On a lower bound of the number of integers in Littlewood's conjecture

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Littlewood's conjecture

Littlewood's conjecture (c.1930).

For <u>every</u> $(\alpha, \beta) \in \mathbb{R}^2$,

$$\liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle = 0,$$

where $\langle x \rangle = \min_{k \in \mathbb{Z}} |x - k|$.

- [Cassles, Swinnerton-Dyer, 1955].
  If α and β are in the same cubic number field, then $\liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle = 0$.
- [Pollington, Velani, 2000].
  For $\forall \alpha \in \mathbf{Bad} := \{\alpha \in \mathbb{R} \,|\, \liminf_{n \to \infty} n \langle n\alpha \rangle > 0\}$, $\exists \mathbf{G}(\alpha) \subset \mathbf{Bad}$ s.t. $\dim_H \mathbf{G}(\alpha) = 1$ and, for $\forall \beta \in \mathbf{G}(\alpha)$,

$$n \langle n\alpha \rangle \langle n\beta \rangle \leq \frac{1}{\log n} \quad \text{for infinitely many } n.$$

The set of exceptions for Littlewood's conjecture has Hausdorff dimension zero.

Theorem [Einsiedler, Katok, Lindenstrauss, 2006].

$$\dim_H \left\{ (\alpha, \beta) \in \mathbb{R}^2 \;\middle|\; \liminf_{n \to \infty} n \langle n\alpha \rangle \langle n\beta \rangle > 0 \right\} = 0.$$

Furthermore, this set is an at most countable union of compact sets of box dimension zero.

This Theorem is obtained as a corollary of some property of the **diagonal action on** $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$.

Quantitative version of Littlewood's conjecture

Littlewood's conjecture says that, for every $(\alpha, \beta) \in \mathbb{R}^2$ and any $0 < \varepsilon < 1$, $n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon$ for infinitely many n.

Problem (Quantitative version of Littlewood's conjecture).

For $(\alpha, \beta) \in \mathbb{R}^2$, $0 < \varepsilon < 1$ and sufficiently large $N \in \mathbb{N}$, **how many integers** $n \in [1, N]$ are there s.t.

$$n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon?$$

Result on quantitative LC / Main Theorem

- [Pollington, Velani, Zafeiropoulos, Zorin, 2022] (quantitative version of [Pollington, Velani, 2000]). For $\forall \alpha \in \mathbf{Bad}, \forall \gamma \in [0,1]$, $\exists \mathbf{G}(\alpha,\gamma) \subset \mathbf{Bad}$ s.t. $\dim_H \mathbf{G}(\alpha,\gamma) = 1$ and, for $\forall \beta \in \mathbf{G}(\alpha,\gamma)$, we have

$$\left| \left\{ n \in [1,N] \,\middle|\, n\langle n\alpha\rangle\langle n\beta - \gamma\rangle \leq \frac{1}{\log n} \right\} \right| \gg \log\log N, \quad N \in \mathbb{N}.$$

Main Theorem [U., 2022+, 2024+].

For $0 < \forall\gamma < 1/72$, there exists an "exceptional set" $Z(\gamma) \subset \mathbb{R}^2$ with $\dim_H Z(\gamma) \le 90\sqrt{2\gamma}$ s.t., for $\forall(\alpha, \beta) \in \mathbb{R}^2 \setminus Z(\gamma)$ and $0 < \forall\varepsilon < 4^{-1}e^{-2}$,

$$\liminf_{N\to\infty} \frac{(\log\log N)^2}{(\log N)^2} |\{n \in [1, N] \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}| \ge \gamma.$$

Corollary.

There exists an "exceptional set" $Z \subset \mathbb{R}^2$ with $\dim_H Z = 0$ s.t., for $\forall(\alpha, \beta) \in \mathbb{R}^2 \setminus Z$ and $0 < \forall\varepsilon < 4^{-1}e^{-2}$,

$$\liminf_{N\to\infty} \frac{(\log\log N)^2}{(\log N)^2} |\{n \in [1, N] \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}| \ge C_{\alpha,\beta},$$

where $C_{\alpha,\beta} > 0$ is a constant depending only on (α, β).

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The diagonal action on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$

We write

$$G := \mathrm{SL}(3,\mathbb{R}),\ \Gamma := \mathrm{SL}(3,\mathbb{Z}),\ X := G/\Gamma.$$

- By the one to one correspondence
  $$X = G/\Gamma \ni g\Gamma \longleftrightarrow g \cdot \mathbb{Z}^3 \in \{\Lambda \subset \mathbb{R}^3 : \text{ lattice of covolume } 1\},$$
  we can identify X as the space of lattices in $\mathbb{R}^3$ of covolume 1.
- $X = G/\Gamma$ admits a unique G-invariant Borel probability measure m_X on X, called the **Haar measure**. However, X is *not compact*.

Proposition (Mahler's criterion).

For a subset $B \subset X$, B is unbounded in X iff

$$0 < \forall\varepsilon < 1,\ \exists v \in \bigcup_{\Lambda\in B} \Lambda \setminus \{0\} \text{ s.t. } \|v\| < \varepsilon.$$

Let

$$A := \left\{ \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{pmatrix} \middle| \ t_1, t_2, t_3 \in \mathbb{R},\ t_1 + t_2 + t_3 = 0 \right\} < G.$$

The left action of A

$$A \times X \ni (a, x) \mapsto ax \in X$$

is called **the (higher rank) diagonal action** on X.
For the application to Littlewood's conjecture, we consider the action of the positive cone A^+ of A:

$$A^+ := \left\{ a_{s,t} := \begin{pmatrix} e^{-s-t} & & \\ & e^{s} & \\ & & e^{t} \end{pmatrix} \middle| s, t \geq 0 \right\}.$$

The relation between the diagonal action and LC

Let $U < G$ be the **unstable subgroup** for conjugation by A^+:

$$U := \left\{ u \in G \,\middle|\, a^{-n} u a^n \xrightarrow[n\to\infty]{} e,\ \forall a \in A^+ \setminus \{e\} \right\}$$
$$= \left\{ u_{\alpha,\beta} := \begin{pmatrix} 1 & & \\ \alpha & 1 & \\ \beta & & 1 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\} \cong \mathbb{R}^2.$$

For $(\alpha, \beta) \in \mathbb{R}^2$, we write

$$\tau_{\alpha,\beta} = u_{\alpha,\beta}\Gamma \in X = G/\Gamma.$$

By Mahler's criterion for $B = A^+ \tau_{\alpha,\beta} \subset X$, we have the following:

Proposition.

For $(\alpha, \beta) \in \mathbb{R}^2$, $\liminf_{n\to\infty} n\langle n\alpha\rangle\langle n\beta\rangle = 0$ iff the A^+ orbit of $\tau_{\alpha,\beta}$ is unbounded in X.

Measure rigidity under positive entropy condition

For an A-invariant probability measure μ and $a \in A$, we write $h_\mu(a)$ for the entropy of the map $X \ni x \mapsto ax \in X$ w.r.t. μ.

Theorem [Einsiedler, Katok, Lindenstrauss, 2006].

If μ is an A-invariant and ergodic Borel probability measure on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ s.t. $h_\mu(a) > 0$ for $\exists a \in A$, then μ is the Haar measure m_X on X.

As a corollary of this Theorem, we obtain that $\dim_H \{ u \in U \,|\, A^+ u\Gamma \subset X \text{ is bounded} \} = 0$ and, by Proposition, this is equivalent to

$$\dim_H \left\{ (\alpha, \beta) \in \mathbb{R}^2 \,\middle|\, \liminf_{n\to\infty} n\langle n\alpha\rangle\langle n\beta\rangle > 0 \right\} = 0.$$

Remarks on measure rigidity for the diagonal action

- Measure rigidity does not hold if $X = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.
- Theorem is the similar to the following measure rigidity for the $\times 2, \times 3$ action on $\mathbb{R}/\mathbb{Z}$ (by Rudolph, Johnson): if $a, b \in \mathbb{Z}_{\geq 2}$ are multiplicatively independent and μ is a $\times a, \times b$-invariant and ergodic Borel probability measure on $\mathbb{R}/\mathbb{Z}$ s.t. $\dim_H \mu > 0$, then μ is the Lebesgue measure.
- The positive entropy condition is believed to be dropped.

Full measure rigidity conjecture [Margulis].

For $n \geq 3$, every A-invariant and ergodic Borel probability measure on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ is homogeneous.

It is known that if Full measure rigidity conjecture is true, then Littlewood's conjecture follows from it.

Main Theorem (review)

Main Theorem [U., 2022+, 2024+].

For $0 < \forall\gamma < 1/72$, there exists an "exceptional set" $Z(\gamma) \subset \mathbb{R}^2$ with $\dim_H Z(\gamma) \leq 90\sqrt{2\gamma}$ s.t., for $\forall(\alpha, \beta) \in \mathbb{R}^2 \setminus Z(\gamma)$ and $0 < \forall\varepsilon < 4^{-1}e^{-2}$,

$$\liminf_{N\to\infty} \frac{(\log\log N)^2}{(\log N)^2} \left|\{n \in [1, N] \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}\right| \geq \gamma.$$

To prove this, we want to define $Z(\gamma)$ properly and show that

- on the outside of $Z(\gamma)$, we can obtain the quantitative result, and
- $Z(\gamma)$ has small Hausdorff dimension.

Empirical measures w.r.t. the diagonal action

For $x \in X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ and $T > 0$, we define the T**-empirical measure of** x **w.r.t.** A^+ by

$$\delta^T_{A^+,x} := \frac{1}{T^2}\int_{[0,T]^2} \delta_{a_{s,t}x}\, dsdt.$$

We are interested in the behavior of $\delta^T_{A^+,x}$ as $T \to \infty$. If $\delta^T_{A^+,x}$ $(T > 0)$ accumulate to a measure μ on X, (w.r.t. the weak*-topology), then μ is A-invariant but it may be that $\mu(X) < 1$ (since X is not compact).

Definition (escape of mass).

For $x \in X$, a sequence $(T_k)_{k=1}^{\infty}$ in $\mathbb{R}_{>0}$ s.t. $T_k \to \infty$ and $0 < \gamma < 1$, we say that $\delta^{T_k}_{A^+,x}$ $(k = 1, 2, \dots)$ exhibit γ**-escape of mass** if $\limsup_{k\to\infty} \delta^{T_k}_{A^+,x}(K) \leq 1 - \gamma$ for any compact subset $K \subset X$.

Outline of the proof

We take $(\alpha, \beta) \in \mathbb{R}^2$ and $(T_k)_{k=1}^{\infty} \subset \mathbb{R}_{>0}$ s.t. $T_k \to \infty$. We consider the sequence of the empirical measures $(\delta_{A^+, \tau_{\alpha,\beta}}^{T_k})_{k=1}^{\infty}$ of $\tau_{\alpha,\beta} \in X$.

Case 1 (large entropy case): For $0 < \gamma < 1$, assume that $(\delta_{A^+, \tau_{\alpha,\beta}}^{T_k})_{k=1}^{\infty}$ converges to a Borel measure μ on X s.t.

$$1 - \gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\widehat{\mu}}(a_1) > \gamma,$$

where $\widehat{\mu} = \mu(X)^{-1}\mu$ and $a_1 = \mathrm{diag}(e^{-1}, e, 1) \in A^+ \setminus \{e\}$.

If we write $\widehat{\mu} = \int_{E_A(X)} \nu \, d\sigma(\nu)$ for the A-ergodic decomposition of $\widehat{\mu}$, then, by the measure rigidity,

$$h_{\widehat{\mu}}(a_1) = \int_{E_A(X)} h_\nu(a_1) \, d\sigma(\nu) = \sigma(\{m_X\}) h_{m_X}(a_1) = 4\sigma(\{m_X\}),$$

and hence,

$$\lim_{k\to\infty} \delta^{T_k}_{A^+,\tau_{\alpha,\beta}} = \mu \geq \mu(X)\cdot\sigma(\{m_X\})m_X \geq \ 4^{-1}(1-\gamma)\gamma\cdot m_X. \qquad (1)$$

(The following "tessellation" idea is from [Björklund, Fregoli, Gorodnik, 2022+].)

For $T > 1$ and $0 < \varepsilon < 4^{-1}e^{-1}$, we define

$$\Omega_{T,\varepsilon} = \{(y,x_1,x_2)\in\mathbb{R}^3 \mid 0 < y < e^{2T},\ 0 < |x_1|,|x_2| < 2^{-1},\ y|x_1||x_2| < \varepsilon\} \subset \mathbb{R}^3.$$

Then

$$\left|\{n < e^{2T} \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\}\right| = \left|\Omega_{T,\varepsilon}\cap(u_{\alpha,\beta}\cdot\mathbb{Z}^3)\right|.$$

If we define

$$\Delta_\varepsilon = \{(y,x_1,x_2)\in\mathbb{R}^3 \mid 0 < y < 1,\ (2e)^{-1} < |x_1|,|x_2| < 2^{-1},\ y|x_1||x_2| < \varepsilon\} \subset \mathbb{R}^3,$$

we have the following partial tessellation of $\Omega_{T,\varepsilon}$:

$$\Omega_{T,\varepsilon} \supset \bigsqcup_{m,n\in\mathbb{Z}_{\geq 0},0\leq m,n\leq T} a_{m,n}^{-1}\Delta_\varepsilon,$$

and hence,

$$\frac{1}{T^2}\left|\left\{n < e^{2T} \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\right\}\right| = \frac{1}{T^2}\left|\Omega_{T,\varepsilon} \cap (u_{\alpha,\beta}\cdot\mathbb{Z}^3)\right|$$
$$\geq \frac{1}{T^2}\sum_{0\leq m,n\leq T}\left|\Delta_\varepsilon \cap (a_{m,n}u_{\alpha,\beta}\cdot\mathbb{Z}^3)\right|. \tag{2}$$

By using (1) and (2) and Siegel integral formula, we obtain:

Theorem (large entropy case).

Under our assumption of Case 1 (large entropy case), for $0 < \forall\varepsilon < 4^{-1}e^{-2}$, we have

$$\liminf_{k\to\infty}\frac{1}{T_k^2}\left|\left\{n < e^{2T_k} \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\right\}\right| \geq \gamma(1-\gamma)\varepsilon.$$

Case 2 (escape of mass): For $0 < \gamma < 1$, assume that $(\delta^{T_k}_{A^+,\tau_{\alpha,\beta}})_{k=1}^{\infty}$ exhibits γ-escape of mass.

For $0 < \varepsilon < 1/2$, we define $X_\varepsilon = \{x \in X \,|\exists v \in x \setminus \{0\} \text{ s.t. } \|v\|_\infty \leq \varepsilon\}$. X_ε is the complement of a bounded subset in X (Mahler's criterion), and hence,

$$\begin{aligned}
&\liminf_{k\to\infty} \delta^{T_k}_{A^+,\tau_{\alpha,\beta}}(X_\varepsilon) \\
=&\liminf_{k\to\infty} \frac{1}{T_k^2} m_{\mathbb{R}^2}\left(\left\{(s,t) \in [0,T_k]^2 \mid a_{s,t}\tau_{\alpha,\beta} \in X_\varepsilon\right\}\right) \\
=&\liminf_{k\to\infty} \frac{1}{T_k^2} m_{\mathbb{R}^2}\left(\bigcup_{\boldsymbol{n}\in\mathbb{Z}^3\setminus\{0\}} d_{\varepsilon,\boldsymbol{n}} \cap [0,T_k]^2\right) \\
\geq& \gamma, \qquad (3)
\end{aligned}$$

where, for $\boldsymbol{n} = {}^t(n, m_1, m_2) \in \mathbb{Z}^3 \setminus \{0\}$ $(n > 0)$,

$$\begin{aligned}
&d_{\varepsilon,\boldsymbol{n}} = \left\{(s,t) \in \mathbb{R}^2 \mid \|a_{s,t}u_{\alpha,\beta}\boldsymbol{n}\|_\infty \leq \varepsilon\right\} \\
=&\left\{(s,t) \in \mathbb{R}^2 \,\middle|\, s \leq \log\frac{\varepsilon}{|n\alpha + m_1|},\ t \leq \log\frac{\varepsilon}{|n\beta + m_2|},\ s + t \geq \log\frac{|n|}{\varepsilon}\right\}.
\end{aligned}$$

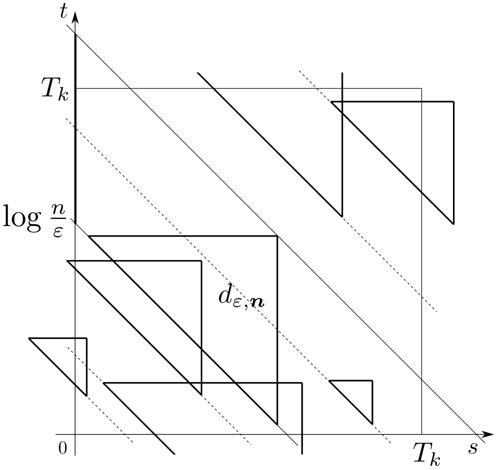

Figure: Illustration of $\bigcup_{\boldsymbol{n}\in\mathbb{Z}^3\setminus\{0\}} d_{\varepsilon,\boldsymbol{n}} \cap [0,T_k]^2$.

For $\boldsymbol{n} = {}^t(n, m_1, m_2) \in \mathbb{Z}^3 \setminus \{0\}$, if $d_{\varepsilon,\boldsymbol{n}} \cap [0,T_k]^2 \neq \emptyset$, we have

$$n \leq \varepsilon e^{2T_k} < e^{2T_k} \quad \text{and} \quad n\langle n\alpha\rangle\langle n\beta\rangle \leq n|n\alpha + m_1||n\beta + m_2| \leq \varepsilon^3 < \varepsilon.$$

Furthermore, if $d_{\varepsilon,\boldsymbol{n}} \cap [0, T_k]^2$ is large, we can see that

$$kn < e^{2T_k} \text{ and } kn\langle kn\alpha\rangle\langle kn\beta\rangle < \varepsilon \text{ for many } k \in \mathbb{N}.$$

Using (3) and a counting method, we can obtain the following:

Theorem (escape of mass).

Under our assumption of Case 2 (escape of mass), for $0 < \forall\varepsilon < 1/2$, we have

$$\liminf_{k\to\infty} \frac{(\log T_k)^2}{T_k^2} \left|\left\{n < e^{2T_k} \mid n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon\right\}\right| \geq \frac{\gamma}{18}.$$

If the sequence of the empirical measures $(\delta^{T_k}_{A^+,\tau_{\alpha,\beta}})_{k=1}^{\infty}$ of $\tau_{\alpha,\beta}$ converges to a measure with large entropy or exhibits escape of mass, then we can obtain a quantitative result on LC for (α, β).

The exceptional case

To prove Main Theorem, we must consider the exceptional case, that is, $(\alpha, \beta) \in \mathbb{R}^2$ s.t. some subsequence $(\delta^{T_k}_{A^+, \tau_{\alpha,\beta}})_{k=1}^{\infty}$ of the empirical measures of $\tau_{\alpha,\beta}$ converges to a measure μ on X s.t.

$$1 - \gamma < \mu(X) \leq 1 \quad \text{and} \quad h_{\widehat{\mu}}(a_1) \leq \gamma.$$

Actually, we can show that *the set of such* (α, β) *has small Hausdorff dimension.*

Theorem (Hausdorff dimension of the exceptional set).

Let $x_0 \in X$ and $0 < \gamma < 1$. We write $Z_{x_0}(\gamma)$ for the set of $u \in \overline{B^U_1} = \{u_{\alpha,\beta} \in U \mid |\alpha|, |\beta| \leq 1\}$ s.t. $\delta^T_{A^+, ux_0}$ $(T > 0)$ accumulate to some A-invariant measure μ on X s.t. $1 - \gamma < \mu(X) \leq 1$ and $h_{\widehat{\mu}}(a_1) \leq \gamma$. Then we have

$$\dim_H Z_{x_0}(\gamma) \leq 15\sqrt{\gamma}.$$

This Theorem is based on the following result by R. Bowen.

Proposition [Bowen, 1973].

Let $T : X \to X$ be a continuous map on a compact metric space X. For $\gamma > 0$, we write $QR(\gamma)$ for the set of $x \in X$ s.t. the empirical measures $N^{-1} \sum_{n=0}^{N-1} \delta_{T^n x}$ $(N > 0)$ accumulate to some T-invariant probability measure μ s.t. $h_\mu(T) \leq \gamma$. Then we have

$$h(T, QR(\gamma)) \leq \gamma,$$

where $h(T, A)$ for $A \subset X$ is Bowen's topological entropy (for an arbitrary subset).

If T is $\times a$ map on $\mathbb{R}/\mathbb{Z}$ $(a \geq 2)$, $h(T, A) = \log a \cdot \dim_H A$. In Bowen's argument and ours, the following combinatorial lemma is important:

Lemma [Bowen, 1973].

For $k \in \mathbb{N}$ and $\gamma > 0$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \left| \left\{ c \in \{1, \ldots, k\}^N \,\middle|\, H(\mathrm{dist}(c)) \leq \gamma \right\} \right| \leq \gamma.$$

In our setting, empirical measures are of two-parameter action, but the entropy is of one-parameter subaction. In addition, the space $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is not compact.
Theorem is based on the following result on the $\times a, \times b$ action.

Theorem [U., 2023].

Let $a, b \in \mathbb{Z}_{\geq 2}$. We take $0 < \gamma < \min\{\log b, (\log a)^2/\log b\}$ and write $K(\gamma)$ for the set of $x \in \mathbb{R}/\mathbb{Z}$ such that the empirical measures $N^{-2}\sum_{m,n=0}^{N-1}\delta_{a^m b^n x}$ $(N \in \mathbb{N})$ accumulate to a probability measure μ s.t. $h_\mu(\times a) \leq \gamma$. Then we have

$$\dim_H K(\gamma) \leq \frac{2\sqrt{\log b}\sqrt{\gamma}}{\log a + \sqrt{\log b}\sqrt{\gamma}}.$$

The A^+-action on each U-orbit is expanding. But, we need more argument because $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ is not compact. This can be done by the assumption that the accumulation measure μ satisfies $\mu(X) > 1-\gamma$.