Extensions of the random beta-transformation

Younès Tierce

Université de Rouen-Normandie, LMRS, supervised by Thierry de la Rue and Jean-Baptiste Bardet

November 9, 2021

Expansions in base β

- $\beta > 1$
- $x \in I_\beta := \left[0; \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$

Expansions in base β

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{\beta^n},\ x_n \in \{0, \ldots, \lfloor \beta \rfloor\}.$$

We will now fix $1 < \beta < 2$.

How to obtain an expansion in base β ?

The greedy map (Rényi, 1957)

$$T_g : \begin{array}{ccc} I_\beta & \to & I_\beta \\ x & \mapsto & \begin{cases} \beta x & \text{if } x < \frac{1}{\beta} \\ \beta x - 1 & \text{otherwise} \end{cases} \end{array} .$$

Provides the greatest expansion of a given number in the lexicographic order (Erdös, Joo).

The lazy map

$$T_\ell : \begin{array}{ccc} I_\beta & \to & I_\beta \\ x & \mapsto & \begin{cases} \beta x & \text{if } x \leqslant \frac{1}{\beta(\beta-1)} \\ \beta x - 1 & \text{otherwise} \end{cases} \end{array} .$$

Provides the smallest expansion of a given number in the lexicographic order.

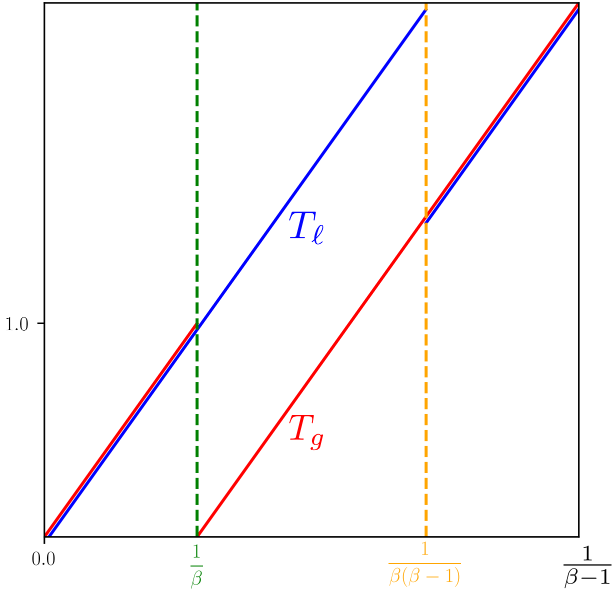


FIGURE – Graphs of T_g and T_ℓ

Ergodic properties of the associated systems

Rényi (1957) and Parry (1960)

The greedy map T_g has a unique absolutely continuous invariant probability measure ν_β, whose density is proportional to the function

$$\sum_{n=0}^{+\infty} \frac{1}{\beta^n} 1_{[0,T_g^n(1)]}(x).$$

Smorodinsky (1973)

The natural extension of the system $(I_\beta, \nu_\beta, T_g)$ is a Bernoulli shift.

Dajani and Kraaikamp (2002)

The systems associated to the greedy and lazy maps are isomorphic.

Generate every expansions in base β ?

- Idea : choose between greedy and lazy at each step.
- Formally : $\Omega = \{g, \ell\}^{\mathbb{N}}$, with the Bernoulli measure m_p.

Random dynamical system

$$K_\beta : \begin{array}{ccc} \Omega \times I_\beta & \to & \Omega \times I_\beta \\ (\omega, x) & \mapsto & (\sigma(\omega), T_{\omega_0}(x)) \end{array} .$$

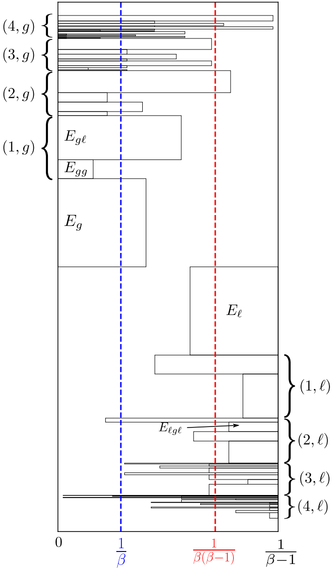
where σ is the left shift.

Questions

- Existence of an absolutely continuous probability measure μ_p on I_β such that $m_p \otimes \mu_p$ is K_β-invariant ? Uniqueness ?
- Explicit expression of its density ?
- Ergodic properties of the associated system ?

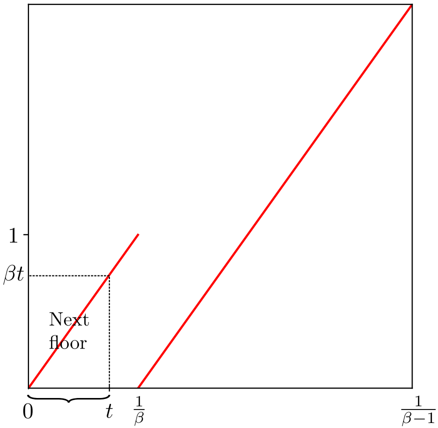
How to get such a measure ?

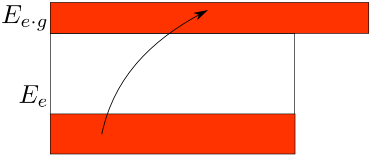
- Idea : Construct an extension of K_β with a "simple" invariant measure, then project this measure on I_β.
- 2014 : extension in the case $p = \frac{1}{2}$ by Kempton.
  $\rightarrow$ Expression of the density of $\mu_{1/2}$.
- 2019 : Generalised expression with transfer operators by Suzuki.
  $\rightarrow \mu_p$ for any p.
- Our work : extension of the system for any p.
  $\rightarrow$ Provides an expression of μ_p, and ergodic properties of the random β-transformation.



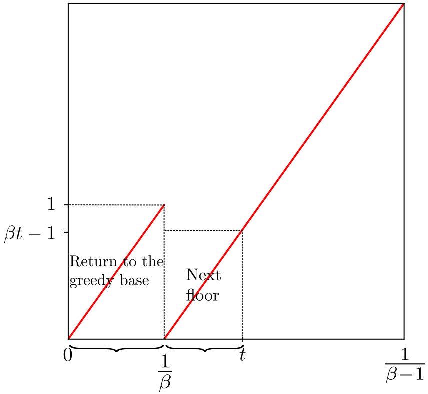
(4, g)
(3, g)
(2, g)
(1, g)
$E_{g\ell}$
E_{gg}
E_g
E_ℓ
(1, ℓ)
$E_{\ell g \ell}$
(2, ℓ)
(3, ℓ)
(4, ℓ)
0
$\frac{1}{\beta}$
$\frac{1}{\beta(\beta-1)}$
$\frac{1}{\beta-1}$

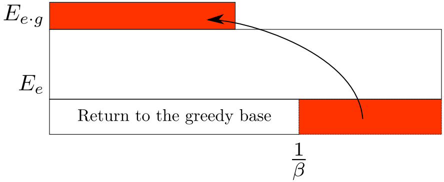
The new dynamics



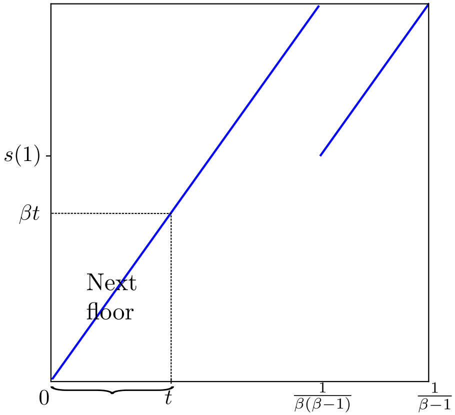


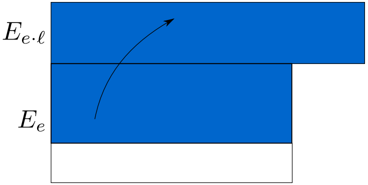
The new dynamics



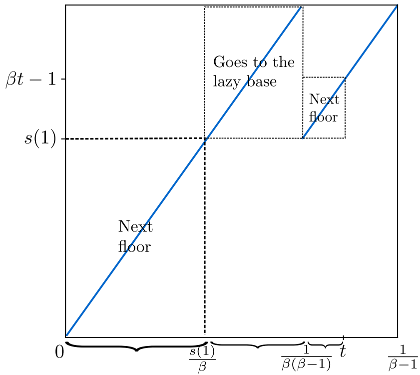


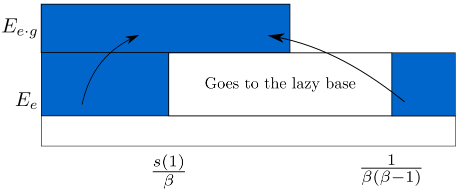
The new dynamics



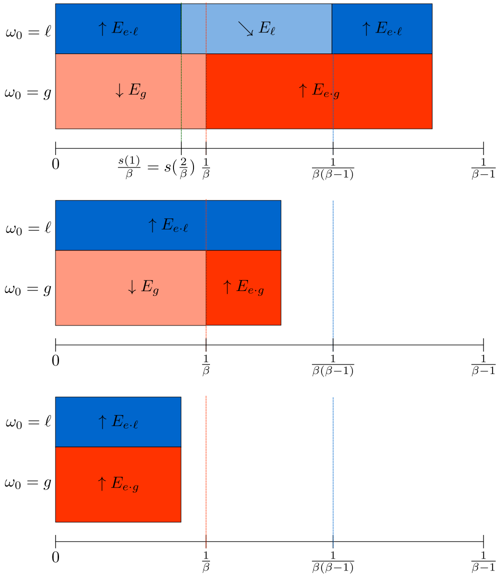


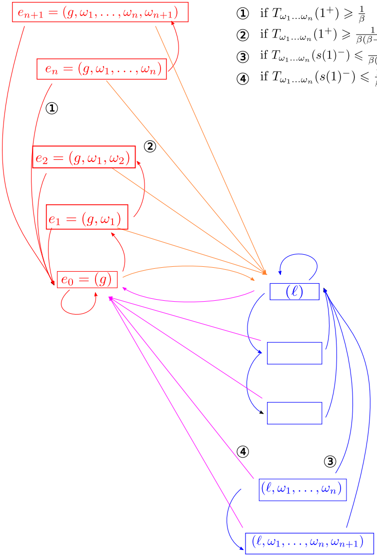
The new dynamics



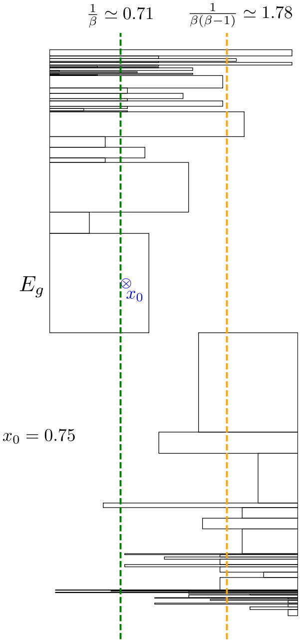


The new dynamics

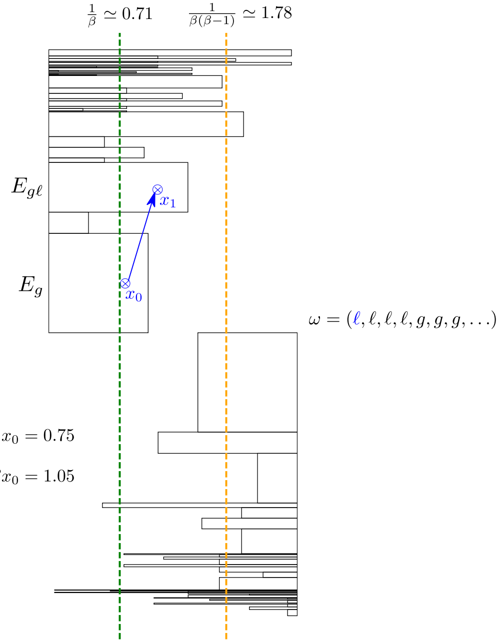




① if $T_{\omega_1 \ldots \omega_n}(1^+) \geqslant \frac{1}{\beta}$
② if $T_{\omega_1 \ldots \omega_n}(1^+) \geqslant \frac{1}{\beta(\beta-1)}$
③ if $T_{\omega_1 \ldots \omega_n}(s(1)^-) \leqslant \frac{1}{\beta(\beta-1)}$
④ if $T_{\omega_1 \ldots \omega_n}(s(1)^-) \leqslant \frac{1}{\beta}$

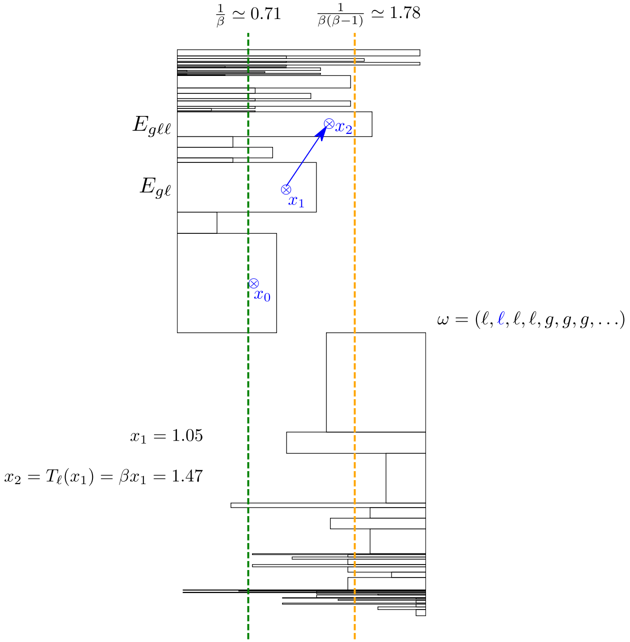


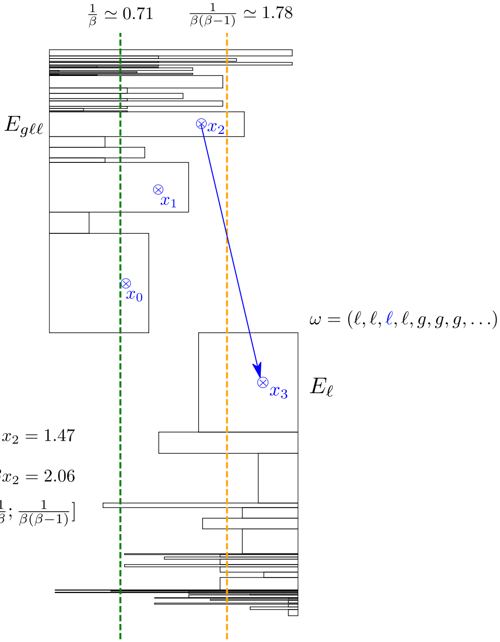
$\frac{1}{\beta} \simeq 0.71$
$\frac{1}{\beta(\beta-1)} \simeq 1.78$
E_g
x_0
$x_0 = 0.75$



$x_0 = 0.75$

$x_1 = T_\ell(x_0) = \beta x_0 = 1.05$

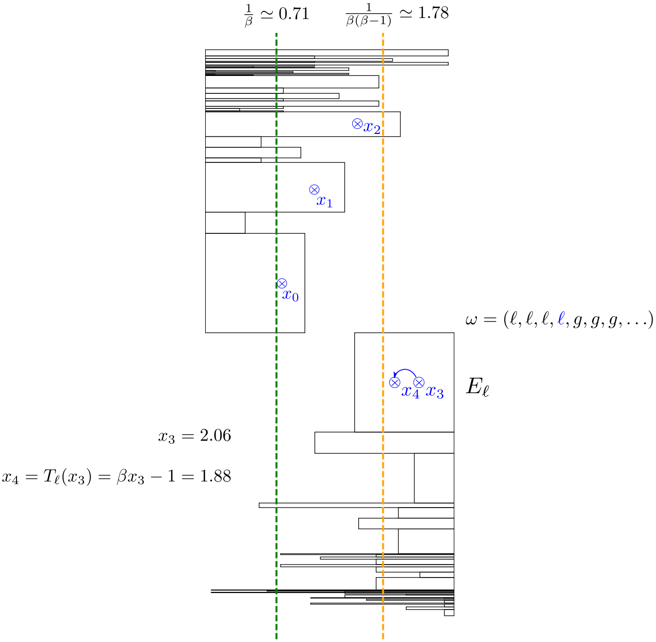

$\frac{1}{\beta} \simeq 0.71$
$\frac{1}{\beta(\beta-1)} \simeq 1.78$
$E_{g\ell\ell}$
x_2
$E_{g\ell}$
x_1
x_0
$\omega = (\ell, \ell, \ell, \ell, g, g, g, \dots)$
$x_1 = 1.05$
$x_2 = T_\ell(x_1) = \beta x_1 = 1.47$

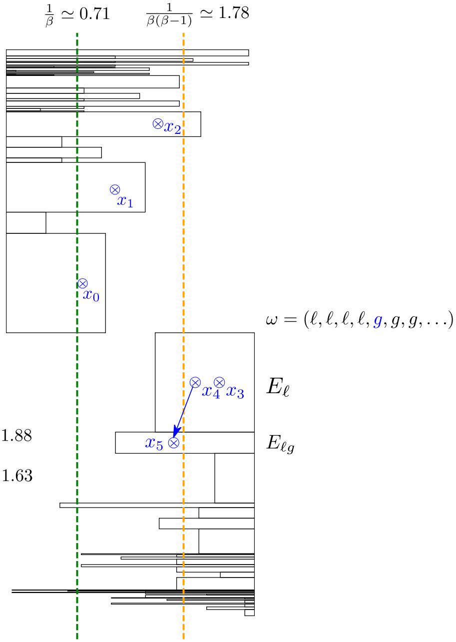


$x_2 = 1.47$

$x_3 = T_\ell(x_2) = \beta x_2 = 2.06$

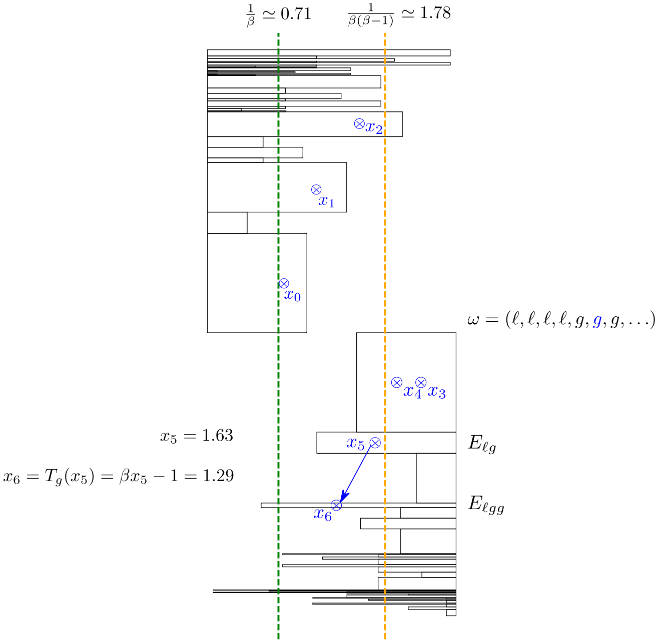
$x_2 \in [\frac{1}{\beta(\beta-1)} - \frac{1}{\beta}; \frac{1}{\beta(\beta-1)}]$

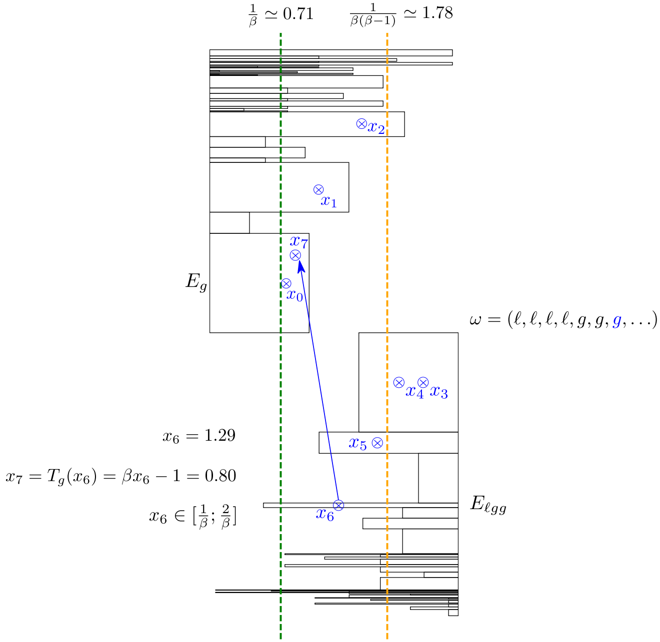

$\frac{1}{\beta} \simeq 0.71$
$\frac{1}{\beta(\beta-1)} \simeq 1.78$
x_2
x_1
x_0
$\omega = (\ell, \ell, \ell, \ell, g, g, g, \ldots)$
x_4 x_3
E_ℓ
$x_3 = 2.06$
$x_4 = T_\ell(x_3) = \beta x_3 - 1 = 1.88$



$x_4 = 1.88$

$x_5 = T_g(x_4) = \beta x_4 - 1 = 1.63$


$\frac{1}{\beta} \simeq 0.71$
$\frac{1}{\beta(\beta-1)} \simeq 1.78$
x_2
x_1
x_0
$\omega = (\ell, \ell, \ell, \ell, g, g, g, \dots)$
x_4 x_3
$x_5 = 1.63$
x_5
$E_{\ell g}$
$x_6 = T_g(x_5) = \beta x_5 - 1 = 1.29$
$E_{\ell g g}$
x_6


$\frac{1}{\beta} \simeq 0.71$
$\frac{1}{\beta(\beta-1)} \simeq 1.78$
x_2
x_1
x_7
E_g
x_0
$\omega = (\ell, \ell, \ell, \ell, g, g, g, \ldots)$
x_4 x_3
$x_6 = 1.29$
x_5
$x_7 = T_g(x_6) = \beta x_6 - 1 = 0.80$
$E_{\ell gg}$
$x_6 \in [\frac{1}{\beta}; \frac{2}{\beta}]$
x_6

Density of μ_p (T.)

$$\rho_p(x) = \sum_{n=0}^{+\infty} \frac{1}{\beta^n} \Bigg(C_g \sum_{\omega_1,\ldots,\omega_n \in \{g,\ell\}^n} m_p([\omega_1,\ldots,\omega_n]_0^{n-1}) 1_{[0, T_{\omega_1,\ldots,\omega_n}(1^+)]}(x)$$

$$+ C_\ell \sum_{\omega_1,\ldots,\omega_n \in \{g,\ell\}^n} m_p([\omega_1,\ldots,\omega_n]_0^{n-1}) 1_{[T_{\omega_1,\ldots,\omega_n}(s(1)^-); \frac{1}{\beta-1}]}(x) \Bigg).$$

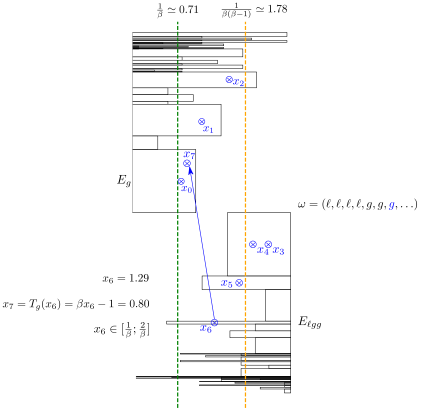
where :

- C_g and C_ℓ are two positive explicit constants,
- $m_p([\omega_1,\ldots,\omega_n]_0^{n-1})$ is the Bernoulli measure of the cylinder $[\omega_1,\ldots,\omega_n]_0^{n-1}$.

Similar form of the density previously obtained by Suzuki.

The induced system on E_g

- Induced transformation of $\mathcal{K}$ on the greedy base E_g : look at the returns to the base E_g.
- Partition E_g according to the encountered floors until the next return to E_g.



$\rightarrow$ Provides a sequence of independent partitions.
$\rightarrow$ Generates the induced system.

The natural extension

- $\mathcal{K}$ not invertible.
- Add the information of the past floors !
- New system : the natural extension of the initial random β-transformation.

Results

- Ergodicity of the natural extension $\Rightarrow$ Ergodicity of the initial random dynamical system + uniqueness of the measure μ_p as an invariant absolutely continuous probability measure.
- Consequence : for $m_p \otimes Leb$ a.e $(\omega, x) \in \Omega \times I_\beta$,

$$\frac{1}{N}\sum_{n=0}^{N-1} \delta_{\pi(K_\beta^n(\omega,x))} \to \mu_p.$$

Bernoulli system (T.)

The natural extension of the system $(\Omega \times I_\beta, m_p \otimes \mu_p, K_\beta)$ is a Bernoulli shift.

Merci ! Thank you !