Finiteness and Periodicity of Continued Fractions over Quadratic Number Fields

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Continued fractions

A continued fraction is an expression of the form

$$[a_0, a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}$$

In the classical setting, we take the a_i to be positive integers for $i > 0$.
When this is the case, it make sense to consider an infinite sequence of a_i's and the corresponding limit of the values p_n/q_n.

The classical continued fraction expansion

Starting with a real number $\alpha = \alpha_0$ we define the iteration

$$a_n = \lfloor \alpha_n \rfloor$$
$$\alpha_{n+1} = (\alpha_n - a_n)^{-1}$$

and the recurrence sequences

$$p_n = a_n p_{n-1} + p_{n-2}, \qquad p_{-1} = 1, p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2}, \qquad q_{-1} = 0, q_{-2} = 1.$$

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The a_n are called *partial quotients*
The α_n are called *complete quotients*
The p_n/q_n are called *convergents*

Classical results

$$\alpha = \frac{\alpha_{i+1} p_i + p_{i-1}}{\alpha_{i+1} q_i + q_{i-1}}$$

$$[a_0, \ldots, a_n] = p_n/q_n \quad \text{and} \quad \lim_{n\to\infty} p_n/q_n = \alpha$$

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The convergents provide very good rational approximations to α.

A question of Rosen

In 1977 David Rosen asked:

> *Is it possible to devise a continued fraction that represents uniquely all real numbers, so that the finite continued fractions represent the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction?*

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Rosen gives one example of such a construction for the field $\mathbb{Q}(\sqrt{5})$ and partial quotients which are integral multiples of $\phi = \frac{1+\sqrt{5}}{2}$. Bernat '06 gives a different construction again for $\mathbb{Q}(\sqrt{5})$.

β-expansions

Let $\beta > 1$ be an algebraic integer. Any real number x can be expanded in base-β as

$$x = \pm \sum_{i=-\infty}^{k} x_i \beta^i.$$

The digits x_i belong to the set $\{0, 1, \ldots, \lceil \beta \rceil - 1\}$, and are selected according to a greedy algorithm.

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Not all expansions are admissible. $\phi^2 = \phi + 1$

β-integers

Consider the set $\mathbb{Z}_\beta$ of the real numbers whose β-expansion uses only non-negative powers of β. These numbers are called β-integers.

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They form a discrete subset of the algebraic integers in the field $\mathbb{Q}(\beta)$.
For some special β's e.g. for Pisot numbers, it is possible to give an algebraic characterization of this set in terms of their algebraic conjugates.

β-fractionary expansion

For a positive real number x, define

$$\lfloor x \rfloor_\beta = \max\{a \in \mathbb{Z}_\beta \mid a \leq x\}.$$

Replace $\lfloor \cdot \rfloor$ by $\lfloor \cdot \rfloor_\beta$ in the definition of the continued fraction expansion.

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Not all expansions are admissible. If $1 < \xi < \phi$, then $(\xi - 1)^{-1} > \phi$.

Bernat's result

In 2006 Julien Bernat proved that every positive element of $\mathbb{Q}(\phi)$ has a finite ϕ-fractionary expansion (this had been conjectured by Akiyama in 2002).
The proof relies on an intricate case-by-case analysis, and it uses that ϕ is a quadratic Pisot number smaller than 2.

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He writes

> *. . . we do not know for which numbers β the [finiteness result] holds, or even for which numbers the weaker result that, for any $p, q \in \mathbb{Z}_\beta^+$ with $q > 0$, the continued β-fraction of p/q is either finite or ultimately periodic, holds.*

Periodicity and finiteness for the β-fractionary expansion

Let $\beta > 1$ be an algebraic integer.

(CFP)

We say that β has the (CFP) property if the β-fractionary expansion of every element of $\mathbb{Q}(\beta)$ if finite or eventually periodic.

(CFF)

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We study these properties when $[\mathbb{Q}(\beta) : \mathbb{Q}] = 2$.

A result on expansions of elements in quadratic fields

Let $K = \mathbb{Q}(\sqrt{D})$ for a positive squarefree D. For any $x \in K$ denote by x' the image of x under the non-trivial Galois automorphism of K.

Theorem (Másáková, Vávra, Veneziano)

Let $\xi = [a_0, a_1, \dots]$ be an infinite continued fraction with $a_n \in \mathcal{O}_K$ such that $a_n \geq 1$ and $|a'_n| \leq a_n$ for $n \geq 0$. Assume that $\xi \in K$. Then the sequence $(a_n)_{n \geq 0}$ is eventually periodic and either all partial quotients in the period belong to $\mathbb{Z}$, or all belong to $\sqrt{D}\mathbb{Z}$. Moreover, the following bounds hold

$$H(\xi_n) \leq \sqrt{3} H(\xi),$$
$$H(a_n) \leq 3H(\xi)^2,$$

and therefore the lengths of the period and preperiod can be effectively bounded.

Weil height

$$H : \overline{\mathbb{Q}} \to [0, +\infty)$$

For all non-zero $x, y \in \overline{\mathbb{Q}}$ we have

- $H(x + y) \leq 2H(x)H(y)$;
- $H(xy) \leq H(x)H(y)$;
- $H(x^n) = H(x)^{|n|}$ for all $n \in \mathbb{Z}$;
- $H(\sigma(x)) = H(x)$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;
- $H(x) = 1$ if and only if x is a root of unity (Kronecker's theorem);

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- for all $B, D \geq 1$ the set

$$\{\xi \in \overline{\mathbb{Q}} \mid H(\xi) \leq B \text{ and } [\mathbb{Q}(\xi) : \mathbb{Q}] \leq D\}$$

is finite (Northcott's theorem).

Main ingredients of the proof

The proof is built in three steps.

- Using the recurrence relations for p_n, q_n and the properties of the Weil height, we establish the height bounds, and thus that the sequence (a_n) is eventually periodic;
- Using an argument of algebraic number theory, we show that if the expansion is purely periodic of period lenght n, then the quantity $p_{n-1} + q_{n-2}$ belongs to either $\mathbb{Z}$ or $\sqrt{D}\mathbb{Z}$;
- Using the representation of p_n and q_n as polinomials ("continuants") in the a_i's we conclude.

The theorem holds for any representation of ξ as a continued fraction satisfying the hypothesis.

Quadratic Perron numbers have the (CFP)

From now on, let $\beta > 1$ be a quadratic integer, and let β' be its algebraic conjugate.

Corollary (Másáková, Vávra, Veneziano)

If $|\beta'| < \beta$ *(i.e.* β *is a Perron number), then* (CFP) *holds.*
Every purely periodic β*-fractionary expansion in* $\mathbb{Q}(\beta)$ *has partial quotients in* $\{1, \ldots, \lfloor \beta \rfloor\}$*.*

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Check that $\mathbb{Z} \cap \mathbb{Z}_\beta^+ = \{1, \ldots, \lfloor \beta \rfloor\}$. We give also a second proof based on an argument of diophantine approximation and a comparison lemma to estimate the relative growhts of the sequences p_n, q_n and their conjugates. This gives a better value for the bound on the length of the preperiodic part.

(CFF) for small quadratic Perron numbers

Theorem (Másáková, Vávra, Veneziano)

The four Perron numbers

$$\frac{1+\sqrt{5}}{2}, \qquad 1+\sqrt{2}, \qquad \frac{1+\sqrt{13}}{2}, \qquad \frac{1+\sqrt{17}}{2}$$

have (CFF), *and are the only quadratic Perron numbers smaller than 3 with property* (CFF).

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$$\frac{164+65\sqrt{17}}{251} = [\overline{1,1,2,1,1,2,2,2,2}]$$

$$\frac{164+65\sqrt{17}}{251} = [1,1,\beta,2\beta^3+\beta^2+1,\beta^3+\beta+1,2,\beta+1]$$

Quadratic Perron numbers smaller than 3

The set of quadratic Perron numbers is discrete.

β	Approximate value	Minimal polynomial	(CFF)
$\frac{1}{2}(1+\sqrt{5})$	1.618033988...	x^2-x-1	yes
$\frac{1}{2}(1+\sqrt{13})$	2.302775637...	x^2-x-3	yes
$1+\sqrt{2}$	2.414213562...	x^2-2x-1	yes
$\frac{1}{2}(1+\sqrt{17})$	2.561552812...	x^2-x-4	yes
$\frac{1}{2}(3+\sqrt{5})$	2.618033988...	x^2-3x+1	no
$1+\sqrt{3}$	2.732050807...	x^2-2x-2	no
$\frac{1}{2}(1+\sqrt{21})$	2.791287847...	x^2-x-5	no

$$\frac{1+\sqrt{5}}{2} = [\overline{1}],$$

$$\frac{11055+10864\sqrt{3}}{18471} = [\overline{1,1,1,1,1,1,1,2,1,1,2,1,2,1,1,1,2,2}],$$

$$\frac{117+44\sqrt{21}}{202} = [\overline{1,1,1,2,1,2,1,2,2,2,1,1,2,2}].$$

Why smaller than 3?

The integers $1, 2, \ldots, \lfloor \beta \rfloor$ belong to $\mathbb{Z}_\beta$.

Remark

If the simple continued fraction expansion of some $\xi > 0$ involves only partial quotients smaller than $\lfloor \beta \rfloor$, then it coincides with the β-fractionary expansion of ξ.

If $K = \mathbb{Q}(\sqrt{D})$ with D squarefree, standard estimates on the continued fraction expansion of $\left\lfloor \sqrt{D} \right\rfloor + \sqrt{D}$ imply that no $\beta > 2 \left\lfloor \sqrt{D} \right\rfloor + 1$ in K can satisfy (CFF)

Can we find continued fractions with small partial quotients in every quadratic field?

Problem (McMullen '08)

Does every real quadratic field contain infinitely many periodic continued fractions with partial quotients equal to 1 or 2?

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Conjecture (Mercat '13)

Every real quadratic number field contains a periodic continued fraction with partial quotients equal to 1 or 2.

Under Mercat's conjecture, no quadratic $\beta > 3$ can have property (CFF).

An unconditional statement

Assuming Mercat's Conjecture, $\frac{1+\sqrt{5}}{2}, 1+\sqrt{2}, \frac{1+\sqrt{13}}{2}, \frac{1+\sqrt{17}}{2}$ are the only quadratic Perron numbers with property (CFF).

Theorem (Másáková,Vávra,Veneziano)

The only quadratic Pisot units with property (CFF) *are* $\dfrac{1+\sqrt{5}}{2}$ *and* $1+\sqrt{2}$.

This unconditional result is obtained by explicitely constructing for every quadratic Pisot unit β an element in $\mathbb{Q}(\beta)$ whose continued fraction expansion only uses small integers. For example $\beta = \frac{5+\sqrt{29}}{2} \approx 5.19$ doesn't have the (CFF) because

$$\frac{7+5\sqrt{29}}{26} = [\overline{1,3,3,1}].$$

Square roots have (CFP)

Corollary (Másáková, Vávra, Veneziano)

Let D be a positive integer with irrational square root β. Then β has (CFP) *and, under Mercat's conjecture, does not have* (CFF).

$$[\overline{4\sqrt{2}}] = 3 + 2\sqrt{2}, \qquad [\overline{8\sqrt{6}, 2\sqrt{6}}] = 2(5 + 2\sqrt{6}),$$

$$[\overline{3, 4}] = \frac{3 + 2\sqrt{3}}{2}, \qquad [\overline{\sqrt{7}, 2\sqrt{7}}] = \frac{3 + \sqrt{7}}{2},$$

$$[\overline{32\sqrt{5}, 2\sqrt{5}}] = 16(9 + \sqrt{5}), \qquad [\overline{2\sqrt{8}}] = 3 + \sqrt{8}$$

Non-Perron quadratic β's with positive conjugate

Theorem (Másáková, Vávra, Veneziano)

Let $\beta > 1$ be a quadratic integer such that $\beta' > \beta$.

- *Every purely periodic β-fractionary expansion in $\mathbb{Q}(\beta)$ has partial quotients in $\{1, \ldots, \lfloor\beta\rfloor\}$;*
- *every $\xi \in \mathbb{Q}(\beta)$ such that $\xi > \beta$ and $-1 < \xi' < 0$ has an infinite, aperiodic β-fractionary expansion;*
- *β does not have either (CFP) or (CFF).*

We use an argument of algebraic number theory to establish the first point of the statement.
It is easy to see that elements with $-1 < \xi' < 0$ can't have a finite β-fractionary expansion because every complete quotient lies in $(-1, 0)$. A characterization of pure periodicity then shows that the expansion of ξ is either aperiodic or purely periodic.

Non-Perron quadratic β's with negative conjugate

The case in which $\beta' < -\beta$ presents additional difficulties because in this case there are nontrivial elements in $\mathbb{Z} \cap \mathbb{Z}_\beta^+$ other than $\{1, \ldots, \lfloor \beta \rfloor\}$. For instance if β is the positive root of $X^2 + 2X - 9 = 0$, then $\beta^2 + 2\beta = 9 \in \mathbb{Z} \cap \mathbb{Z}_\beta^+$.

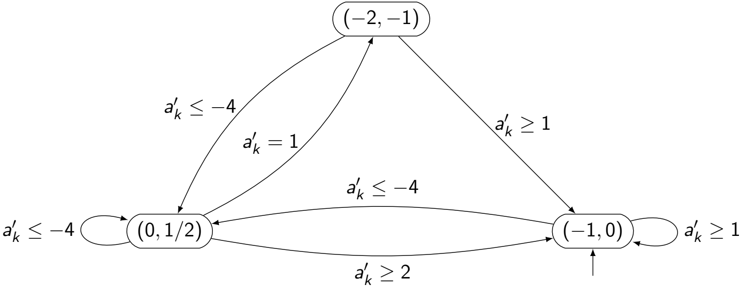
Theorem (Másáková, Vávra, Veneziano)

Let $\beta > 1$ be quadratic integer with conjugate β' satisfying $\beta' \leq -\beta - 4$. Let $\xi \in \mathbb{Q}(\beta)$ be such that $\xi' \in (-1, 0)$. Then ξ does not have a finite β-expansion.
In particular, β does not satisfy property (CFF).

The idea is similar, but more intricate, than what was done for $\beta' > \beta$.

Lemma (Másáková,Vávra,Veneziano)

Assume that $\beta' \leq -\beta - 4$. *Then for every* $x \in \mathbb{Z}_\beta^+$ *we have that either* $x = 1$, *or* $x' \geq 2$, *or* $x' < -4$.



(CFP) and (CFF) for quadratic integer $\beta > 1$

	(CFP)	(CFF)
$\beta' > \beta$	NO	NO
$\lvert\beta'\rvert < \beta$	YES	$\frac{1+\sqrt{5}}{2}, 1+\sqrt{2}, \frac{1+\sqrt{13}}{2}, \frac{1+\sqrt{17}}{2}$ and no other
$\beta' = -\beta$	YES	NO
$-\beta - 3 \le \beta' < -\beta$	Open	Probably NO, but still open for 20 values of β
$\beta' \le -\beta - 4$	Open	NO

Entries in blue assume the validity of Mercat's Conjecture.