Analogues of Khintchine's theorem for random attractors.

Sascha Troscheit

(sascha.troscheit@univie.ac.at)

23 November 2021 One World Numerations Seminar (Joint with Simon Baker)





Der Wissenschaftsfonds.

Part 1: Khintchine's theorem

Given a function $\Psi: \mathbb{N} \to [0,\infty)$, we define

 $J(\Psi) = \{x \in \mathbb{R} : \|x - \frac{p}{q}\| \le \Psi(q) \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N}\}$

An application of the Borel-Cantelli lemma shows that

$$\mathcal{L}(J(\Psi))=0 \quad ext{if} \quad \sum_{q\in \mathbb{N}} q\Psi(q)<\infty.$$

Khintchine proved a partial converse.

Theorem (Khintchine, 1926) Assume that $\Psi : \mathbb{N} \to [0, \infty)$ is decreasing and $\sum_{q \in \mathbb{N}} q \Psi(q) = \infty.$ Then, \mathcal{L} -almost every $x \in \mathbb{R}$ is in $J(\Psi)$. The monotonicity assumption cannot be removed (Duffin & Schaeffer, 1941) which motivated the (now proven) conjecture:

Theorem (Koukoulopoulos & Maynard, 2020) Let $\Psi : \mathbb{N} \to [0, \infty)$. Then, \mathcal{L} -a.e. $x \in \mathbb{R}$ is in $J(\Psi) \iff \sum_{q \in \mathbb{N}} \Psi(q) \phi(q) = \infty$.

Note that the monotonicity condition shows that there are subtleties in the geometry of rational numbers that can be explored by different Ψ .

We will use this approach to study Diophantine sets on "fractals".

101 on fractal sets.

Let \mathcal{A} be a finite alphabet, $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}^n$ be all finite words over \mathcal{A} , and $\mathcal{A}^{\mathbb{N}}$ be all infinite words.

Let $\Phi = {\phi_a}_{a \in \mathcal{A}}$ be a (finite) collection of strict contractions on \mathbb{R}^d indexed by \mathcal{A} . We write $\phi_{w} = \phi_{w_1} \circ \phi_{w_2} \circ \cdots \circ \phi_{w_n}$ for $w = w_1 \ldots w_n \in \mathcal{A}^n$.

There exists a unique, non-empty, compact set $X = X(\Phi) \subset \mathbb{R}^d$ that satisfies

$$X = \bigcup_{w \in \mathcal{A}} \phi_w(X).$$

The invariant set X is also called the attractor of X. In fact, for any fixed $x \in \mathbb{R}^d$,

$$d_H\left(igcup_{{\mathtt w}\in {\mathcal A}^n}\phi_{{\mathtt w}}(x),X
ight)
ightarrow 0 \ \ {\hbox{as}} \ \ n
ightarrow\infty.$$



Sierpinski triangle for similarities with Lipschitz constants 1/2 and 11/20, and similarity dimensions $\log 3/\log 2 = 1.584...$ and $\log 3/\log(20/11) = 1.837...$, respectively.

Some pictures



Sierpinski triangle with Lipschitz constant 3/5 and similarity dimension $\log 3/\log(5/3) = 2.150...$

We emulate Diophantine approximation by replacing the role of rational numbers with those in the dynamical/iterative structure. Let $\Psi : \mathcal{A}^* \to [0, \infty)$ and $z \in \mathbb{R}^d$. We define

$$W_{\Phi}(z, \Psi) = \{x \in \mathbb{R}^d : \|x - \varphi_{\mathtt{w}}\| \leq \Psi(\mathtt{w}) \text{ for i.m. } \mathtt{w} \in \mathcal{A}^*\}.$$

We ask:

Motivating Question

Are there similar dichotomies with divergence conditions for the natural volume, e.g. does the following hold:

$$\sum_{n\in\mathbb{N}}\sum_{\mathsf{w}\in\mathcal{A}^n}\Psi(\mathsf{w})^{\dim_H X}\Longrightarrow\mathcal{H}^{\dim_H}(W_{\Phi}(z,\Psi))=\mathcal{H}^{\dim_H X}(X)?$$

Diophantine approximation on fractal sets

The implication holds, e.g. when φ_i are similarities or conformal mappings under separation conditions.

The behaviour above appears for suitable classes of Ψ in a variety of settings. It is closely linked to the general shrinking target problem. Recent important progress: Allen and Bárány; Baker; Persson and Reeve; and Levesly, Salp, and Velani;....

Studying the classes of Ψ for which such a statement holds provides information on how "spread out" the points in X are.

Similarity dimension, affinity dimension, etc.: The similarity dimension, affinity dimension, are the zero of a suitable pressure

$$\mathcal{P}(s) = \lim_{n o \infty} \log \sum_{\mathtt{w} \ \mathcal{A}^n} \sup_{x \in X} \| \phi_{\mathtt{w}}'(x) \|^s$$

[The derivative being replaced by a "singular value function" in the affine case.]

Exceeding expectations

The zero of the pressure is the "best guess" to the dimension of the attractor X. They coincide in many cases.

The interesting case is when the value exceeds the ambient space dimension. Then we would expect $\dim_H X = d$ and X to have positive Lebesgue measure.

To give such a result we need some more notation. Let $B \subset \mathbb{N}$. Recall the upper density

$$\overline{d}(B) = \limsup_{n \to \infty} \frac{\#\{1 \le j \le n : j \in B\}}{n}$$

and write $\mathit{G} = igcup_{\gamma \in (0,1)} \mathit{G}_{\gamma}$, where

$$\mathcal{G}_{\gamma} = \left\{ g: \mathbb{N}
ightarrow [0,\infty) : \sum_{n \in B} g(n) = \infty, orall B \subseteq \mathbb{N} ext{ with } \overline{d}(B) > \gamma
ight\}.$$

A Diophantine fractal example

Let $\Phi_{\mathbf{t}} = \{\phi_i(x) = \lambda \cdot Ox + t_i\}_{i \in \mathcal{A}}$ be a finite collection of equicontractive similarities on \mathbb{R}^d , where $\mathbf{t} = (t_i)_{i \in c\mathcal{A}}$ is a collection of translation vectors. Write $X_{\mathbf{t}}$ for the invariant set.

Proposition (Baker, 2019)

Suppose $\log \# \mathcal{A} / \log(1/\lambda) > d$. Then, for Lebesgue almost every $\mathbf{t} \in \mathbb{R}^{\# \mathcal{A} d}$, for any $g \in G$ and $z \in X_{\mathbf{t}}$, the set

$$\left\{x \in \mathbb{R}^d : |x - \phi_{\mathtt{W}}(z)| \le \left(\frac{g(|\mathtt{W}|)}{(\#\mathcal{A})^{|\mathtt{W}|}}\right)^{1/d} \text{ for i.m. } \mathtt{W} \in \mathcal{A}\right\}$$

has positive Lebesgue measure.

Using different test functions g (such as 1/n) allows one to get detailed information on the "bunching" of these typical attractors. Another consequence is that any such X_t has (typically) positive Lebesgue measure.

The last result already made use of the common observation that randomisation enables a "smoothing" of the object under consideration. Adding a random translation goes back to Falconer (1988) where the dimension of generic self-affine sets was calculated.

Jordan, Pollicott, and Simon considered self-affine attractors with random perturbations, whereas Peres, Simon, and Solomyak considered random constructions where the contraction rates at every level of the construction are randomly chosen.

We follow a similar approach but consider stochastically self-similar and self-affine sets. These were first introduced (independently) by Falconer and Graf in the 80s and satisfy an invariance in distribution.

Stochastic self-similarity - Intuition

A stochastically self-similar set is one where each (random) image looks the same (in distribution) as the entire set:

$$F_{\omega} \equiv_{d} \bigcup_{i=1}^{N} \phi_{\omega,i}(F_{\omega',i})$$

...



		tin o Bi tin k Bi R Cot z Tin Z Tin w Di
552 853 11 12 12 55 15 12 55 15 12 55 15 12 55	200 2001 11 22 14 55 15 11 35 14 15 11 35 14 15 11 35 14	

Let M_d denote the set of invertible $d \times d$ matrices with ||A|| < 1for all $A \in M_d$. Write $S_d \subset M_d$ for those which are similarities (scalar multiple of orthogonal matrices). For all $i \in A$ we let $\Omega_i \subset M_d$ be a subset with measure η_i supported on Ω_i .

We define a product measure on $\Omega = \prod_{\mathbf{w} \in \mathcal{A}^*} \Omega_{\ell(\mathbf{w})}$ by $\eta = \prod_{\mathbf{w} \in \mathcal{A}^*} \mu_{\ell(\mathbf{w})}$ where $\ell(\mathbf{w})$ is the last letter of $\mathbf{w} \in \mathcal{A}^*$. A particular realisation $\omega \in \Omega$ is a collection of randomly chosen matrices, indexed by $\mathbf{w} \in \mathcal{A}^*$. We write $A_{\omega,\mathbf{w}}(x) = \omega_{\mathbf{w}} \cdot x$ to highlight the matrix/linear component associated with address \mathbf{w} and realisation ω .

Note that for distinct $v, w \in A^*$, the matrices $A_{v,\omega}$ and $A_{w,\omega}$ are independent though only identical in distribution if $\ell(w) = \ell(v)$.

Random framework, continued

Let t_i for $i \in A$ be a finite choice of distinct translations in \mathbb{R}^d . For every $w \in A^*$ we define the random maps

$$f_{\omega, \mathtt{w}}(x) = A_{\omega, \mathtt{w}}(x) + t_{\ell(\mathtt{w})}$$

and

$$\phi_{\omega,\mathbf{w}}(\mathbf{x}) = f_{\omega,\mathbf{w}_1} \circ \cdots \circ f_{\omega w_{|\mathbf{w}|}}.$$

Given a realisation $\omega \in \Omega$, and an infinite word $\mathbf{w} \in \mathcal{A}^{\mathbb{N}}$, we define its projection $\Pi_{\omega}(\mathbf{w}) : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}^{d}$ by

$$\Pi_{\omega}(\mathbf{w}) = \lim_{n \to \infty} \phi_{\omega, \mathbf{w}|_n}(0) = \lim_{n \to \infty} f_{\omega, w_1} \circ \cdots \circ f_{\omega w_n}(0)$$

and the random attractor by

$$F_{\omega} = \bigcup_{\mathtt{w} \in \mathcal{A}^N} \Pi_{\omega}(\mathtt{w}).$$

Random framework, continued

By definition, we have

$$F_{\omega} \equiv_d igcup_{i \in \mathcal{A}} f_{\omega',i}(F_{\omega''_i})$$

where $\omega, \omega', \omega''_1, \ldots, \omega''_{\#\mathcal{A}}$ are independent realisations in (ω, η) . Given $\Psi : \mathcal{A}^* \to [0, \infty)$, $v \in \mathcal{A}^{\mathbb{N}}$, and $\omega \in \Omega$ we want to investigate

$$W_\omega({\mathtt v},\Psi)=\left\{x\in {\mathbb R}^d: |x-{\sf \Pi}_\omega({\mathtt w}\,{\mathtt v})|\leq \Psi({\mathtt w}) ext{ for infinitely many }{\mathtt w}\in {\mathcal A}^*
ight\}$$

Doing this directly is difficult. Instead we consider an auxiliary family to deduce results about $W_{\omega(v,\Psi)}$.

Let μ be a slowly decaying measure defined on $\mathcal{A}^{\mathbb{N}}$ such that

$$\mu([w_1,\ldots,w_{n+1})/\mu([w_1,\ldots,w_n]) \geq c$$

for all *n* and μ almost all $w \in \mathcal{A}^{\mathbb{N}}$.

Random framework, continued

Let $L_{\mu,n}$ be all the finite words w such that $\mu([w]) \sim c^n$. We investigate

$$U_{\omega}(\mathbf{v}, \mu, g) = \Big\{ x \in \mathbb{R}^d : |x - \Pi_{\omega}(\mathbf{w} \, \mathbf{v})| \le (\mu([\mathbf{w}])g(n))^{1/d}$$

for some $\mathbf{w} \in L_{\mu, n}$ for i.m. $n \Big\}.$

We will also write

$$\lambda(\eta,\mu) = \sum_{i\in\mathcal{A}} \mu([i]) \cdot \lambda'(\eta_i),$$

where

$$\lambda'(\eta_i) = -\int_{\Omega_i} \log(|\operatorname{\mathsf{Det}}(A)|) d\eta_i(A)$$

for the Lyapunov exponent of the random system with respect to $\boldsymbol{\mu}.$

Assumptions

We need to make certain assumptions on the random matrices such as the logarithmic moment condition to allow use of Cramér's theorem on large deviations:

$$\log \int_{\Omega_i} \exp(s \log |\operatorname{Det}(A)|) d\eta_i(A) < \infty.$$

We say that our RIFS is **non-singular** if there exists C > 0 such that for all $i \in A, x \in \bigcup_{\omega \in \Omega} \prod_{\omega} (A^{\mathbb{N}})$ and B(y, r),

$$\eta_i (A \in \Omega_i : A \cdot x \in B(y, r)) \leq Cr^d.$$

We say that our RIFS is **distantly non-singular** if there exists C > 0 such that for all $i \in \mathcal{A}, x \in \bigcup_{\omega \in \Omega} \prod_{\omega} (\mathcal{A}^{\mathbb{N}})$ and $y \in \mathbb{R}^d \setminus B(0, \min_{i \neq j} |t_i - t_j|/8)$,

$$\eta_i(A \in \Omega_i : A \cdot x \in B(y, r)) \leq Cr^d.$$

Inspiration for condition

Peres, Solomyak, and Simon considered a similar question for random similarities in \mathbb{R} such that (in our notation) $\mathcal{A}_{\omega, \mathbb{W}} = Y_{|\mathbb{W}|} c_{\ell(\mathbb{W})}$, where c_i only depends on the last letter of $\mathbb{W} \in \mathcal{A}^*$ and Y is a random variable depending only on the length of the word $\mathbb{W} \in \mathcal{A}^*$.

The random perturbation Y is therefore "homogeneously" applied everywhere at the same level in the construction.

Theorem (Peres, Simon, Solomyak 2006)

Let Y be an absolutely continuous random variable with distribution ν satisfying, for some C > 0,

$$\frac{d\nu}{dx} \le C\frac{1}{x}$$

Let μ be an ergodic shift invariant measure on $\mathcal{A}^{\mathbb{N}}$. Assume further that $h(\mu)/\lambda(\eta,\mu) > 1$. Then, F_{ω} has positive Lebesgue measure.

Result

Theorem (Baker-T., 2020)

Let $({\{\Omega_i\}}_{i\in\mathcal{A}}, {\{\eta_i\}}_{i\in\mathcal{A}}, {t_i\}}_{i\in\mathcal{A}})$ be a RIFS and assume one of:

- A. Assume $\Omega_i \subset S_d$ for all $i \in A$ and the RIFS is distantly non-singular.
- B. Assume $\Omega_i \subset M_d$ for all $i \in \mathcal{A}$ and the RIFS is non-singular.

Suppose μ is a slowly decaying shift invariant ergodic probability measure with $h(\mu)/\lambda(\eta,\mu) > 1$. Then the following hold:

- 1. For any $v \in \mathcal{A}^{\mathbb{N}}$, for η almost every $\omega \in \Omega$, for any $g \in G$, the set $U_{\omega}(v, \mu, g)$ has positive Lebesgue measure.
- For any v ∈ A^N, for η almost every ω ∈ Ω, for any Ψ : A* → [0,∞) the set W_ω(v, Ψ) has positive Lebesgue measure if there exists g ∈ G such that Ψ(w) ≈ (m([w])g(n))^{1/d}.

Some Corollaries

Corollary

Let $({\{\Omega_i\}}_{i\in\mathcal{A}}, {\{\eta_i\}}_{i\in\mathcal{A}}, {t_i\}}_{i\in\mathcal{A}})$ be a RIFS and assume one of:

- A. Assume $\Omega_i \subset S_d$ for all $i \in A$ and that the RIFS is distantly non-singular.
- B. Assume $\Omega_i \subset M_d$ for all $i \in \mathcal{A}$ and the RIFS is non-singular.

Let $(p_i)_{i \in \mathcal{A}}$ be a probability vector satisfying $\frac{-\sum p_i \log p_i}{\sum p_i \lambda'(\eta_i)} > 1$. Then for all $v \in \mathcal{A}^{\mathbb{N}}$, for η -almost every $\omega \in \Omega$ the set

$$\left\{x\in\mathbb{R}^d:|x-\Pi_\omega({\tt w}\,{\tt v})|\leq \left(\frac{\prod_{k=1}^{|\,{\tt w}\,|}p_{{\tt w}_k}}{|\,{\tt w}\,|}\right)^{1/d} \text{ for i.m. } {\tt w}\in\mathcal{A}^*\right\}$$

has positive Lebesgue measure.

The compactness of F_{ω} implies that $U_{\omega}(\mathbf{v}, \mu, g) \subseteq F_{\omega}$ whenever g is bounded. This gives

Corollary

- Let $({\{\Omega_i\}}_{i\in\mathcal{A}}, {\{\eta_i\}}_{i\in\mathcal{A}}, {t_i\}}_{i\in\mathcal{A}})$ be a RIFS and assume one of:
 - A. Assume $\Omega_i \subset S_d$ for all $i \in A$ and that the RIFS is distantly non-singular.
 - B. Assume $\Omega_i \subset M_d$ for all $i \in A$ and the RIFS is non-singular.

If there exists a slowly decaying shift invariant ergodic probability measure μ satisfying $h(\mu)/\lambda(\eta,\mu) > 1$, then for η -almost every $\omega \in \Omega$ the set F_{ω} has positive Lebesgue measure.

Self-similar. For each $i \in A$ let $0 \le r_i^- < r_i^+ < 1$ and set

$$\Omega_i = \left\{ \lambda \cdot \mathcal{O} : \lambda \in [r_i^-, r_i^+], \ \mathcal{O} \in \mathcal{O}(d)
ight\},$$

where $\mathcal{O}(d)$ is the set of orthogonal $d \times d$ matrices. For each $i \in \mathcal{A}$ let η_i be the product measure of the Haar measure and the Lebesgue measure, restricted and normalised to $[r_i^-, r_i^+]$.

It can be checked that this setup satisfies all conditions, and letting # A be sufficiently large and r^+ uniformly bounded away from zero, we can find Bernoulli measures μ such that $h(\mu)/\lambda(\eta,\mu) > 1$ and our Theorem and its Corollaries apply.

Self-affine. Similar examples can be created when considering self-affine constructions. For instance, letting $Z_i \subset M_d$ be compact with

$$\Omega_i = \left\{ \lambda \cdot \textit{OB} : \lambda \in [r_i^-, r_i^+], \ \textit{O} \in \mathcal{O}(\textit{d}), \ \textit{B} \in \textit{Z}_i
ight\},$$

and assuming $\{t_i\}$ are large enough such that

$$B(0,\delta)\cap igcup_{\omega\in\Omega} {\sf \Pi}_\omega({\mathcal A}^{\mathbb N})=arnothing$$

we can apply our results under the non-singular condition.

In the stochastic self-similar / self-affine setting one usually expects the "correct" Lyapunov exponent to be

$$\log \int_{\Omega_i} |\operatorname{Det}(A)| d\eta_i(A)$$

instead of

$$\int_{\Omega_i} \log |\operatorname{Det}(A)| d\eta_i(A).$$

This e.g. applies when calculating the almost sure Hausdorff dimension.

In our case the near optimal use of large deviations suggests that the latter exponent is the correct to use. We suspect that this has to do with needing "level specific" information for our Diophantine results, as opposed to "eventually averaging".

Thank you



Image by www.traveljunction.com

To use a transversality argument we need to show for distinct $\mathtt{w}, \mathtt{w}' \in \mathcal{A}^*$ that

$$\begin{split} \int_{\Omega} \chi_{[-r,r]} \left(|\Pi_{\omega}(\mathbf{w}\,\mathbf{v}) - \Pi_{\omega}(\mathbf{w}'\,\mathbf{v})| \right) \\ & \cdot \chi \left(\omega : \mathsf{Det}(\widehat{A}_{\omega,w_1\dots w_n}) \sim e^{-sn}, \; \forall 1 \le n \le |\,\mathbf{w}\,| \right) \\ & \cdot \chi \left(\omega : \mathsf{Det}(\widehat{A}_{\omega,w_1'\dots w_n'}) \sim e^{-sn}, \; \forall 1 \le n \le |\,\mathbf{w}'\,| \right) d\eta \\ & = O(r^d e^{|\,\mathbf{w}\,\wedge\,\mathbf{w}'\,|s}) \end{split}$$

We need to make sure that $\Pi_{\omega}(wv)$ and $\Pi_{\omega}(wv')$ are not "too close". [Tablet time!]