{c} \ominus \boldsymbol{c}' := (c_1 - c_1')(c_2 - c_2') \cdots .$$

Then we have the *carry formula*.

Obs: Let $\ell \in \mathbb{N}$ and $\boldsymbol{c} := c_1 c_2 \cdots \in [\![0, \lfloor \beta \rfloor]\!]^{\mathbb{N}}$. Then

$$\boldsymbol{c} \overset{\nu}{=} \boldsymbol{c}[1, \ell-1](c_\ell + 1)(\boldsymbol{c}[\ell+1, \infty] \ominus d^*_\beta(1)).$$

Def

Let $\boldsymbol{c} = c_1c_2\cdots \in d_\beta([0,1))$. $\{k_i\}_{i\geq 1} \subset \mathbb{N}$ gives a block decomposition (BD for short) of $\boldsymbol{c}$ if $\{k_i\}_{i\geq 1}$ satisfies $c_{k_{i+1}} < d_{k_{i+1}-k_i}$ and

$$\boldsymbol{c} = d_\beta^*(1)[1, k_1 - 1]c_{k_1} d_\beta^*(1)[1, k_2 - k_1 - 1]c_{k_2}\cdots.$$

<u>**Remark 1**</u>: Let $\boldsymbol{c} = c_1c_2\cdots$ and $\ell \in \mathbb{N}$ satisfy $\nu(\boldsymbol{c}_0^+) < 1$ where

$$\boldsymbol{c}_0^+ = \boldsymbol{c}[1, \ell - 1](c_\ell + 1)\boldsymbol{c}[\ell + 1, \infty].$$

Let $\{k_j\}$ give the BD of $\boldsymbol{c}$ and $k_i < \ell \leq k_{i+1}$. Then

$$\boldsymbol{c}_0^+ \stackrel{\nu}{=} \boldsymbol{c}_0^{carry} := \boldsymbol{c}[1, k_i - 1](c_{k_i} + 1)0^{\ell - k_i - 1}\theta\boldsymbol{c}_1'$$

where

$$\theta := c_\ell + 1 - d_{\ell - k_i} \ \& \ \boldsymbol{c}_1' := \boldsymbol{c}[\ell + 1, \infty] \ominus d_\beta^*(1)[\ell - k_i + 1, \infty].$$

<u>**Proof**</u>: Note $\boldsymbol{c}[k_i + 1, \ell - 1] = d_\beta^*(1)[1, \ell - k_i - 1]$.

The following example explains how to use Remark 1.

Example: Let $\beta > 1$ be the algebraic integer with minimal polynomial $x^3 - x^2 - x - 1$. Then $d_\beta^*(1) = (110)^\infty$. Let

$$\boldsymbol{c} = c_1 c_2 \cdots := 10(110)^2 10^\infty \text{ and } \ell = 9.$$

Then the BD of $\boldsymbol{c}$ is $\{k_j\}_{j \geq 1} = \{2, 10, 11, 12, \cdots\}$. So by Remark 1,

$$\boldsymbol{c}_0^+ := 10(110)^2 20^\infty \overset{\nu}{=} 110^6 1 \boldsymbol{c}_1' \text{ where } \boldsymbol{c}_1' := 0^\infty \ominus d_\beta^*(1)[8, \infty].$$

Moreover, since $1 + \nu(\boldsymbol{c}_1') = 1 - \beta^{-1} - \beta^{-2} \in [0, 1)$, we have

$$\boldsymbol{c}_0^+ \overset{\nu}{=} \boldsymbol{c}_1^+ := 110^7 d_\beta(1 + \nu(\boldsymbol{c}_1'))).$$

In order to prove Key Lemma, we need the repeated use of the carry formula.

Lemma: Let $\boldsymbol{c}' \in d_\beta([0,1))$ and $\boldsymbol{c} = c_1 c_2 \cdots \in d_\beta([0,1))$ with its BD $\{k_j\}_j$. Suppose that $i \in \mathbb{N}$ and $\ell \in \mathbb{N}$ with $\ell > k_i$. Define

$$\tilde{\boldsymbol{c}} = \begin{cases} \boldsymbol{c}[1, \ell - 1](c_\ell + 1)\boldsymbol{c}' & \text{if } k_i < \ell < k_{i+1} \\ \boldsymbol{c}[1, k_{i+1} - 1](c_{k_{i+1}} + 1)0^{\ell - k_{i+1}}\boldsymbol{c}' & \text{if } \ell \geq k_{i+1} \end{cases}$$

$$\& \ \tilde{\boldsymbol{c}}^{carry} = \boldsymbol{c}[1, k_i - 1](c_{k_i} + 1)0^{\ell - k_i - 1}\theta(\boldsymbol{c}' \ominus d_\beta^*(1)[\ell - k_i + 1, \infty])$$

where 0^0 is empty and

$$\theta = \begin{cases} 1 & \text{if } k_i < \ell < k_{i+1} \\ 0 & \text{if } \ell \geq k_{i+1}. \end{cases}$$

Then

$$\tilde{\boldsymbol{c}} \notin d_\beta([0,1)) \Rightarrow \tilde{\boldsymbol{c}} \stackrel{\nu}{=} \tilde{\boldsymbol{c}}^{carry}$$

where $\theta + \nu(\boldsymbol{c}') - \nu(d_\beta^*(1)[\ell - k_i + 1]) \geq 0$.

Example: Let $\beta > 1$ be the algebraic integer with minimal polynomial $x^3 - x - 1$. Then $d^*_\beta(1) = (10000)^\infty$. Let

$$\boldsymbol{c} = c_1 c_2 \cdots := 00010^\infty \text{ and } \ell = 5.$$

Then the BD of $\boldsymbol{c}$ is $\{k_j\}_{j\geq 1} = \{1, 2, 3, 9, 10, \cdots\}$. So by Remark 1,

$$\boldsymbol{c}_0^+ := 000110^\infty \overset{\nu}{=} 00101\boldsymbol{c}'_1 \text{ where } \boldsymbol{c}'_1 := 0^\infty \ominus d^*_\beta(1)[3, \infty].$$

Moreover, since $1 + \nu(\boldsymbol{c}'_1) = 1 - \beta^{-3} = \beta^{-2} \in [0, 1)$, we have

$$\boldsymbol{c}_0^+ \overset{\nu}{=} \boldsymbol{c}_1^+ := 00100 d_\beta(1 + \nu(\boldsymbol{c}'_1)).$$

Here notice that $\boldsymbol{c}_1^+ = 001000010^\infty \notin d_\beta([0, 1))$. So by Lemma,

$$\boldsymbol{c}_1^+ \overset{\nu}{=} \tilde{\boldsymbol{c}}^{carry} := 01000\boldsymbol{c}'_2$$

where $\boldsymbol{c}'_2 = d_\beta(1 + \nu(\boldsymbol{c}'_1)) \ominus d^*_\beta(1)[4, \infty]$. Hence we have

$$\boldsymbol{c}_1^+ \overset{\nu}{=} \boldsymbol{c}_2^+ := 010^\infty$$

because $d_\beta(\nu(\boldsymbol{c}'_2)) = 0^\infty$.

Sketch of the proof of Key Lemma: Let $x \in [0, \infty)$, $\ell = L(x+1)$ and

$$\boldsymbol{c} = c_1 c_2 \cdots := 0^{\ell - L(x)} d_\beta(\beta^{-L(x)} x)$$

and

$$\boldsymbol{c}_0^+ := \boldsymbol{c}[1, \ell - 1](c_\ell + 1)\boldsymbol{c}[\ell + 1, \infty].$$

Note $\{x\}_\beta = \nu(\boldsymbol{c}[\ell - L(x) + 1, \infty])$ and $\{x+1\}_\beta = \nu(\boldsymbol{c}_0^+[\ell + 1, \infty])$. So by Lemma, we get

$$\{x+1\}_\beta - \{x\}_\beta = \theta - \sum_{j=1}^{q} \xi(k_j) \text{ for some } k_j \in \mathbb{N} \text{ and } q \in \mathbb{N}.$$

where $\xi(n) := \nu(d_\beta^*(1)[n, \infty])$. So, since $T^k(1) = \xi(n)$ for some k, we can choose ω_j and r such that

$$\{x+1\}_\beta - \{x\}_\beta = \theta - \textstyle\sum_{j=0}^{r} \omega_j T^j(1).$$

Cor 1

For each $N \in \mathbb{N}$, there is $\{\omega_n\}_{n=0}^{r}$ $(\omega_n \in \mathbb{N}_0)$ such that

$$\{N\}_\beta = \left\{ -\sum_{n=1}^{r} \omega_n T^n(1) \right\}.$$

Proof: Suppose that

$$\{N\}_\beta = \{-\textstyle\sum_{n=1}^{r} \omega_n T^n(1)\}.$$

Here, we may take r to be sufficiently large. By Key Lemma,

$^\exists\theta' \in \{0,1\}$, $^\exists\omega_n' \in \mathbb{N}_0$ s.t. $\{N+1\}_\beta - \{N\}_\beta = \theta - \sum_{n=1}^{r} \omega_n' T^n(1)$.

Note $\{N+1\}_\beta \in [0,1)$. So

$$\begin{aligned}\{N+1\}_\beta = \{\{N+1\}_\beta\} &= \{\theta - \textstyle\sum_{n=1}^{r} \omega_n' T^n(1) + \{-\sum_{n=1}^{r} \omega_n T^n(1)\}\} \\ &= \{-\textstyle\sum_{n=1}^{r} (\omega_n' + \omega_n) T^n(1)\}.\end{aligned}$$

Sufficient condition for (F_1)

Let $\beta > 1$ be an algebraic integer with minimal polynomial

$$x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0.$$

For $\boldsymbol{\ell} = (l_1, l_2, \cdots, l_{d-1}) \in \mathbb{Z}^{d-1}$, define λ, τ by

$$\lambda(\boldsymbol{\ell}) := \sum_{j=1}^{d-1} l_j \left(\sum_{i=1}^{j} a_{j-i}\beta^{-i} \right), \qquad \tau(\boldsymbol{\ell}) := (l_2, \cdots, l_{d-1}, -\lfloor \lambda(\boldsymbol{\ell}) \rfloor).$$

Let $\{\lambda\}(\boldsymbol{\ell}) := \{\lambda(\boldsymbol{\ell})\}$. Then we have the following comutative diagram.

$$\begin{array}{ccc} \mathbb{Z}^{d-1} & \xrightarrow{\tau} & \mathbb{Z}^{d-1} \\ \{\lambda\}\big\downarrow & & \big\downarrow\{\lambda\} \\ \mathbb{Z}[\beta] \cap [0,1) & \xrightarrow[T]{} & \mathbb{Z}[\beta] \cap [0,1) \end{array}$$

Example: Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^3 - ax^2 - bx - c$. Then

$$\lambda(l_1, l_2) = l_1 c\beta^{-1} + l_2(b\beta^{-1} + c\beta^{-2}) = (l_1 c + l_2 b)\beta^{-1} + l_2 c\beta^{-2}.$$

So $\tau(l_1, l_2) = (l_2, -\lfloor \lambda(l_1, l_2) \rfloor) = (l_2, -\lfloor (l_1 c + l_2 b)\beta^{-1} + l_2 c\beta^{-2} \rfloor)$.

Let

$$\boldsymbol{e} := (0, \cdots, 0, 1) \in \mathbb{Z}^{d-1}.$$

Remark 2: $T^n(1) = \{\lambda\}(\tau^{n-1}(\boldsymbol{e}))$ for all $n \in \mathbb{N}$.

Proof: Since $\beta^d - a_{d-1}\beta^{d-1} - \cdots - a_1\beta - a_0 = 0$, we have

$$\beta - a_{d-1} = a_{d-2}\beta^{-1} + \cdots + a_1\beta^{-d+2} + a_0\beta^{-d+1} = \lambda(\boldsymbol{e}).$$

Thus $\{\lambda(\boldsymbol{e})\} = \beta - \lfloor \beta \rfloor$. So we have

$$T(1) = \beta - \lfloor \beta \rfloor = \{\lambda\}(\boldsymbol{e}).$$

Hence by commutative diagram, we have the desired result.

Let $F_\beta := \{\boldsymbol{\ell} \in \mathbb{Z}^{d-1} \mid {}^\exists k \geq 0;\ \tau^k(\boldsymbol{\ell}) = \mathbf{0}\}$.

Then by commutative diagram, $\{\lambda\}(F_\beta) \subset \mathrm{Fin}(\beta)$. Let

$$V := \left\{ -\sum_{n=1}^{r} \omega_n \tau^{n-1}(\boldsymbol{e}) \,\middle|\, r \in \mathbb{N},\ \omega_n \in \mathbb{N}_0 \right\}.$$

<u>**Remark 3**</u>: $V \subset F_\beta \Rightarrow \beta$ has (F_1).

Now we aim to get a sufficient condition for $V \subset F_\beta$. Define

$$\tau^*(\boldsymbol{\ell}) = -\tau(-\boldsymbol{\ell}).$$

Notice that if $\boldsymbol{\ell} := (l_1, \cdots, l_{d-1}) \neq \mathbf{0}$ then

$$\tau^*(\boldsymbol{\ell}) = (l_2, \cdots, l_{d-1}, -\lfloor \lambda(\boldsymbol{\ell}) \rfloor - 1).$$

<u>**Note**</u>: For $\boldsymbol{\ell}, \boldsymbol{k} \in \mathbb{Z}^{d-1}$,

$$\tau(\boldsymbol{\ell} + \boldsymbol{k}) \in \{\tau(\boldsymbol{\ell}) + \tau(\boldsymbol{k}), \tau(\boldsymbol{\ell}) + \tau^*(\boldsymbol{k})\}$$

$$\tau(\boldsymbol{\ell} - \boldsymbol{k}) \in \{\tau(\boldsymbol{\ell}) - \tau(\boldsymbol{k}), \tau(\boldsymbol{\ell}) - \tau^*(\boldsymbol{k})\}.$$

Define

$$Q_\beta = \left\{ \boldsymbol{\ell} \in \mathbb{Z}^{d-1} \;\middle|\; \begin{array}{l} {}^\exists N \in \mathbb{N},\ {}^\exists\{\boldsymbol{\ell}_n\}_{n=1}^N \text{ s.t. } \boldsymbol{\ell}_1 = \boldsymbol{e} \\ \boldsymbol{\ell}_N = \boldsymbol{\ell},\ \boldsymbol{\ell}_{n+1} \in \{\tau(\boldsymbol{\ell}_n), \tau^*(\boldsymbol{\ell}_n)\} \end{array} \right\}.$$

Note: β is a Pisot number $\Rightarrow \sharp Q_\beta < \infty$

Obs

Let $\boldsymbol{k} = \tau^j(\boldsymbol{e})$ $(j \in \mathbb{N}_0)$. Then we have

$$\boldsymbol{k}' := \tau(\boldsymbol{\ell}) - \tau(\boldsymbol{\ell} - \boldsymbol{k}) \in \{\tau(\boldsymbol{k}), \tau^*(\boldsymbol{k})\} \text{ and so } \boldsymbol{k}' \in Q_\beta$$

Def: The set $U \subset \mathbb{Z}^{d-1}$ satisfies backward invariant (BI for short) if

$$\tau^{-1}(U) = \{\boldsymbol{\ell} \in \mathbb{Z}^{d-1} \mid \tau(\boldsymbol{\ell}) \in U\} \subset U.$$

Let

$$P_\beta := \{\boldsymbol{\ell} \in Q_\beta \setminus \{\boldsymbol{0}\} \mid {}^\exists m \in \mathbb{N};\ \tau^m(\boldsymbol{\ell}) = \boldsymbol{\ell}\} \ \&\ -Q_\beta := \{-\boldsymbol{\ell} \mid \boldsymbol{\ell} \in Q_\beta\}$$

Lemma: Let β be a Pisot number and P_β satisfy BI. Then $-Q_\beta = Q_\beta$

Sketch of the proof: First, we note

$$(Q_\beta \setminus \{\mathbf{0}\}) \cap F_\beta \neq \emptyset \Rightarrow -Q_\beta = Q_\beta.$$

Indeed, we can choose $\zeta \neq 0$ with $\tau(\zeta, \mathbf{0}) = \mathbf{0}$. Then $\tau^*(\zeta, \mathbf{0}) = -\boldsymbol{e} \in -Q_\beta$ and so $-Q_\beta = Q_\beta$. Consider the case $(Q_\beta \setminus \{\mathbf{0}\}) \cap F_\beta = \emptyset$. Since $\boldsymbol{e} \in P_\beta$ by BI, we can choose $(\zeta, \eta) \neq \mathbf{0}$ such that $(\zeta, \eta, \mathbf{0}) \in P_\beta$ and $\tau^2(\zeta, \eta, \mathbf{0}) = \boldsymbol{e}$. Notice that $(\eta, \mathbf{0}, -1) = \tau^*(\zeta, \tau, \mathbf{0}) \in P_\beta$ and

$$\tau(\eta, \mathbf{0}, -1) \in \{\tau(\eta, \mathbf{0}) + \tau(-\boldsymbol{e}), \tau(\eta, \mathbf{0}) + \tau^*(-\boldsymbol{e})\}.$$

Now we can show

$$\tau(\eta, \mathbf{0}, -1) = \tau(\eta, \mathbf{0}) + \tau(-\boldsymbol{e}) \Rightarrow P_\beta \ni \tau^*(\eta, \mathbf{0}, -1) = \tau(-\boldsymbol{e})$$

$$\tau(\eta, \mathbf{0}, -1) = \tau(\eta, \mathbf{0}) + \tau^*(-\boldsymbol{e}) \Rightarrow P_\beta \ni \tau(\eta, \mathbf{0}, -1) = \tau(-\boldsymbol{e}).$$

Thus $-\boldsymbol{e} \in Q_\beta$ and hence $-Q_\beta \subset Q_\beta$.

Thm

Suppose that a Pisot number β satisfies

(i) P_β satisfies BI, that is, $\tau^{-1}(P_\beta) \subset P_\beta$.

(ii) $R_0 := [\![-\delta, \delta]\!]^{d-1} \cap V \subset F_\beta$ where

$$\delta := \max\{|l_j| \mid (l_1, \cdots, l_{d-1}) \in P_\beta,\ j \in [\![1, d-1]\!]\}.$$

Then β has (F_1).

<u>**Sketch of the proof**</u>: Let $R = \cup_{n \geq 0} R_n$ where

$$R_n := \{\boldsymbol{\ell} - \tau^j(\boldsymbol{e}) \in V \mid \boldsymbol{\ell} \in R_{n-1}, j \in \mathbb{N}_0\}.$$

Then $V = R$. Assume $R_{n-1} \subset F_\beta$ and $\boldsymbol{\ell} \in R_n \setminus F_\beta$. Then $\boldsymbol{\ell} = \boldsymbol{\ell}' - \tau^j(\boldsymbol{e})$ for some $\boldsymbol{\ell}' \in R_{n-1}$ and j. Since $\boldsymbol{\ell}'$ ends up at $\mathbf{0}$ and $-\tau^j(\boldsymbol{e})$ ends up P_β, we get $\boldsymbol{\ell} \in P_\beta$. Contradiction. So by induction on n, $R_n \subset F_\beta$.

Proof of Main Theorem

Let $\beta > 1$ be an algebraic integer with minimal polynomial $x^3 - 2tx^2 + 2tx - t$ ($t \in \mathbb{N} \cap [2, \infty)$). So

$$1 = 2t\beta^{-1} - 2t\beta^{-2} + t\beta^{-3}.$$

Then β is a Pisot number because

Thm (Akiyama): Let $\beta > 1$ be an cubic number with minimal polynomial $x^3 - ax^2 - bx - c$. Then

$$\beta \text{ is a Pisot number} \Leftrightarrow |b - 1| < a + c \;\&\; (c^2 - b) < \operatorname{sgn}(c)(1 + ac).$$

Note: $\tau(l_1, l_2) = (l_2, -\lfloor t(l_1 - 2l_2)\beta^{-1} + tl_2\beta^{-2} \rfloor)$.

Fact: (1) $P_\beta = \{(1,1)\}$ (2) $\lambda(0,1) < -1$ and $\lambda(2,1) > 0$.

Proof of Main Theorem: It suffices to show that β satsfies the conditions of Thm-(i) and (ii).

(i) : Since $\lambda(0,1) < -1$ and $\lambda(2,1) > 0$ and $\lambda(x,1)$ is increasing in x, we have

$$\tau^{-1}(1,1) = \{(x,1) \mid x \in \mathbb{Z}, -1 \leq \lambda(x,1) < 0\} = \{(1,1)\}$$

(ii) : Since

$$(0,1) \xrightarrow{\tau} (1,2) \xrightarrow{\tau} (2,2) \xrightarrow{\tau} (2,1) \xrightarrow{\tau} (1,0) \xrightarrow{\tau} (0,0),$$

we have $V \subset ((-\infty,0] \cap \mathbb{Z})^2$ and so

$$R_0 \subset [\![-1,0]\!]^2.$$

Moreover

$$(0,-1) \xrightarrow{\tau} (-1,-1) \xrightarrow{\tau} (-1,0) \xrightarrow{\tau} (0,1).$$

So $[\![-1,0]\!]^2 \subset F_\beta$. Hence $R_0 \subset F_\beta$.

<u>**Example**</u>: Let

$$x := (2t-2) + (2t-2)\beta^{-1} + (t-1)\beta^{-2} + (t-1)\beta^{-4} + (t-1)\beta^{-5} + (t-1)\beta^{-6}.$$

Then x is in $\mathbb{N}_0[1/\beta]$ and does not have the finite β-expansion. Indeed, by using $1.000 \overset{\nu}{=} 0.4\bar{4}2$,

	$d_\beta(\beta^{-2}x)$											
$\overset{\nu}{=} 0$	$(2t-2)$	$(2t-2)$	$(t-1)$	0	$(t-1)$	$(t-1)$	$(t-1)$	0	0	0	0	0^∞
$\overset{\nu}{=} 1$	(-2)	$(4t-2)$	(-1)	0	$(t-1)$	$(t-1)$	$(t-1)$	0	0	0	0	0^∞
$\overset{\nu}{=} 1$	0	(-2)	$(4t-1)$	$(-2t)$	$(t-1)$	$(t-1)$	$(t-1)$	0	0	0	0	0^∞
$\overset{\nu}{=} 1$	0	0	(-1)	$(2t)$	$(-t-1)$	$(t-1)$	$(t-1)$	0	0	0	0	0^∞
$\overset{\nu}{=} 1$	0	0	0	0	$(t-1)$	(-1)	$(t-1)$	0	0	0	0	0^∞
$\overset{\nu}{=} 1$	0	0	0	0	$(t-2)$	$(2t-1)$	$(-t-1)$	t	0	0	0	0^∞
$\overset{\nu}{=} 1$	0	0	0	0	$(t-2)$	$(2t-2)$	$(t-1)$	$(-t)$	t	0	0	0^∞
$\overset{\nu}{=} 1$	0	0	0	0	$(t-2)$	$(2t-2)$	$(t-2)$	t	$(-t)$	t	0	0^∞
$\overset{\nu}{=} 1$	0	0	0	0	$(t-2)$	$(2t-2)$	$(t-2)$	$(t-1)$	t	$(-t)$	t	0^∞.

and so

$$\text{a beta-expansion of } x = 10.000(t-2)(2t-2)(t-2)(t-1)^\infty.$$

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Thank you!