# Computing Hausdorff dimension of Gauss-Cantor sets and its applications to the study of classical Markov spectrum

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June 2021

*A computation is a temptation that should be resisted as long as possible.*

*J.P. Boyd*

# Markov and Lagrange spectra

The continued fraction of $x \in (0,1)$ is an expression

$$x = [0; \alpha_1, \ldots, \alpha_n, \ldots] := \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \cfrac{1}{\alpha_3 + \ddots}}}, \quad \alpha_n \in \mathbb{N}$$

Consider a set of bi-infinite sequences $(\mathbb{N}^*)^{\mathbb{Z}}$ and Bernoulli shift

$$\sigma\colon (\mathbb{N}^*)^{\mathbb{Z}} \to (\mathbb{N}^*)^{\mathbb{Z}}; \qquad \sigma\left((\alpha_n)_{n\in\mathbb{Z}}\right) = (\alpha_{n+1})_{n\in\mathbb{Z}}.$$

Introduce a map

$$\lambda\colon (\mathbb{N}^*)^{\mathbb{Z}} \to \mathbb{R} \qquad \lambda(\underline{\alpha}) = [\alpha_0; \alpha_1, \alpha_2, \ldots] + [0; \alpha_{-1}, \alpha_{-2}, \ldots].$$

## Definition (Perron, 1921)

The Lagrange value of $\alpha \in (\mathbb{N}^*)^{\mathbb{Z}}$ is $\ell(\alpha) := \limsup_{n\to\infty} \lambda(\sigma^n \alpha)$

The Markov value of $\alpha \in (\mathbb{N}^*)^{\mathbb{Z}}$ is $m(\alpha) := \sup_{n\in\mathbb{Z}} \lambda(\sigma^n \alpha)$.

The collection of Lagrange (Markov) values is called the Lagrange (Markov) spectrum.

$$\mathcal{L} := \left\{\ell(\alpha) \mid \alpha \in (\mathbb{N}^*)^{\mathbb{Z}}\right\} \quad \text{and} \quad \mathcal{M} := \left\{m(\alpha) \mid \alpha \in (\mathbb{N}^*)^{\mathbb{Z}}\right\}.$$

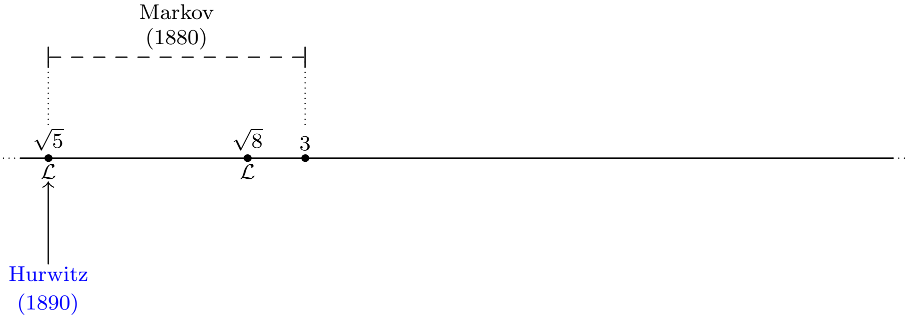
# Study of the spectra — I


Markov, 1880

$$\mathcal{L} \cap \left(\sqrt{5}, 3\right) = \mathcal{M} \cap \left(\sqrt{5}, 3\right) = \left\{\sqrt{5} < \sqrt{8} < \sqrt{221}/5 < \dots\right\}$$

is a countable set. (Proof uses Markov triples).

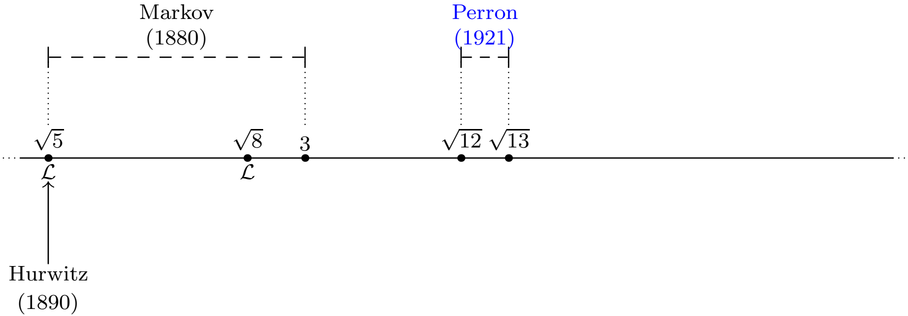
# Study of the spectra — II




## Hurwitz, 1890

$\min \mathcal{L} = \sqrt{5}$. (Proof uses a more classical definition via best approximants).

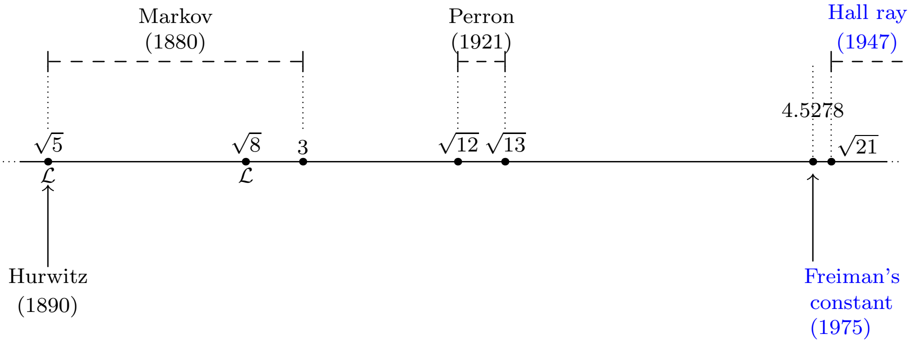
# Study of the spectra — III




## Perron, 1921

$\left(\sqrt{12}, \sqrt{13}\right) \cap \mathcal{M} = \varnothing$, while $\sqrt{12}, \sqrt{13} \in \mathcal{L}$. Moreover, $\mathcal{L} \subset \mathcal{M}$ and both sets are closed subsets of $\mathbb{R}$.

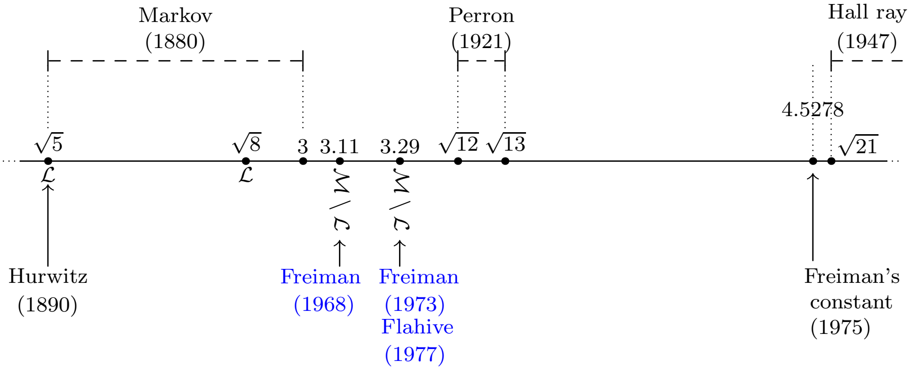
# Study of the spectra — IV




Hall, 1947: There exists $c \in \mathbb{R}$ such that $[c, +\infty) \subset \mathcal{L} \subset \mathcal{M}$.
Schecker & Freiman, 1963: One can take $c = \sqrt{21}$ above.
Freiman, 1975: The smallest possible $c$ is

$$c_F = \frac{2221564096 + 283748\sqrt{462}}{491993569}.$$

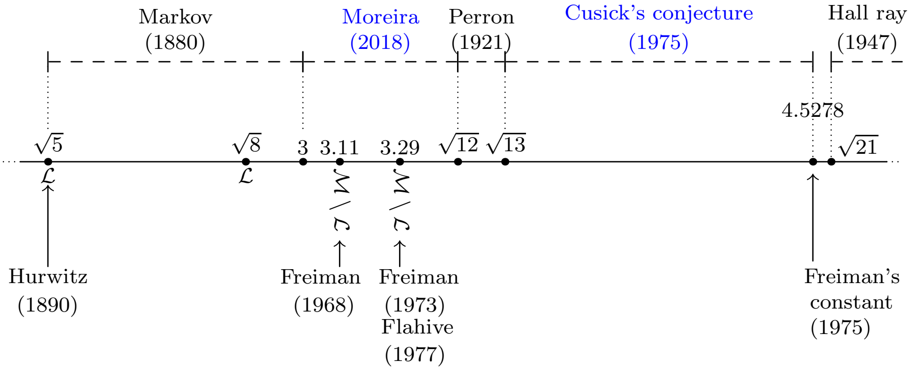
# Study of the spectra — V




## Freiman 1968, 1973; Flahive 1977

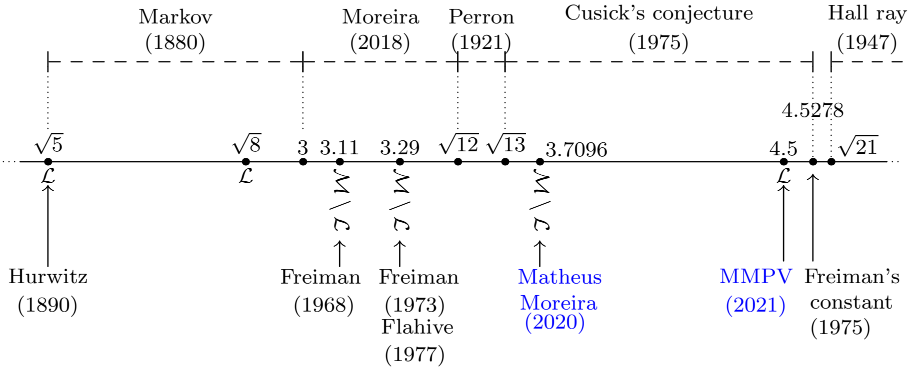
Near 3.11 and 3.29 the set $\mathcal{M} \setminus \mathcal{L}$ contains two countable subsets.

# Study of the spectra — VI




Cusick's conjecture, 1975: $(\mathcal{M} \smallsetminus \mathcal{L}) \cap [\sqrt{12}, +\infty) = \varnothing$.
Bernstein's conjecture, 1973: $[4.1, 4.52] \subset \mathcal{L}$.
Moreira, 2018: $\dim(\mathcal{L} \cap (-\infty, t)) = \dim(\mathcal{M} \cap (-\infty, t))$ and the function $f\colon t \mapsto \dim(\mathcal{M} \cap (-\infty, t))$ is continuous. Moreover, $f(\sqrt{12}) = 1$ and $f(3+\varepsilon) > 0$ for any $\varepsilon > 0$.

# Study of the spectra — VII

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# Intermission

Today we are concerned with the function $f$ from the last statement:

Moreira, 2018: $\dim(\mathcal{L} \cap (-\infty, t)) = \dim(\mathcal{M} \cap (-\infty, t))$ and the function $f : t \mapsto \dim(\mathcal{M} \cap (-\infty, t))$ is continuous. Moreover, $f(\sqrt{12}) = 1$ and $f(3 + \varepsilon) > 0$ for any $\varepsilon > 0$.

Clearly, $f$ is a monotone increasing function.

Define the first transition:

$$t_1 := \inf \{t \in \mathbb{R} \mid \dim(\mathcal{M} \cap (-\infty, t)) = 1\}$$

Can we determine the value of $t_1$? (By Moreira's result $3 < t_1 < \sqrt{12}$)

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THE END OF PART 1: INTRODUCTION

NEXT IS PART 2: A METHOD FOR COMPUTING $t_1$

# First transition

## Goal for today's talk

To give a rigourous and accurate estimate on the first transition point

$$t_1 := \inf \{t \in \mathbb{R} \mid \dim(\mathcal{M} \cap (-\infty, t)) = 1\}$$

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In 1982 Bumby gave heuristic bounds: $3.33437 < t_1 < 3.33440$.
We shall follow the same technique.

## Preliminaries

- Consider a set of continued fractions of 1's and 2's:

  $$E_2 := [a = [0; \alpha_1, \alpha_2, \dots] \mid \alpha_j \in \{1, 2\},\ j \geq 1]$$

  Then $\min E_2 = \frac{1}{2}(\sqrt{3} - 1)$, $\max E_2 = \sqrt{3} - 1$ and $\dim E_2 > 0.53128$.
- Let $\alpha \in \{1, 2\}^{\mathbb{Z}}$. Then $m(\alpha) \geq \sqrt{5}$ and $m(\alpha) = \sqrt{5}$ if and only if $\alpha = 1, 1, 1, \dots$.

# Approach to lower bound

Fix $T > 0$ and construct a finite set $F$ of finite strings of 1's and 2's with a property that if $\alpha \in \{1,2\}^{\mathbb{Z}}$ doesn't contain a string from $F$, then $m(\alpha) < T$.

Claim. Let $K \subset E_2$ be such that for any $x \in K$ its continued fraction expansion doesn't contain a string from $F$. Then

$$M \cap (\sqrt{5}, T) \subset 2 + K + K.$$

Proof: $m = \lambda(\alpha) = [2; \alpha_1, \alpha_2, \dots] + [0; \alpha_{-1}, \alpha_{-2} \dots]$.
Since $\dim_H K + K = \dim_H K + \overline{\dim}_B K$, $\dim_H K \leq 0.5$ gives $t_1 \geq T$.

## Example (Hall, 1971)

If $\alpha \in \{1,2\}^{\mathbb{Z}}$ doesn't contain a substring 121, then $m(\alpha) \leq \sqrt{10}$.

$$\dim \left( \{ [0; \alpha_1, \alpha_2, \dots] \mid \alpha_j \in \{1,2\}, \, (\alpha_j \alpha_{j+1} \alpha_{j+2}) \neq (121), \, j \geq 1 \} \right) \leq 0.45,$$

therefore $t_1 \geq \sqrt{10} = 3.16 \dots$.

# Approach to upper bound

Let $S$ be the maximal Markov value of strings which do not contain a substring from a finite set of finite strings $F$:

$$S = \max m(\alpha), \text{ where } \alpha \in \{1,2\}^{\mathbb{Z}} \text{ doesn't contain a string from } F$$

and let $K \subseteq E_2$ be such that for any $x \in K$ its continued fraction expansion doesn't contain a string from $F$. It was shown by Moreira that

$$\min\{2 \cdot \dim_H K, 1\} \le \dim_H((\sqrt{5}, S) \cap M),$$

Therefore $\dim_H K \ge 0.5$ implies $t_1 \le S$.

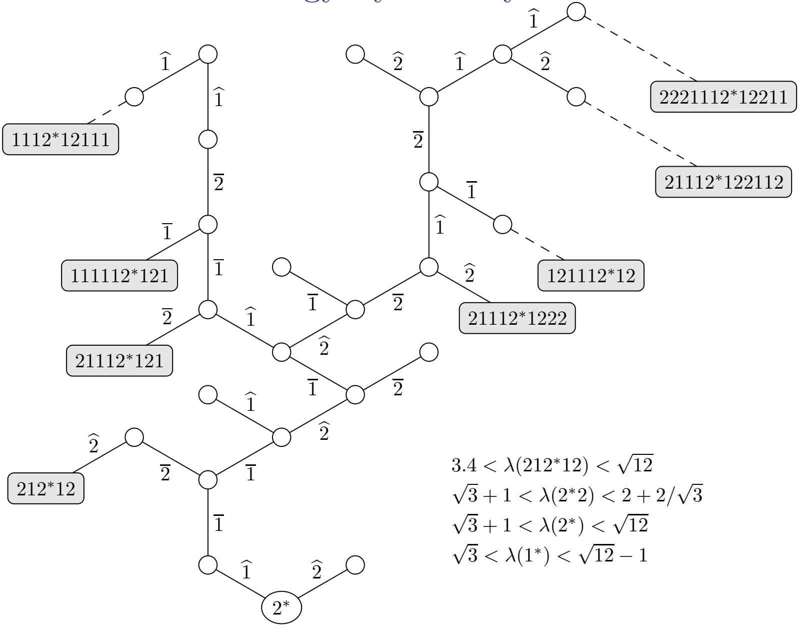
## Example (Perron )

Note that $m(\alpha) \le \sqrt{12}$ if and only if $\alpha \in \{1,2\}^{\mathbb{Z}}$. Therefore we may choose $F = \varnothing$ and $K = E_2$. Then

$$\dim_H((\sqrt{5}, \sqrt{12}) \cap M) \ge \min\{2 \cdot \dim_H E_2, 1\} = \min(2 \cdot 0.54318, 1) = 1,$$

and conclude that $t_1 \le \sqrt{12}$.

# Strategy by Bumby $T = 3.334369$




$3.4 < \lambda(212^*12) < \sqrt{12}$

$\sqrt{3} + 1 < \lambda(2^*2) < 2 + 2/\sqrt{3}$

$\sqrt{3} + 1 < \lambda(2^*) < \sqrt{12}$

$\sqrt{3} < \lambda(1^*) < \sqrt{12} - 1$

# Lower bound: $t_1 \geq 3.334384009$

Following this strategy with $T = 3.334384009$ we get the set $F$ of 24 words of length up to 24:

- The 14 words proposed by Bumby: 21212, 21112121, 211121222, 111112121, 12111212, 111212111, 21112122112, 222111212211, 12211112121122, 112211121221111, 22111121211221, 21122111212211, 2111212211112, 2222111121212112222;
- The 10 additional words: 1222211112121122212112, 12122221111212112221 1, 2122221111212112221111;

## Question

How to estimate the dimension of the set $X$ which we obtain from $E_2$ after removing all numbers whose continued fraction expansion contains these strings? (It turns out $0.5 - 10^{-6} < \dim X < 0.5 - 10^{-8}$.)

# Intermission

The usability of the method of computing the first transition point $t_1$ depends on our ability to estimate the Hausdorff dimension of the Gauss–Cantor set of continued fractions.

$$
\begin{aligned}
X_{\bar r} := \Big\{ [0;a_1, a_2, \ldots] \mid\ & a_n \in \{1,2\},\ \text{with extra restrictions} \\
& a_j a_{j+1} \ldots a_{j+r_1} \neq d_{i_1} d_{i_2} \ldots d_{i_{r_1}},\ i_1 i_2 \ldots i_{r_1} \in \{1,2\}^{r_1} \\
& a_j a_{j+1} \ldots a_{j+r_2} \neq d_{i_1} d_{i_2} \ldots d_{i_{r_2}},\ i_1 i_2 \ldots i_{r_2} \in \{1,2\}^{r_2} \\
& * \qquad * \qquad * \\
& a_j a_{j+1} \ldots a_{j+r_k} \neq d_{i_1} d_{i_2} \ldots d_{i_{r_k}},\ i_1 i_2 \ldots i_{r_k} \in \{1,2\}^{r_k} \Big\} \subsetneq E_2
\end{aligned}
$$

with $k \leq 24$ and $r_j \leq 24$ for all $1 \leq j \leq k$.

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THE END OF PART 2: A METHOD FOR COMPUTING $t_1$

NEXT IS PART 3: COMPUTATION OF DIMENSION

# Step 1: Introduce a dynamical system

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Idea

To compute the Hausdorff dimension of a bounded set $X \subset B \subset \mathbb{R}$ we want to realise it as a limit set of an iterated function scheme.

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More precisely, we want to find a finite family of uniformly contracting maps $\mathcal{T} = \{T_1, \ldots, T_k\}$ such that $T_j(B) \subset B$ for all $1 \leq j \leq k$ and $X$ is the limit set for $\mathcal{T}$:

$x \in X \Longleftrightarrow$

there exists $y \in B$ and a sequence $\{j_n\} \in \{1, \ldots, k\}^{\mathbb{N}}$ such that

$$x = \lim_{n \to \infty} T_{j_1} \circ \ldots \circ T_{j_{n-1}} \circ T_{j_n}(y)$$

In fact, since all $T_j$ are uniformly contracting, i.e. $|T_j'| < 1 - \varepsilon$ for some $\varepsilon > 0$, the limit depends only on the sequence $j_n$, and not on the reference point $y$.

# Toy example

$$X = \Big\{ [0; a_1, a_2, \ldots, ] \mid a_j \in \{1,2\},\ a_j a_{j+1} a_{j+2} \neq 121, 212 \Big\}$$

We define a Markov iterated function scheme of 4 maps parametrised by strings $\bar{j} \in \mathcal{A} = \{1,2\}^2$ and a transition matrix $M$

$$T_{j_1 j_2}(x) = \cfrac{1}{j_1 + \cfrac{1}{j_2 + x}} \qquad M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Columns and rows encoded by $\mathcal{A} = \{11, 12, 21, 22\}$

$$M_{i_1 i_2, j_1 j_2} = 1 \iff j_1 j_2 i_1 i_2 \text{ doesn't contain 121 or 212.}$$

The limit set of $\{T_j\}_{\bar{j} \in \mathcal{A}}$ with respect to $M$ is

$$\Big\{ \lim_{n \to +\infty} T_{\bar{j}_1} \circ \cdots \circ T_{\bar{j}_n}(0) \mid \bar{j}_k \in \mathcal{A}, M_{\bar{j}_k, \bar{j}_{k+1}} = 1, 1 \leq k \leq n-1 \Big\} = X$$

# Step 2: Introduce the operators

## Idea

The estimates on the Hausdorff dimension of the limit set of an iterated function scheme of uniform contractions come from the study of associated bounded linear operators.

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Let $S := [0,1] \times \{1, \dots, |\mathcal{A}|\}$. Define the maps $T_j : S \to S$, $T_j(x,k) = (T_j(x), j)$, consider the Banach space of functions $C^2(S)$ and the family of linear operators $\mathcal{L}_t : C^2(S) \to C^2(S)$:

$$[\mathcal{L}_t \bar{w}]_k(x,k) = \sum_{j=1}^{|\mathcal{A}|} M(j,k) \cdot |T_j(x,k)'|^t \cdot \bar{w}_j(T_j(x,k)) \qquad (t > 0)$$

The operator is called the transfer operator for the iterated function scheme. Let $\rho(t)$ be the spectral radius of $\mathcal{L}_t$.

## Lemma (after Bowen and Ruelle, from 1980s)

*The map $t \mapsto \rho(\mathcal{L}_t)$ is strictly monotone decreasing and the solution to $\rho(\mathcal{L}_t) = 1$ is $t = \dim_H(X)$.*

# Step 3: Estimates on $\rho(\mathcal{L}_t)$

The spectral radius is an isolated eiegenvalue and we can use

## Lemma (M. Pollicott & P.V., 2020)

*Let* $|\mathcal{A}| =: d$ *and choose* $t_0 < t_1$

1. *If there exist* $d$ *positive polynomials* $f_j : [0,1] \to \mathbb{R}^+$ *such that*

$$\inf_x \frac{[\mathcal{L}_{t_0} \bar{f}]_j(x)}{f_j(x)} > 1 \text{ for all } j = 1, \dots, d \implies \text{ then } \rho(\mathcal{L}_{t_0}) > 1.$$

2. *If there exist* $d$ *positive polynomials* $g_j : [0,1] \to \mathbb{R}^+$ *such that*

$$\sup_x \frac{[\mathcal{L}_{t_1} \bar{g}]_j(x)}{g_j(x)} < 1 \text{ for all } j = 1, \dots, d \implies \text{ then } \rho(\mathcal{L}_{t_1}) < 1.$$

This lemma gives us a way to estimate the dimension.

## Corollary

*If we can find* $f_j, g_j$ *as above then* $t_0 < \dim_H(X) < t_1$.

## Step 4: Collocation method for test functions

- Fix a small atural number $m$ (e.g., $m = 8$ works).
- We can introduce
  1. $p_k(x) \in C([0,1])$ — the Lagrange polynomials ($1 \le k \le m$ ), and
  2. $x_k \in [0,1]$ — the Chebyshev nodes ($1 \le k \le m$ )

  so that $p_i(x_j) = \delta_{ij}$, for all $1 \le i, j \le m$
- Introducing $d = |\mathcal{A}|$ small $m \times m$ matrices

$$B^{j,t}(i,l) := |T_j'(x_i)|^t \cdot p_l(T_j(x_i))$$

we get a $dm \times dm$ matrix $A^t$ given by

$$A^t = \begin{pmatrix} M_{1,1} \cdot B^{1,t} & M_{2,1} \cdot B^{2,t} & \dots & M_{d,1} \cdot B^{d,t} \\ M_{1,2} \cdot B^{1,t} & M_{2,2} \cdot B^{2,t} & \dots & M_{d,2} \cdot B^{d,t} \\ \vdots & \vdots & \ddots & \vdots \\ M_{1,d} \cdot B^{1,t} & M_{2,d} \cdot B^{2,t} & \dots & M_{d,d} \cdot B^{d,t} \end{pmatrix}.$$

- Let $w_t = (w_t^1, \cdots, w_t^{dm})$ be the (left) eigenvector for the largest eigenvalue.
- Finally, set $f_j(x) = \sum_{k=1}^{m} w_t^{(j-1)m+k} p_k(x)$.

## Step 5: Verification

To apply the “min-max” principle, we need to confirm that

1. $f_j > 0$; and
2. $\sup_x \frac{[\mathcal{L}_t f]_j(x)}{f_j(x)} < 1$ (or $\inf_x \frac{[\mathcal{L}_t g]_j(x)}{g_j(x)} > 1$ )

Fortunately, $f_j$ is a polynomial, so its derivative can be computed with arbitrary precision, this allows us to verify the first inequality. To verify the second inequality, we differentiate

$$\left(\frac{[\mathcal{L}_t f]_j}{f_j}\right)' = \frac{([\mathcal{L}_t f]_j)' \cdot f_j - (f_j)' \cdot [\mathcal{L}_t f]_j}{(f_j)^2}$$

In the case of $X$, the numerator is sum a rational functions with coefficients $\left(\frac{x+n_1}{x+n_2}\right)^t$, $n_1, n_2 \in \mathbb{N}$. It turns out that

$$([\mathcal{L}_t f_j])' \cdot f_j - (f_j)' \cdot [\mathcal{L}_t f_j] \to 0 \text{ as } m \to \infty$$

exponentially fast (so that $m = 8$ is sufficient).

# Numerical challenges

1. The construction of matrix $M$ which gives Markov condition requires analysing of $2^{2n}$ words of length $2n$ looking for forbidden substrings
2. For $n = 17$ the matrix $M$ would take $2GB$ (and we need $n = 24$)
3. The matrix $A^t$ is even larger: for $n = 17$ and $m = 6$ it would take $1512GB$ to store (and we need its eigenvector!)
4. The best method for the computation of the eigenvector is the power method, it has complexity a bit more than $O(n^{2.5})$

## Lemma (Matheus, Moreira, Pollicott, & V. 2021)

*Assume that the columns $j_1$ and $j_2$ of the Markov matrix $M$ are identical, i.e. for all $1 \le k \le md$ we have that $M_{k,j_1} \equiv M_{k,j_2}$. Then any eigenvector $\bar{f}$ of $A^t$ lies in the subspace of $S$ for which $f_{j_1} = f_{j_2}$.*

This is a huge help: In the case of the set $X$ the Markov matrix has 3940388 columns of which only 429 are pairwise distinct. A reduction procedure allows to replace the matrix $A^t$ of size $\approx 31 \cdot 10^8$ with a matrix of size $429 \cdot 8 = 3432$ only!

# Summary

1. We show that the first transition point

$$t_1 := \inf\{t \in \mathbb{R} \mid \dim(\mathcal{M} \cap (-\infty, t)) = 1\} = 3.334384\ldots$$

2. We improve the dimension bounds

$$0.537152 < \dim(\mathcal{M} \setminus \mathcal{L}) < 0.796445.$$

3. We identify several non-affine Cantor sets in $\mathcal{M} \setminus \mathcal{L}$ and demonstrate that $\mathcal{M} \setminus \mathcal{L}$ has a rich structure.
4. We give a method for computing Hausdorff dimension of fairly complicated Gauss–Cantor sets.

Thank you for your time