Extending the Theory of Substitutions to Compact Alphabets.

1. Finite subs.

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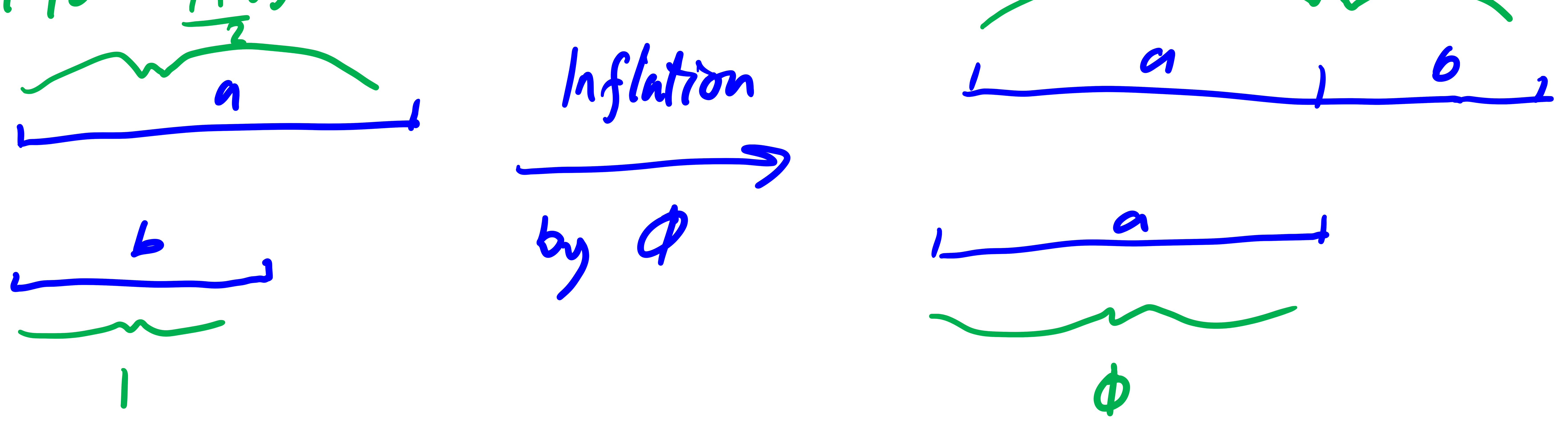
Exp: Fibonacci $\varphi = \begin{cases} a \to ab \\ b \to a \end{cases}$

$$a \xrightarrow{\varphi} ab \xrightarrow{\varphi} aba \xrightarrow{\varphi} abaab \longrightarrow \dots$$

φ <u>primitive</u> $:= \exists\, p \in \mathbb{N}$ s.th $\forall a \in \mathcal{A}$, $\varphi^p(a)$ "$\supseteq$" $\mathcal{A}$.

Primitive $\Longrightarrow$ 1 unique (up to rescaling) "natural tile length function" "ntlf"

Fib: $\phi = \frac{1+\sqrt{5}}{2}$ $\qquad \phi + 1 = \phi^2$



(if non-primitive, still $\exists$ $n^+(f)$).

② Associated subshift is uniquely ergodic.

2. Infinite subs.

How do we extend the above to infinite alphabets?

Can just continue purely symbolically [see Ferenczi].

Instead, we want to retain some structure by weakening <u>finite</u> to <u>compact</u> [see Durand-Ormes-Petite, Queffélec]

$\mathcal{A}$ = cpt, HD space ("alphabet")

$\mathcal{A}^+ :=$ space of non-empty finite words

$$:= \bigsqcup_{n=1}^{\infty} \mathcal{A}^n$$

prod. top.

dis. un. top.

<u>Substitution</u> := continuous$^{(!)}$ map

(note: $\mathcal{A}$ connected $\Rightarrow$ φ const. length)

$\varphi : \mathcal{A} \longrightarrow \mathcal{A}^+ \rightsquigarrow$

$\varphi : \mathcal{A}^+ \longrightarrow \mathcal{A}^+$.

<u>Exp</u>: Any finite sub ($\mathcal{A}$ with discrete top.)

<u>Exp</u>: $\mathcal{A} = S^1 = \{x \in \mathbb{C} \mid |x| = 1\}$

$\varphi(x) = 0 \quad \alpha x$ (Fix $\alpha \in S^1$)

or $\varphi(x) = x \quad \alpha x$.

<u>Exp</u>: (X, f) your favorite D.S.,

$p \in X$ any point,

$\varphi(x) = p \quad f(x)$.

Exp: Let $\varphi : S \to S^+$ be *any* function (S a set), with $\max |\varphi(a)| < \infty$

Extends to a cts map

$\varphi : A \to A^+$, where $A =$ Stone–Čech cpt^n,

$S \subset A$ dense subset of isolated pts.

Exp: $A = \mathbb{N}_0 \cup \{\infty\}$ (1-pt cpt^n)

$0 \quad 1 \quad 2\; 3\; 4 \cdots\cdots \infty$

$$\varphi = \begin{cases} 0 \longrightarrow 0\,1 & \\ n \longrightarrow 0\,(n-1)\,(n+1) & (n>0) \\ \infty \longrightarrow 0\,\infty\,\infty & \end{cases}$$

φ primitive $:=$ $\forall$ open $U \subset A$, $U \neq \emptyset$, $\exists\, p \geq 0$ s.th.

$\forall a \in A$, at least one letter of $\varphi^p(a)$ is in U. (see [Q, D-O-P, Priebe-Frank – Sadun])

<u>Subshift</u> := non-empty, closed, shift invar. subspace of $\mathcal{A}^{\mathbb{Z}}$.

$\updownarrow$ (even when $\mathcal{A}$ has a top.!)

"Languages".

$u \in \mathcal{A}^n$ <u>generated by φ</u> :=

$u \trianglelefteq \varphi^k(a)$ for some $a \in \mathcal{A}$, $k \geq 0$.

"is a subword of"

<u>Language of φ</u> :=

$\mathcal{L}(\varphi) := \{$ words generated by $\varphi \}$

"legal words"

<u>Subshift associated to φ</u> :=

$$X_\varphi := \left\{ w \in \mathcal{A}^{\mathbb{Z}} \;\middle|\; \begin{array}{l} u \trianglelefteq w \implies \\ u \in \mathcal{L}(\varphi) \end{array} \right\}.$$

Prop: X_φ is a subshift $\Longleftrightarrow$

$\mathcal{L}(\varphi)$ contains arbitrarily long words.

$$\cancel{\Updownarrow}$$

$\exists a \in \mathcal{A}$ s.th $\lim\limits_{n\to\infty} |\varphi^n(a)| = \infty.$

Exp $\circledast$: $\mathcal{A} = (\mathbb{N} \cup \{\infty\}) \times (\mathbb{N} \cup \{\infty\})$.

$$\varphi(n,m) = \begin{cases} (n, m-1) & m > 0 \\ (n-1, n)(0,n) & m = 0,\ n > 0 \\ (0,0) & m = n = 0 \end{cases}$$

$(1,2) \to (1,1) \to (1,0) \to$

$(0,1)(0,1) \to (0,0)(0,0)$ etc.

$\left(\lim\limits_{i\to\infty} \varphi^i(n,m) = (0,0)^{n+1}\right)$

$X_\varphi = \{ \cdots (0,0)\,(0,0) \cdot (0,0)\,(0,0) \cdots \}.$

see also [PF-S]

φ primitive ($\Leftarrow$ if φ recog.)

$\Downarrow$

$$\boxed{\begin{array}{l} X_\varphi \text{ minimal} \iff \\ \text{each orbit dense} \iff \\ \text{each } w \in X_\varphi \text{ is} \\ \text{"repetitive"} \end{array}} \implies \exists\, w \in X_\varphi \text{ with dense orbit}$$

$\Updownarrow$

$\mathcal{A}$ seperable & X_φ top. transitive.

(so S-Č cpt^n exps never repetitive!) .

Q: Does φ always have a non-zero natural tile length fn? ✗

No (C-exp above).

Q: Primitive $\Rightarrow$ nat. tile lengths? ?

Q: "——" + ctbl $\Rightarrow$ "——" ?

Q: Primitive $\Rightarrow$ X_φ uniq. ergodic
(& recog.)

✗ C.E. in [D-O-P]

Q: Prim. + recog ? + $\mathcal{A}$ ctbl $\Rightarrow$ X_φ uniq. erg.

3. The sub. operator

Let $E = C(\mathcal{A})$ (Banach space)
$= \{\text{cts } f : \mathcal{A} \to \mathbb{R}\}$.

$M : E \to E$,

$$(Mf)(a) = \sum_{b \triangleleft \varphi(a)} f(b).$$

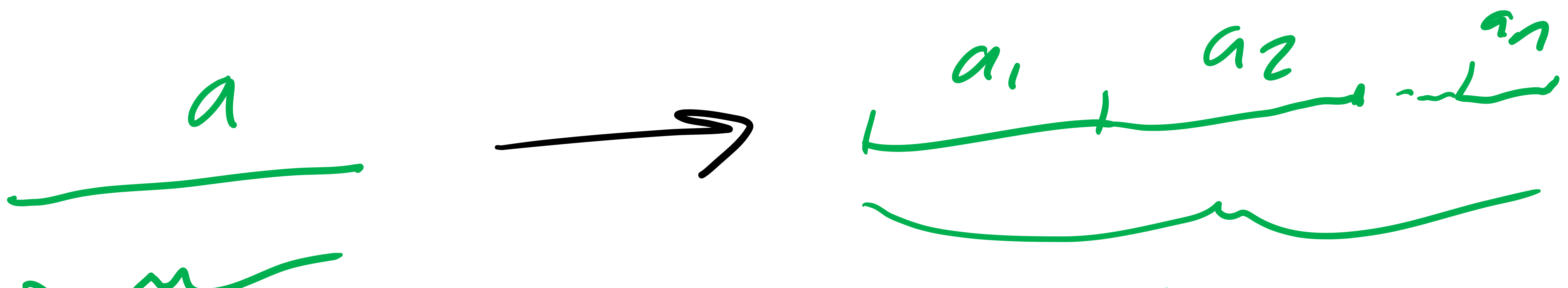
$K = \{ f \in E \mid f \geq 0 \} \subset E$, *positive cone*. (proper, closed, generating, normal, decomp. p.)

ℓ is a *natural tile length function* (ntlf) :=

$\ell \neq 0$, $\ell \in K$, $M\ell = \lambda \ell$

(ℓ is e.vector of M in K).

$\gamma(a) = a_1 \cdots a_n$



$\ell(a)$

$$\lambda \cdot \ell(a) = \ell(a_1) + \ell(a_2) + \cdots + \ell(a_n)$$

$M \rightsquigarrow$ transpose of pop. matrix in finite case.

- $\|M^n\| = \max\limits_{a \in A} |\varphi^n(a)|$
- $r = $ sp. rad of $M = \lim\limits_{n \to \infty} \sqrt[n]{\|M^n\|}$.
- φ primitive $\iff \forall f \in K, \exists$ $p \geq 0$ s.th.

$$M^n(f) > 0$$

$\left(\iff M^n(f) \text{ "quasi-interior" to } K \right)$

4. A sufficient condition for unique ergodicity.

$T = \frac{M}{r}$ is <u>quasi-compact</u> $:=$ $\|T^n - K\| < 1$ for some cpt operator K, $n \in \mathbb{N}$.

$\left(= T(U) \subseteq V \text{ for some open nhd } U \text{ of } 0, \text{ and } V \text{ cpt} \right)$

Reg. Borel Measures $\longleftrightarrow$ Elements of dual cone $K' \subset E'$

E' = cts linear functionals $\phi : E \to \mathbb{R}$

$$\int \cdot \, d\mu \longleftrightarrow \phi(\cdot)$$

Call $\phi \in K'$ an <u>eigenmeasure</u> if $T'\phi = \phi$

<u>Thm</u>: For φ recognisable, with ntlf $\ell > 0$ (and $\lambda > 1$):

(normalised) eigenmeasures $\longleftrightarrow$ invariant reg. prb. measures on X_φ.

Formalism of [PF-S] very useful for this!

<u>Theorem</u> [MRW] Suppose:

(C) $\exists$ finite $P \subseteq \mathcal{A}$ and $k \in \mathbb{N}$ s. th.

$\#\{b \triangleleft \varphi^k(a) \mid b \notin P\} < r^k$

$\forall a \in \mathcal{A}$. Then:

1. M is quasi-cpt
2. φ has a ntlf ℓ
3. r is a pole of the resolvent of M [Karlin - Krein-Rutman - Abdulazid - Schafer] ...

4. If φ is primitive then:

   - $\{r\} = \sigma_{per}(T) = \sigma_{per}^{e}(T)$, useful for discrepancy results

   & r is a simple pole of res.

   - $M^n \xrightarrow{\text{uniformly}}$ projection to eigenline of ℓ.

   - If φ recognisable then (X_φ, σ) is uniquely ergodic.

Exp: (C) holds if φ is constant length r, and $\exists$ finite $P \subseteq \mathcal{A}$, $k \in \mathbb{N}$ s.th. $\varphi^k(a) \cap P \neq \emptyset \quad \forall a \in \mathcal{A}$.

e.g.: $\mathcal{A} = S^1$, $\varphi(x) = 0\,\alpha x$

So (X_φ, σ) uniq. erg. $\left(\alpha = e^{i\beta},\ \beta \in \mathbb{R} \setminus \mathbb{Q}\right)$

Non-exp: $\mathcal{A} = S^1$, $\varphi(x) = x\,\alpha x$

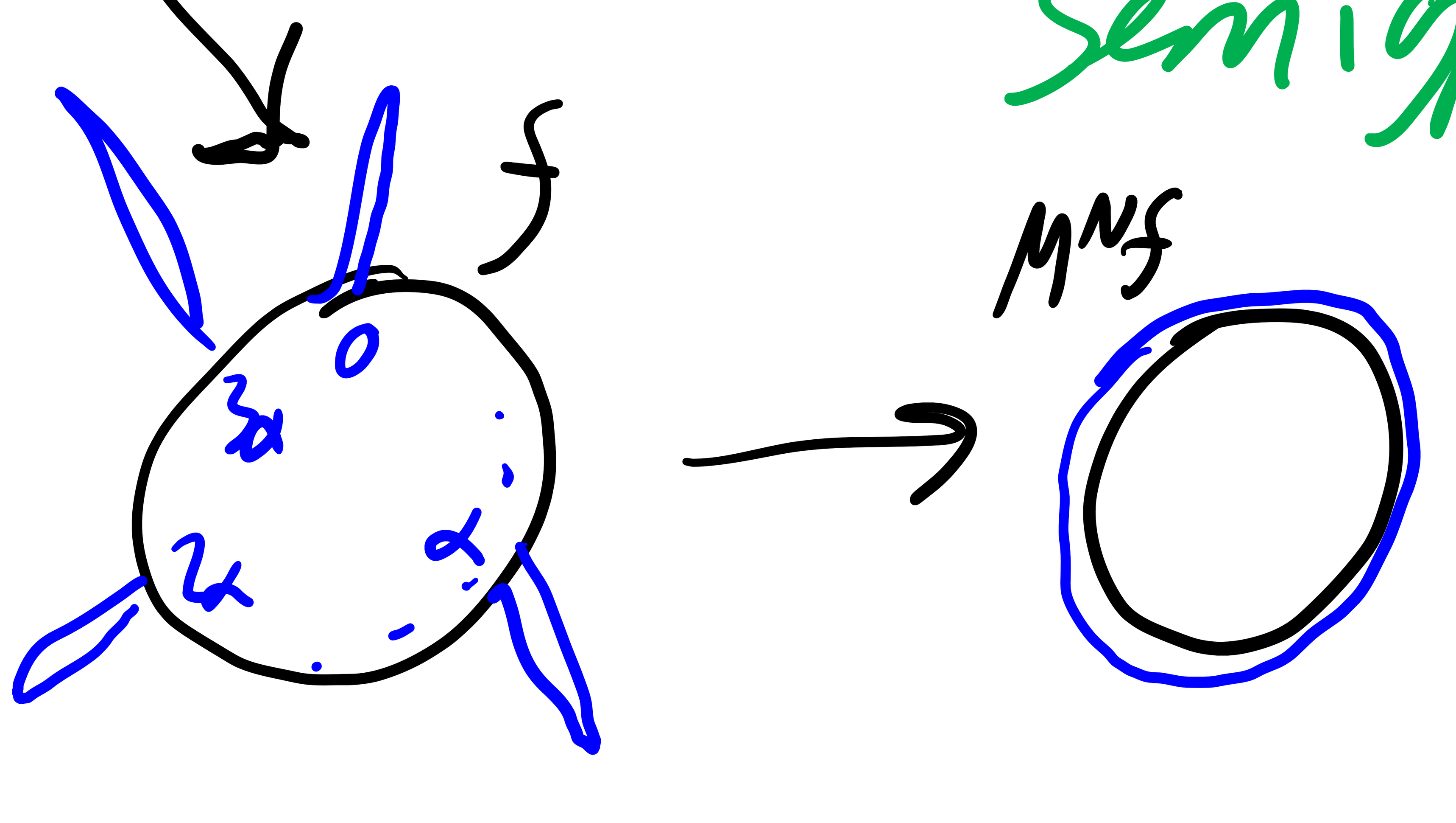
In this case, M is not quasi-cpt!

$\left(M^n \overset{\text{uniformly}}{\not\longrightarrow} \text{projection to } \langle e \rangle\right)$

However: columns are gp rot.

$\Longrightarrow$ generate unif. equicont. semigp $\Longrightarrow$ (X_φ, σ) uniq. erg.

[Q]



Exp: $\varphi : \begin{cases} 0 \to 01 \\ n \to 0\,(n-1)\,(n+1) \\ \infty \to 0\,\infty\,\infty \end{cases}$

Let $P = \{0\}$, $K=1$. Then

Ever 1-supertile contains ≤ 2

letters not in P, and

$2 < r' = r$

$$\varphi^2(0) = |01002| = 5$$
$$\varphi^2(1) = |0101013| = 7$$
$$\varphi^2(n) = |010(n-2)(n+1)0n(n+2)| = 8$$
$$\Rightarrow r \geq \sqrt{5} > 2.$$

Alt.: take $P = \{0,1\}$, $K=2$.

Each $\varphi^2(a)$ has ≤ 4 letters not in P, and $4 < 5 \leq r^2$.

So (X_φ, σ) uniq. erg. (can explicitly find $r = \frac{5}{2}$, $\ell(n) = 1 - 2^{-(n+1)}$)

5. Further Thoughts

- Spectral gap of M $\rightsquigarrow$ disc. estimates

  (Infinite Pisot subs?)

  $\uparrow$

  |Expected − Actual|

  sum of f over k-supertiles

- Q: Does primitive + ctbl $\Rightarrow$ (C)?

  What if $\varphi(I) \subset I^+$?

  (I = isolated points).

- Frettlöh – Garber have family of subs on 0-dim $\mathcal{A}$ realising any inflation factor $r \in [5/2, \infty)$, and

  - prim
  - recog
  - (C)

  $\Big\} \Longrightarrow \exists$ (cts) ntlf, (X_φ, σ) uniq. erg.

<u>Sketch proof of Thm:</u>

Consider $V : E \to E$,

$$Vf = f|_P \ (0 \text{ on } \mathcal{A} \setminus P).$$

V cpt $\Longrightarrow C := V \circ M^k$ cpt.

$M^k - C = (I - V) M^k =$ sub k times then discard any $b \in P$.

$$\|M^k - C\| = \sup_{a \in A} \# \{ b \triangleleft \varphi^k(a) \mid b \notin P \}$$

$$< r^k$$

$\square$