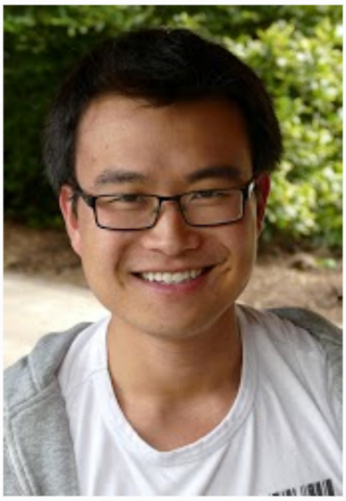
Littlewood and Duffin-Schaeffer type of problems in Diophantine approximation

on joint work with



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NOTATION

Symbols	Meaning
$\lVert \cdot \rVert = \lVert \cdot \rVert_{\mathbb{R}/\mathbb{Z}}$	distance to nearest integer, i.e. $\lVert x \rVert = \min\{ \lvert x-a \rvert : a \in \mathbb{Z}\}$
ψ	approximation function, a map $\mathbb{N} \longrightarrow \mathbb{R}_{>0}$
μ_k , $\mu = \mu_1$	Lebesgue measure on $\mathbb{R}^k$
$k \geq 1$	dimension of the ambient space
i.o.	infinitely often

1 Introduction

1.1 Classical DA

Question: How dense is $\mathbb{Q}$ in $\mathbb{R}$?

How small is $\left|\alpha - \frac{a}{n}\right|$ in terms of n?

$\|n\alpha\|$

Theorem D (Dirichlet, 1842)

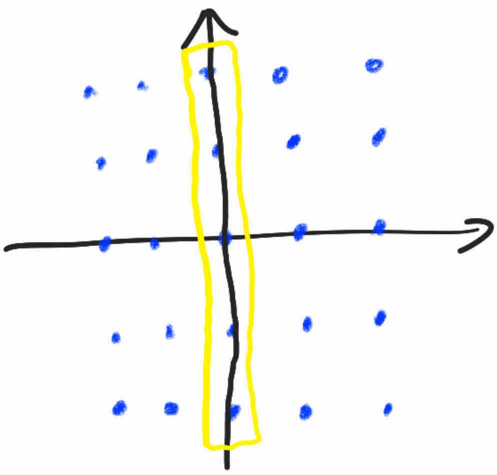
Given $\alpha \in \mathbb{R}$ and $N > 1$, there exist $a, n \geq 1$ such that

$$\left|\alpha - \frac{a}{n}\right| < \frac{1}{nN} \leq \frac{1}{n^2}.$$

$$\iff \|n\alpha\| < \frac{1}{N} \leq \frac{1}{n}.$$

Proof: Apply Minkowski's first theorem to

$$\Lambda = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2 \text{ and } B = \left[-\tfrac{1}{N}, \tfrac{1}{N}\right] \times [-N, N].$$



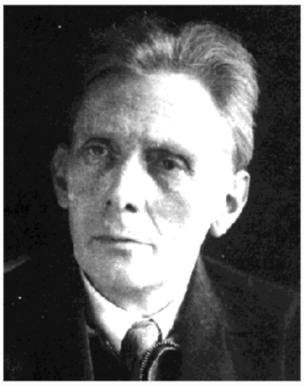
Remark 1 a) Thm. D is optimal (up to constants) since badly approximable numbers α, i.e. $\alpha \in \mathbb{R}$ so that $\underline{\lim}_{n\to\infty} n\|n\alpha\| > 0$, exist.

b) $\sqrt{2}$ is badly approximable: note

$$|n\sqrt{2} - a|\,|n\sqrt{2}+a| = |\underbrace{n^2 2 - a^2}_{\in \mathbb{Z}\setminus\{0\}}| \geq 1;$$

thus $n\,|n\sqrt{2} - a| \geq \dfrac{n}{|n\sqrt{2}+a|}$. ∎

By ignoring μ-null sets, Thm D is improvable



Theorem K (Klintchine, 1926)

Suppose φ is non-increasing.

Then $\{\alpha \in [0,1] : \|n\alpha\| < \varphi(n) \text{ i.o.}\}$

has μ-measure $\begin{cases} 1, & \text{if } \sum_{n\geq 1} \varphi(n) = \infty, \\ 0, & \text{if } \sum_{n\geq 1} \varphi(n) < \infty. \end{cases}$

Corollary 1: For a.e. $\alpha \in [0,1]$ we have

$\|n\alpha\| < (n \log n)^{-1}$ i.o. and $\|n\alpha\| \gg \dfrac{1}{n(\log n)^{1+\varepsilon}}$

1.2 Elemantary tools

The set in Thm. K is the lim-sup set of

$$\mathcal{E}_n = \{ \alpha \in [0,1] : \| n\alpha \| < \psi(n) \}.$$

Note $\mu(\mathcal{E}_n) \asymp \psi(n)$.

We have:

Lemma BC (1st Borel–Cantelli lemma)

Let $\Omega_n \subseteq [0,1]$ be measurable.

If

$$(\text{Div}) \sum_{n \geq 1} \mu(\Omega_n) < \infty,$$

then

$$\overline{\lim_{n \to \infty}} \Omega_n := \{ \alpha \in [0,1] : \alpha \in \Omega_n \text{ i.o.} \}$$

is a null set.

Lemma $(BC)^{-1}$ (2nd Borel–Cantelli lemma)

Let $\Omega_n \subseteq [0,1]$ be measurable. If $\sum_{n \geq 1} \mu(\Omega_n) = \infty$, then

$$\mu\left(\overline{\lim_{n \to \infty}} \Omega_n \right) \geq \overline{\lim_{N \to \infty}} \frac{\left(\sum_{n \leq N} \mu(\Omega_n) \right)^2}{\sum_{m,n \leq N} \mu(\Omega_n \cap \Omega_m)}.$$

1.3 Higher dimensional DA

Planar analogues of Thm D can be

i) smallness of $\|n\alpha_1\| + \|n\alpha_2\|$ in terms of n.

ii) smallness of $\|n\alpha_1 + n\alpha_2\|$ in terms of n.

iii) smallness of $\|n\alpha_1\| \cdot \|n\alpha_2\|$ in terms of n.

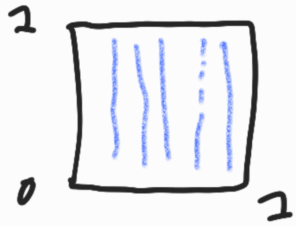
<u>Conjecture</u> (Littlewood, 1932)

Do planar mult. badly approximable vectors exist: is

$$(L) \qquad \lim_{n\to\infty} n\,\|n\alpha_1\|\,\|n\alpha_2\| = 0$$

for all $\underline{\alpha} \in \mathbb{R}^2$?

Verifying (L) for $(\sqrt{2}, \sqrt{3})$ is open.



<u>Theorem G</u> (Gallagher, 1962)

Suppose ψ is non-increasing. Then

$$\{\underline{\alpha} \in [0,1]^k : \|n\alpha_1\| \cdots \|n\alpha_k\| < \psi(n) \text{ i.o.}\}$$

has μ_k-measure

$$\begin{cases} 1, & \text{if } \sum_{n\geq 1} \psi(n)(\log n)^{k-1} = \infty, \\ 0, & \text{if } \sum_{n=1} \psi(n)(\log n)^{k-1} < \infty. \end{cases}$$

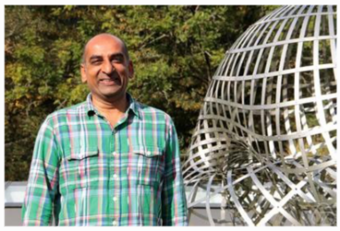
<u>Remark 2</u> a) We have $\|n\alpha_1\| \cdots \|n\alpha_k\| < \frac{1}{n(\log n)^k}$ i.o. for a.e. $\underline{\alpha} \in \mathbb{R}^k$.

b) By Einsiedler, Katok and Lindenstrauss (2006) we know that all $\underline{\alpha} \in \mathbb{R}^2$ not satisfying (L) are a set of zero Hausdorff dimension.

c) Badziahin (2013, 2016) showed that there exist $\underline{\alpha} \in \mathbb{R}^2$ so that

$$n \, \|n\alpha_1\| \, \|n\alpha_2\| \gg \frac{1}{\log n \, \log\log n}.$$

d) Beresnevich, Haynes, and Velani



proved in 2015+/2020 the following refinement of Thm G (in the plane):

Theorem BHV

I) Fix $\alpha_1 \in \mathbb{R}$. Then for a.e. α_2 we have

$$\lim_{n \to \infty} n (\log n)^2 \, \|n\alpha_1\| \cdot \|n\alpha_2\| = 0.$$

II) Assume the Duffin-Schaeffer conjecture. Let $\alpha_1 \in \mathbb{R}$ be not a Liouville number or rational. Then for fixed $\gamma_1 \in \mathbb{R}$,

$$\liminf_{n \to \infty} n (\log n)^2 \|n\alpha_1 - \gamma_1\| \cdot \|n\alpha_2\| = 0$$

for a.e. $\alpha_2 \in \mathbb{R}$.

There were several refinements:

- In 2018, Chow proved II) unconditionally and made it quantitatively sharp.
- In 2019, Chow and T. proved a higher dimensional generalisation.
- In 2019+, Chow and Yang proved a version for planar lines.

2 Results

Theorem CT (Chow, T. 2020+)

Fix $\underline{\gamma} \in \mathbb{R}^k$, ψ non-increasing. Then the set

$$\{\underline{\alpha} \in [0,1]^k : \|n\alpha_1 - \gamma_1\| \cdots \|n\alpha_k - \gamma_k\| < \psi(n) \text{ i.o.}\}$$

has μ_k- measure

$$\begin{cases} 1, & \text{if } \sum_{n \geq 1} \psi(n) (\log n)^{k-1} = \infty, \\ 0, & \text{if } \sum_{n \geq 1} \psi(n) (\log n)^{k-1} < \infty. \end{cases}$$

To state the strong version, let

$$\omega^\times(\underline{\alpha}) = \sup\{ w > 0 : \|n\alpha_1\| \cdots \|n\alpha_d\| < n^{-w} \text{ i.o.}\}.$$

(Generically, $\omega^\times(\underline{\alpha}) = 1$.)

Theorem CTii (Chow, T. 2020+)

Fix $\underline{\gamma} \in \mathbb{R}^k$, ψ non-increasing. Then the set

$$\{ \alpha_k \in [0,1] : \|n\alpha_1 - \gamma_1\| \cdot \ldots \cdot \|n\alpha_k - \gamma_k\| < \psi(n) \text{ i.o.} \}$$

has μ - measure

$$\begin{cases} 1, & \text{if } \sum_{n \geq 1} \psi(n) (\log n)^{k-1} = \infty, \\ 0, & \text{if } \sum_{n \geq 1} \psi(n) (\log n)^{k-1} < \infty \end{cases}$$

Provided

$$\omega^*((\alpha_1, \ldots, \alpha_{k-1})) < \frac{k-1}{k-2}.$$

The set of $\underline{\tilde{\alpha}} \in \mathbb{R}^{k-1}$ with $\omega^*(\underline{\tilde{\alpha}}) \geq \frac{k-1}{k-2}$ has Hausdorff dimension

$$k - 1 - \frac{1}{2k-3}$$

by work of Beresnevich–Velani (2015) and Hussain–Simmons (2018).

Theorem CTiii ((Chow, T. 2020+)

Let $\xi : \mathbb{N} \to [1,\infty)$ be increasing and unbounded.
There exist continuum many $(\alpha_1, \gamma_1) \in \mathbb{R} \times \mathbb{R}$
such that for any $\gamma_2 \in \mathbb{R}$ and almost all $\alpha_2 \in \mathbb{R}$
we do not have

$$\|n\alpha_1 - \gamma_1\| \cdot \|n\alpha_2 - \gamma_2\| < \frac{1}{n(\log n)^2 \xi(n)} \quad \text{i.o.}$$

3 Tools and Outline

For expository reasons, let $k = 2$.
We discuss Thm CTii as Thm CTi will follow
and Thm CTiii uses similar arguments.

Goal: For almost every $\alpha_2 \in [0,1]$ we have

$$\|n\alpha_2 - \gamma_2\| < \Psi(n) := \frac{\psi(n)}{\|n\alpha_1 - \gamma_1\|} \quad \text{i.o.}$$

(For $\gamma_2 = 0$, Chow and T. showed this in 2019.)

Theorem KM (Koukoulopoulos, Maynard 2020) (formerly: Duffin-Schaeffer conj. 1941)

If $\psi: \mathbb{N} \to [0,\infty)$ and $\sum_{n\geq 1} \frac{\varphi(n)}{n} \psi(n) = \infty$,

then for almost all $\alpha \in \mathbb{R}$ we have

$$|n\alpha - a| < \psi(n) \text{ and } \gcd(a,n)=1 \text{ i.o.}$$

Conjecture R (Ramírez 2016, hinted in BHV)

If $\psi: \mathbb{N} \to [0,\infty)$ and $\sum_{n\geq 1} \frac{\varphi(n)}{n} \psi(n) = \infty$,

then for any fixed $\gamma \in \mathbb{R}$ and

for almost all $\alpha \in \mathbb{R}$ we have

$$|n\alpha - \gamma - a| < \psi(n) \text{ and } \gcd(a,n)=1 \text{ i.o.}$$

In earnest, we prove Conjecture R for a special class of approximation functions. To this end, we use

Definition 1:

Let $\eta \in (0,1)$ and c_t/q_t the largest convergent to γ so that

$$q_t \leq n^{\eta}.$$

A pair $(a,n) \in \mathbb{Z} \times \mathbb{N}$ is (γ,η)-shift reduced if $\gcd(q_t a + c_t, n) = 1$.

Note that if $\gamma = 0$ then $q_t = 1$ and $c_t = 0$ so we recover the classical reduced fractions.

We argue morally for proving Thm CT ii that sets of the shape

$$\mathcal{E}_n = \bigcup_{\substack{a \leq n \\ (a,n) \text{ is } (\eta,\gamma_2) \\ \text{shift reduced}}} \mathrm{Ball}\left(\frac{a}{n}, \frac{\psi(n)}{n}\right)$$

satisfy Lemma $(BC)^{-1}$ for "good n".

What are good n?

G1) Want to know size of $\Psi(n)$, so

$$\|n\alpha_1 - \gamma_1\| \asymp \rho \approx N^{-\varepsilon} \text{ for some } \varepsilon > 0.$$

This essentially means that

$$n \in B_\gamma^\alpha(N,\rho) := \{ n \le N : \|n\alpha_1 - \gamma_1\| < \rho \}.$$

From additive combinatorics we expect $B_\gamma^\alpha(N,\rho)$ should be essentially a generalised arithmetic progression, i.e. to be of the form

$$b_0 + A_1 n_1 + A_2 n_2 \quad \text{with} \quad n_i \le N_i.$$

G2) lower bound on $\varphi(n)$.

G3) n to have somewhat large divisor, we restrict to n of the form

$$n = 4^{f(m)} m =: \hat{m}$$ ("Four-ier transform")

where f increases very slowly so that

$$\sum_{m\geq 1} \varphi(\hat{m}) \log \hat{m} = \infty .$$

We need this to compensate for the lack of a zero-one law!

We bound $\mu(\mathcal{E}_n \cap \mathcal{E}_m)$ in fixed dyadic Bohr sets $B(N,\rho)$ and $B(M,\delta)$ by counting the number of solutions to certain linear diophantine inequalities. There are two different regimes depending on $d := \gcd(n,m)$:

A) small d: use congruences (via the structure of Bohr sets from CT2019) & lattice point counting (in some subcases).

B) large d: here the diophantine nature of γ plays a role.

If γ is diophantine, then use a repulsion principle with a strategic choice of ε.

If γ is a rational or a Liouville number then use shift reduction.

Thank you for your attention!!