The bifurcation locus for numbers of bounded type

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Summary

1. Continued fractions
2. Numbers of generalized bounded type
3. Topological bifurcations
4. Hausdorff dimension
5. Tuning operators

Joint with C. Carminati

Continued fractions

Consider

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n + \ddots}}}$$

the (regular) continued fraction expansion of x

A number x is of <u>bounded type</u> if

$$\mathcal{B} := \{x \in [0,1] \ : \ \sup a_i < +\infty\}$$

Given $N \geq 1$, one defines the set

$$\mathcal{B}_N := \{x \in [0,1] \ : \ a_i \leq N \qquad \forall i \geq 1\}$$

of N-bounded type numbers.

Bounded continued fractions

Given $N \geq 1$, one defines the set

$$\mathcal{B}_N := \{x \in [0,1] \ : \ a_i \leq N \qquad \forall i \geq 1\}$$

- $\mathcal{B}_1 := \{\frac{\sqrt{5}-1}{2}\}$
- $\mathcal{B}_N$ uncountable Cantor set for $N \geq 2$
- H.dim $\mathcal{B}_N > 0$ for $N \geq 2$
- (Jarník, 1928)

$$\lim_{N\to\infty} \text{H.dim } \mathcal{B}_N = 1$$

- (Hensley, '92)

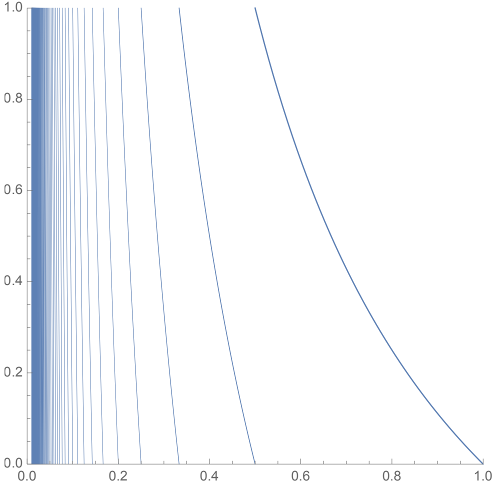
$$\text{H.dim } \mathcal{B}_N = 1 - \frac{6}{\pi^2 N} - \frac{72 \log N}{\pi^4 N^2} + O\left(\frac{1}{N^2}\right) \text{ as } N \to \infty$$

Also:Hensley, Jenkinson-Pollicott, Das-Fishman-Simmons-Urbański...

Dynamical interpretation

Recall the Gauss map

$$G(x) := \left\{ \frac{1}{x} \right\}$$



Dynamical interpretation

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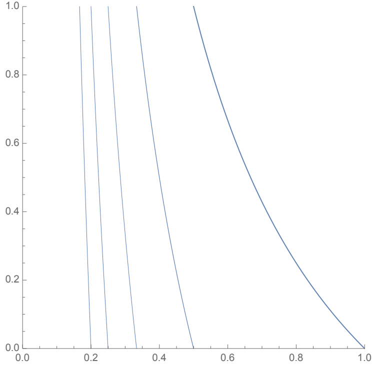


Remove all points whose orbit intersects $[0, \frac{1}{N+1}]$

$$\mathcal{B}_N = \left\{ x \in [0,1] \ : \ G^n(x) \geq \frac{1}{N+1} \qquad \forall n \geq 0 \right\}$$

Numbers of generalized bounded type

Fix $t \in [0,1]$. Then define

$$\mathcal{B}(t) := \{x \in [0,1] \ : \ G^n(x) \geq t \qquad \forall n \geq 0\}$$

Properties.

- $\mathcal{B}(0) = [0,1]$
- $\mathcal{B}(1) = \emptyset$
- If $t < t'$, then
$$\mathcal{B}(t') \subseteq \mathcal{B}(t).$$
- If $t = \frac{1}{N+1}$, then
$$\mathcal{B}\left(\frac{1}{N+1}\right) = \mathcal{B}_N.$$

Numbers of generalized bounded type

Fix $t \in [0,1]$. Then define

$$\mathcal{B}(t) := \left\{ x \in [0,1] \ : \ G^n(x) \geq t \qquad \forall n \geq 0 \right\}$$

Examples.

- $t = g = [0; \overline{1}] = \frac{\sqrt{5}-1}{2}$. Then

$$\mathcal{B}(g) = \{g\}$$

  In fact, $\mathcal{B}(t) = \{g\}$ for all $t \in ([0;\overline{2}],[0,\overline{1}])$.
- $\alpha = [0;\overline{2,1}]$, then

$$\mathcal{B}(\alpha) = \mathcal{B}_2 = \mathcal{B}(1/3)$$

  is the set of numbers whose c.f. contains only 1 and 2.
- For $t = g^2 = [0;2,\overline{1}]$,

$$\mathcal{B}(g^2) = \{\text{in between any two 2s there is an even number of 1s}\}$$

Topological bifurcations

Question. How does the (set-valued) function

$$t \mapsto \mathcal{B}(t)$$

behave with t?

Definition

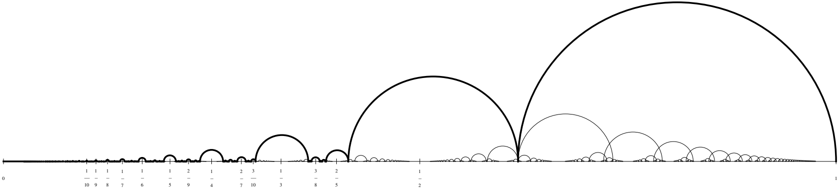
We call t_0 a bifurcation parameter if the function $t \mapsto \mathcal{B}(t)$ is not locally constant in a neighborhood of t_0. The set of bifurcation parameters is the bifurcation locus.

Theorem

The bifurcation locus for the set-valued function $t \mapsto \mathcal{B}(t)$ is the set

$$\mathcal{E} := \{x \in [0,1] \ : \ G^n(x) \geq x \qquad \text{for all } n \geq 0\}$$

The bifurcation locus $\mathcal{E}$



$$\mathcal{E} := \{x \in [0,1] \ : \ G^n(x) \geq x \qquad \text{for all } n \geq 0\}$$

1. $\mathcal{E}$ is closed, with $\text{Leb}(\mathcal{E}) = 0$.
2. $\mathcal{E}$ contains countably many isolated points, which are quadratic irrationals.
3. H.dim $\mathcal{E} = 1$

Theorem

The points of discontinuity of the set-valued function $t \mapsto \mathcal{B}(t)$ are precisely the isolated points of $\mathcal{E}$.

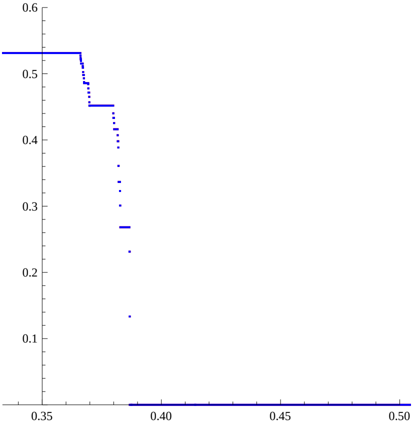
Main results: Hausdorff dimension

Let $\eta(t) := \text{H.dim}\, \mathcal{B}(t)$ the Hausdorff dimension function.

Question. How does

$$\eta(t) = \text{H.dim}\, \mathcal{B}(t)$$

vary with t?



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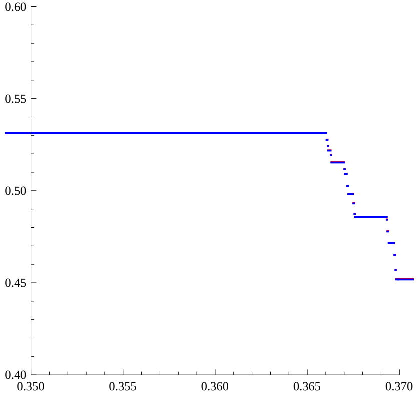

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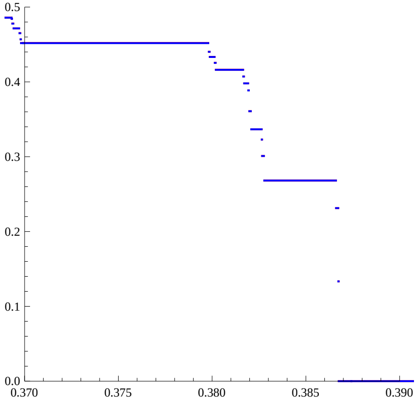


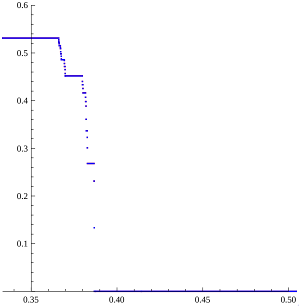
Main results: Hausdorff dimension

Theorem

1. *The function $\eta(t)$ is continuous;*
2. *If $\mathcal{E}(t) := \mathcal{E} \cap [t, 1]$. Then for any t*

$$\text{H.dim } \mathcal{B}(t) = \text{H.dim } \mathcal{E}(t).$$

3. *The function $\eta(t)$ is constant on any tuning window W_r with $r \in \mathbb{Q} \cap (0, c_F)$ extremal.*



Main results: Hausdorff dimension

Theorem

Let $\eta(t) := \text{H.dim}\,\mathcal{B}(t)$*. Then:*

1. *The function* $\eta(t)$ *is continuous;*
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3. *The function* $\eta(t)$ *is constant on any tuning window* W_r *with* $r \in \mathbb{Q} \cap (0, c_F)$ *extremal.*

The structure of $\mathcal{E}$

FACT: Every rational r admits exactly two C.F. expansions.

$$\frac{3}{10} = \cfrac{1}{3 + \cfrac{1}{3}} = \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{1}}}$$

$$\frac{3}{10} = [0; 3, 3] = [0; 3, 2, 1].$$

So any $r \in \mathbb{Q} \cap (0,1)$ will have two C.F. expansions of the type

$$r = [0; S_1] = [0; S_0]$$

where S_0 has even length and S_1 odd length.

The structure of $\mathcal{E}$

Every rational $r \in [0,1]$ has two continued fraction expansions.

$$r = [0; S_0] = [0; S_1]$$

Let S_0 be the one of even length, S_1 the one of odd length.

Given r, we define the quadratic interval

$$I_r := ([0; \overline{S_1}], [0; \overline{S_0}])$$

E.g.:

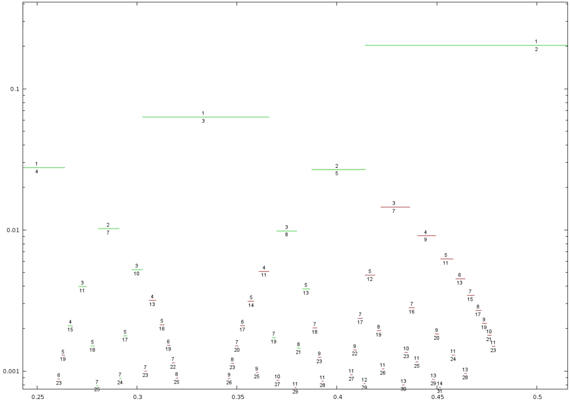
$$\begin{aligned} I_{3/10} &:= ([0; \overline{3,2,1}], [0; \overline{3,3}]) \\ &= \left(\frac{\sqrt{37}-4}{7}, \frac{\sqrt{13}-3}{2} \right) \end{aligned}$$

Thickening $\mathbb{Q}$

Proposition

The exceptional set $\mathcal{E}$ equals

$$\mathcal{E} = [0,1] \setminus \bigcup_{r \in \mathbb{Q} \cap (0,1]} I_r$$



The structure of $\mathcal{E}$

Definition

A rational r is extremal if, is $r = [0; S_1]$ is its c.f. expansion of odd length, then for any splitting

$$S_1 = XY$$

we have

$$XY < YX$$

where $<$ is the alternate lexicographic order.

Example.

$$21 < 11, \qquad 21 < 22$$

$$[0; 3, 2, 1] \text{ is extremal, } [0; 1, 2, 3] \text{ is not extremal}$$

Lemma

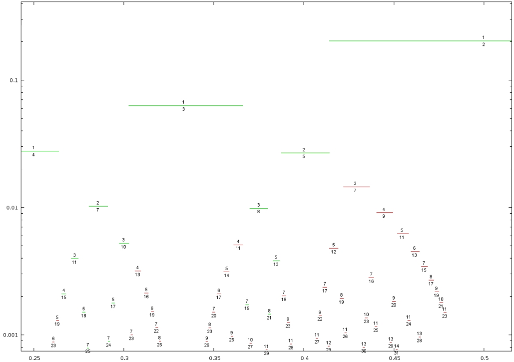
An interval I_r is maximal (i.e. not contained in another $I_{r'}$) if and only if r is extremal.

The structure of $\mathcal{E}$

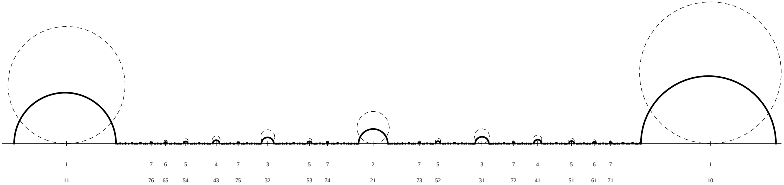
Proposition

The connected components of the complement of $\mathcal{E}$ are precisely I_r for r extremal:

$$\mathcal{E} = [0,1] \setminus \bigsqcup_{r \in \mathbb{Q}_E} I_r$$



$\mathcal{E}$ vs horoballs

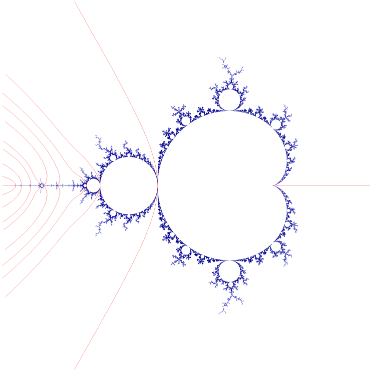


$\mathcal{E} \subseteq \{$numbers of bounded type$\} \Rightarrow$ measure zero

The real slice of the Mandelbrot set

$$\mathcal{R} := \{\theta \in \mathbb{R}/\mathbb{Z} : 2^n\theta \notin (\theta, 1-\theta)\ \forall n \in \mathbb{N}\}$$

The set $\mathcal{R}$ equals the set of angles of external rays which 'land' on the real slice of the Mandelbrot set.



Identity of bifurcation sets

Theorem (Bonanno-Carminati-Isola-T, '10)

The sets $\mathcal{R} \cap [0,1/2]$ and $\mathcal{E}$ are homeomorphic. The map $\varphi : [0,1] \to [0,1/2]$ given by

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \mapsto \varphi(x) = 0.0\underbrace{11\ldots1}_{a_1}\underbrace{00\ldots0}_{a_2}\underbrace{11\ldots1}_{a_3}\ldots$$

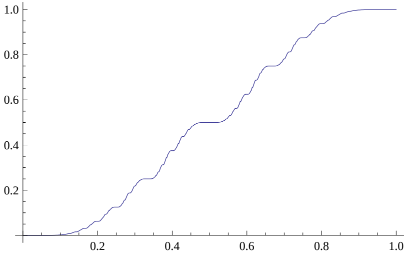
is a homeomorphism which maps $\mathcal{E}$ onto $\mathcal{R} \cap [0,1/2]$.

Question. Is there a renormalization scheme for sets of numbers of bounded type?

Minkowski's question mark function

Let $\alpha := [0; a_1, a_2, a_3, ...]$, define

$$?(\alpha) := 0.\underbrace{00\ldots0}_{a_1-1}\underbrace{11\ldots1}_{a_2}\underbrace{00\ldots0}_{a_3}\cdots$$

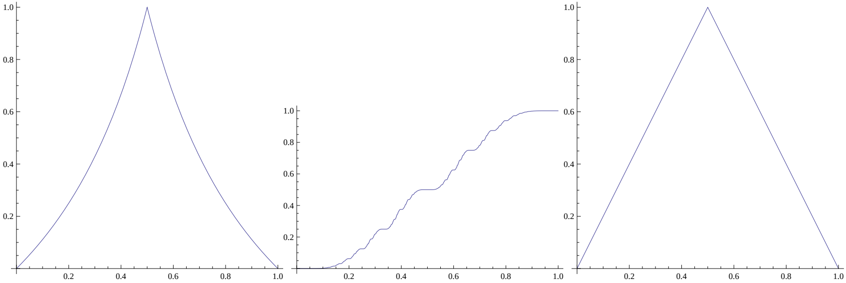


The dictionary

Continued fractions	$\Leftrightarrow$	**Binary expansions**
$\mathcal{E}$	$\leftarrow ? \rightarrow$	Λ

From Farey to the tent map, via ?

Minkowski's question-mark function conjugates the Farey map with the tent map



Period doubling cascades

Let Z be a finite string of natural numbers. Let Z' be the conjugate of Z, i.e. so that

$$[0; Z] = [0; Z']$$

Example.

$$(3)' = (2, 1) \qquad (2, 1)' = 3.$$

Define, for $r \in \mathbb{Q} = [0; S_0] = [0; S_1]$,

$$\begin{cases} Z_0(r) := S_0 \\ Z_1(r) := S_1 \\ Z_{n+1}(r) := Z_n Z_n' \end{cases}$$

Proposition

Every isolated point of $\mathcal{E} \setminus \{g\}$ is of the form $[0; \overline{Z_n(r)}]$ with $n \geq 1$ and r extremal.

Discontinuity points

Define, for $r \in \mathbb{Q} = [0; S_0] = [0; S_1]$,

$$\begin{cases} Z_0(r) := S_0 \\ Z_1(r) := S_1 \\ Z_{n+1}(r) := Z_n Z_n' \end{cases}$$

Proposition

Every isolated point of $\mathcal{E}$ is of the form $[0; \overline{Z_n(r)}]$ with $n \geq 1$ and r extremal.

Proposition

The points of discontinuity of the set-valued function $t \mapsto \mathcal{B}(t)$ are precisely the isolated points of $\mathcal{E}$.

The Feigenbaum point

Let us start with $Z_0 = (1, 1)$, $Z_1 = (2)$.
Then the cascade is

$$\alpha_0 = [0; \overline{1, 1}], \quad \alpha_1 = [0; \overline{2}], \quad \alpha_2 = [0; \overline{2, 1, 1}], \quad \alpha_3 = [0; \overline{2, 1, 1, 2, 2}]$$

Then

$$c_F := \lim_{n \to \infty} \alpha_n = [0; 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, \dots]$$

Its continued fraction expansion is the fixed point of the substitution

$$\begin{cases} 1 \mapsto 2 \\ 2 \mapsto 211. \end{cases}$$

(Compare Thue-Morse sequence)

Question. For which t_0 is such that

$$\text{H.dim}\, \mathcal{B}(t) = 0?$$

The Feigenbaum point

Let us start with $Z_0 = (1, 1)$.
Then the cascade is

$$\alpha_0 = [0; \overline{1, 1}], \quad \alpha_1 = [0; \overline{2}], \quad \alpha_2 = [0; \overline{2, 1, 1}], \quad \alpha_3 = [0; \overline{2, 1, 1, 2, 2}]$$

Then

$$c_F := \lim_{n \to \infty} \alpha_n = [0; 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, \ldots]$$

Its continued fraction expansion is the fixed point of the substitution

$$\begin{cases} 1 \mapsto 2 \\ 2 \mapsto 211. \end{cases}$$

Theorem

We have

$$\text{H.dim } \mathcal{B}(t) = 0$$

if and only if $t \geq c_F$.

Main results: Hausdorff dimension

Theorem

Let $\eta(t) := \text{H.dim}\,\mathcal{B}(t)$*. Then:*

1. *The function* $\eta(t)$ *is continuous;*
2. *If* $\mathcal{E}(t) := \mathcal{E} \cap [t, 1]$*. Then for any* t

$$\text{H.dim}\,\mathcal{B}(t) = \text{H.dim}\,\mathcal{E}(t).$$

3. *The function* $\eta(t)$ *is constant on any tuning window* W_r *with* $r \in \mathbb{Q} \cap (0, c_F)$ *extremal.*

Tuning operators and windows

Starting with a rational number

$$r = [0; S_0] = [0; S_1]$$

the two continued fraction expansions of r.
Given $r \in \mathbb{Q}$, let us define $\tau_r : [0,1] \to [0,r]$

$$[0; a_1, a_2, \dots] \mapsto [0; S_1 S_0^{a_1 - 1} S_1 S_0^{a_2 - 1} \dots]$$

Then:

1. $\tau_r(\mathcal{E}) \subseteq \mathcal{E}$
2. $\mathcal{E}(\tau_r(x)) = \mathcal{E}(\alpha) \cup \tau_r(\mathcal{E}(x))$

Definition

Define the tuning window

$$W_r := [\omega, \alpha)$$

with

$$\alpha := [0; \overline{S_0}] \qquad \omega := [0; S_1, \overline{S_0}]$$

Tuning operators and cascades

Given $r \in \mathbb{Q}$, let us define $\tau_r : [0,1] \to [0,r]$

$$[0; a_1, a_2, \ldots] \mapsto [0; S_1 S_0^{a_1 - 1} S_1 S_0^{a_2 - 1} \ldots]$$

Proposition

For any extremal rational number r, the cascade generated by r is just the sequence

$$\{\tau_r \tau_{1/2}^n(g), \quad n \geq 0\}$$

Theorem

The function $\eta(t) = \mathrm{H.dim}\, \mathcal{B}(t)$ is constant on each tuning window W_r.

Corollary

The bifurcation locus for the dimension function $\eta(t)$ is strictly smaller than the bifurcation locus for the set-valued function.

Scaling at the Feigenbaum point

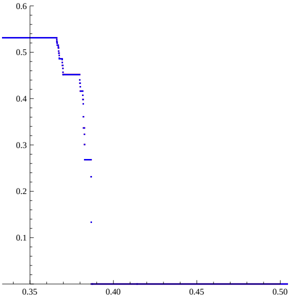
Theorem

There exist $c_1, c_2 > 0$ such that for any $t < c_F$,

$$\frac{c_1}{-\log|t - c_F|} \leq \text{H.dim}\,\mathcal{B}(t) \leq \frac{c_2}{-\log|t - c_F|}.$$

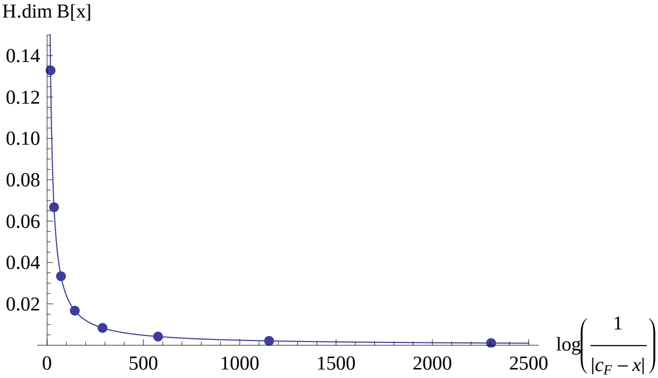
Moreover,

$$\lim_{n\to\infty} \text{H.dim}\,\mathcal{B}(\alpha_n)\log\left(\frac{1}{|c_F - \alpha_n|}\right) = 5\log\frac{\sqrt{5}+1}{2}.$$



Scaling at the Feigenbaum point

$$\text{H.dim}\, \mathcal{B}(t) \approx \frac{1}{-\log|t - c_F|}$$



Note. However,

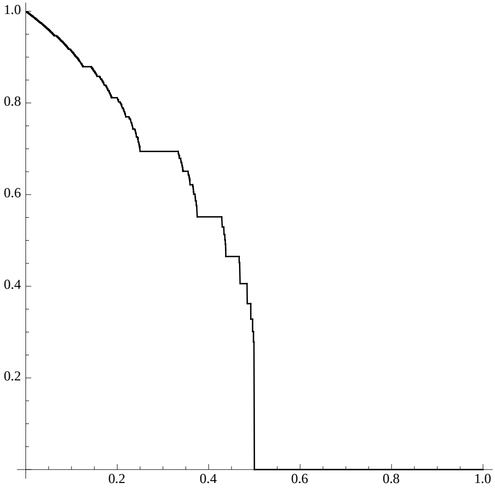
$$\lim_{t \to c_F^+} \text{H.dim}\, \mathcal{B}(t) \log\left(\frac{1}{|t - c_F|}\right)$$

does not exist.

Comparison with the linear (doubling) case

$$K(t) := \{x \in [0,1] \ : \ 2^n x \notin [0,t) \quad \forall n \geq 0\}$$

$$\eta'(t) := \text{H.dim } K(t) \qquad \text{(Urbański, '86)}$$



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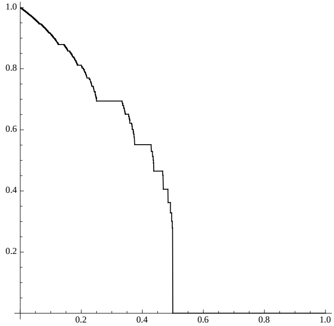
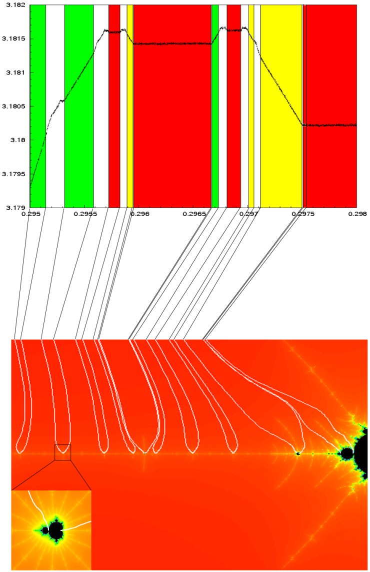


- Bifurcation locus is a Cantor set
- The modulus of continuity at $t = 1/2$ is $\omega(x) \approx \frac{\log\log(1/x)}{\log(1/x)}$

Further connections

1. $\mathcal{E}$ is the bifurcation locus for α-continued fractions
   (Carminati-T., Kraaikamp-Schmidt-Steiner...)
2. The recurrence spectrum for Sturmian sequences
   (Cassaigne)
3. Markov and Lagrange spectra
   (Moreira), but different
4. Bifurcation locus for unimodal maps
   (Milnor-Thurston, Isola-Politi, Bonanno-Carminati-Isola-T.)
5. The set $\mathcal{U}$ of univoque numbers
   (Erdos-Horvath-Joo, Komornik-Loreti, Allaart-Kong)
6. Open dynamical systems: doubling map with "holes"
   (Urbański, Carminati-T.)

Correspondence between α-continued fractions and the Mandelbrot set



A unified approach

The dictionary yields a unified proof of the following results:

1. The set of matching intervals for α-continued fractions has zero measure and full Hausdorff dimension
(Nakada-Natsui conjecture, CT 2010)
2. The real part of the boundary of the Mandelbrot set has Hausdorff dimension 1

$$H.dim(\partial\mathcal{M} \cap \mathbb{R}) = 1$$

(Zakeri, 2000)

The end


Köszönöm
Merci
Dankjewel
Toda
Sagolun
Gracias
ありがとう
Sagolun
Спасибо
Merci
Xièxie
Gracias
Tak
Mahalo
Grazie
Matondo
Mercé
Mahalo
Efharisto
شكرا
Takk
Terima Kasih
Dankie
Kiitos
Toda
Tak
Faleminderit
Hvala
Danke
Mercé
Xièxie
Dankjewel
Misaotra
Grazie
Спасибо
Thank You
Tak
Rahmat
شكرا
Xièxie
Takk
Kiitos
Köszönöm
Grazie
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Misaotra
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Dankie
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Obrigado
Toda
Obrigado
Merci
Terima Kasih
Rahmat
Maururu
Efharisto
Terima Kasih
Danke
Mercé
Rahmat