# How inhomogeneous Cantor sets can pass a point

Zhiqiang Wang
zhiqiangwzy@163.com

Department of Mathematics
East China Normal University, Shanghai, China

One World Numeration Seminar
November 9, 2021

joint work with Wenxia Li

# Contents

# $\beta$-expansion

- $\beta$-expansion: for $1 < \beta \le 2$ and $x \in I_\beta := [0, 1/(\beta - 1)]$, there exists a sequence $(\varepsilon_n)_{n=1}^\infty \in \Omega := \{0,1\}^{\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\beta^n}. \tag{1.1}$$

  Every sequence $(\varepsilon_n)_{n=1}^\infty \in \Omega$ satisfying (1.1) is called an expansion of $x$ in base $\beta$.

- We are concerned with two sets

$$\mathcal{U}_\beta := \{x \in I_\beta : x \text{ has a unique expansion in base } \beta\}, \ 1 < \beta \le 2;$$
$$\mathscr{U}(x) := \{1 < \beta \le 2 : x \in \mathcal{U}_\beta\}, \ x \ge 0.$$

# Homogeneous IFS family

The $\beta$-expansion can be viewed as an iterated function system (IFS)

$$\Phi_\beta := \left\{ f_{\beta,0}(x) = \frac{x}{\beta},\ f_{\beta,1}(x) = \frac{x+1}{\beta} \right\},\ 1 < \beta \leq 2.$$

- The interval $I_\beta$ is the invariant set for the IFS $\Phi_\beta$.
- We define the natural projection $\pi_\beta : \Omega \to I_\beta$ by

$$\pi_\beta((\varepsilon_n)) := \lim_{n\to\infty} f_{\beta,\varepsilon_1} \circ f_{\beta,\varepsilon_2} \circ \cdots \circ f_{\beta,\varepsilon_n}(0) = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\beta^n}.$$

Then we have

$$\mathcal{U}_\beta = \left\{ x \in I_\beta : \#\pi_\beta^{-1}(\{x\}) = 1 \right\}.$$

# Inhomogeneous IFS family

Fix $\alpha > 0$. We consider the inhomogeneous IFS

$$\Phi_\lambda = \Phi_{\lambda,\alpha} := \{f_{\lambda,0}(x) = \lambda x,\ f_{\lambda,1}(x) = \lambda^\alpha(x+1)\},\ 0 < \lambda \le \gamma, \tag{1.2}$$

where $\gamma := \gamma(\alpha)$ satisfying the equation $\gamma + \gamma^\alpha = 1$.

- Let $K_\lambda := K_{\lambda,\alpha}$ be the invariant set of the IFS $\Phi_\lambda$.
- When $0 < \lambda < \gamma$, every $x \in K_\lambda$ has a unique coding; when $\lambda = \gamma$, $K_\gamma = [0, \gamma^{\alpha-1}]$ and except for countably many points which has two codings, every $x \in K_\gamma$ has a unique coding.

For $0 \le x \le \gamma^{\alpha-1}$, we define

$$\Lambda(x) = \Lambda_\alpha(x) := \{0 < \lambda \le \gamma : x \in K_\lambda\}, \tag{1.3}$$

which is analogous to $\mathscr{U}(x)$ considered in $\beta$-expansion.

**Question**: How about the set $\Lambda(x)$ ?

- When $\alpha = 1$, the IFS is homogeneous,

$$\Phi_\lambda = \{f_{\lambda,0}(x) = \lambda x,\ f_{\lambda,1}(x) = \lambda(x+1)\},\ 0 < \lambda \le 1/2.$$

### Remark

The set $\Lambda(x)$ has been studied in [K. Jiang, D. Kong, W. Li, How likely can a point be in different Cantor sets, arXiv:2102.13264] for the case $\alpha = 1$.

- Note that

$$\Lambda(0) = (0, \gamma],\ \Lambda(\gamma^{\alpha-1}) = \{\gamma\},$$

and

$$\gamma \in \Lambda(x) \quad \text{for} \quad 0 \le x \le \gamma^{\alpha-1}.$$

We shall focus on the set $\Lambda(x)$ for $0 < x < \gamma^{\alpha-1}$.

# Main results-1

**Theorem**

*For $0 < x < \gamma^{\alpha-1}$, the set $\Lambda(x)$ is a Cantor set. Moreover, the mapping $x \mapsto \Lambda(x)$ is continuous on $\left(0, \gamma^{\alpha-1}\right)$ with respect to the Hausdorff metric.*

It is easy to calculate that

$$\min \Lambda(x) = \left(\frac{x}{1+x}\right)^{1/\alpha} \text{ and } \max \Lambda(x) = \gamma.$$

# Main results-2

**Theorem**

*For* $0 < x < \gamma^{\alpha-1}$ *and* $\lambda \in \Lambda(x)$, *we have*

$$\lim_{\delta \to 0^+} \dim_H \big(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)\big) = \frac{\log \gamma}{\log \lambda} = \dim_H K_\lambda.$$

**Corollary**

*For* $0 < x < \gamma^{\alpha-1}$, *the set* $\Lambda(x)$ *is a Lebesgue null set with full Hausdorff dimension in* $\mathbb{R}$.

# Main results-3

**Theorem**

*For* $x_1, x_2, \cdots, x_\ell \in \big(0, \gamma^{\alpha-1}\big)$, *we have*

$$\dim_H \left( \bigcap_{i=1}^{\ell} \Lambda(x_i) \right) = 1.$$

- Symbolic space: $\Omega = \{0,1\}^{\mathbb{N}}$. The *lexicographical order* on $\Omega$ is denoted by $\prec$, this is, $i_1 i_2 i_3 \cdots \prec j_1 j_2 j_3 \cdots$ if there exists a $n \geq 1$ such that $i_1 = j_1, \cdots, i_{n-1} = j_{n-1}, i_n < j_n$. The metric $\rho$ on $\Omega$ is defined by

$$\rho(i_1 i_2 i_3 \cdots, j_1 j_2 j_3 \cdots) := 2^{-\min\{n \geq 1 : i_n \neq j_n\}}.$$

- Projection mapping: for $0 < \lambda \leq \gamma$, the mapping $\pi_\lambda : \Omega \to K_\lambda$ is defined by

$$\begin{aligned}\pi_\lambda(i_1 i_2 i_3 \cdots) &:= \lim_{n\to\infty} f_{\lambda,i_1} \circ f_{\lambda,i_2} \circ \cdots \circ f_{\lambda,i_n}(0) \\ &= \sum_{n=1}^{\infty} i_n \lambda^{n+(\alpha-1)(i_1+i_2+\cdots+i_n)}.\end{aligned}$$

For $0 < x < \gamma^{\alpha-1}$, we define the mapping $\Psi_x : \Lambda(x) \to \Omega$ by

$$\Psi_x(\lambda) = \pi_\lambda^{-1}(x) \text{ for } \lambda \in \Lambda(x). \tag{3.1}$$

- For $\lambda \in \Lambda(x) \setminus \{\gamma\}$, the mapping $\pi_\lambda : \Omega \to K_\lambda$ is one-to-one correspondence and so $\pi_\lambda^{-1}(x) \in \Omega$ is well-determined.
- For $\lambda = \gamma$, the mapping $\pi_\gamma : \Omega \to K_\gamma = [0, \gamma^{\alpha-1}]$ is not injective. We use $\pi_\gamma^{-1}(x)$ to denote either the unique sequence or the bigger one when there exist two distinct sequences in $\Omega$ which maps to $x$ by $\pi_\gamma$.

# Symbolic sets of $\Lambda(x)$

## Proposition

*For* $0 < x < \gamma^{\alpha-1}$*, the mapping* $\Psi_x$ *is strictly decreasing with respect to the lexicographical order and bijective between* $\Lambda(x)$ *and the set*

$$\Omega(x) = \{i_1 i_2 i_3 \cdots \in \Omega : \Psi_x(\gamma) \preceq i_1 i_2 i_3 \cdots\}.$$

*Moreover, the mapping* $\Psi_x : \Lambda(x) \to (\Omega(x), \rho)$ *is a homeomorphism.*

Then we can prove the first main result.

## Theorem (Main results-1)

*For* $0 < x < \gamma^{\alpha-1}$*, the set* $\Lambda(x)$ *is a Cantor set. Moreover, the mapping* $x \mapsto \Lambda(x)$ *is continuous on* $\left(0, \gamma^{\alpha-1}\right)$ *with respect to the Hausdorff metric.*

## Proposition

*For* $0 < x < \gamma^{\alpha-1}$ *and* $0 < \lambda \le \gamma$*, we have*

$$\dim_H \big(\Lambda(x) \cap [0, \lambda]\big) \le \dim_H K_\lambda = \frac{\log \gamma}{\log \lambda}. \tag{3.2}$$

## Proof

We assume that $(\frac{x}{x+1})^{1/\alpha} < \lambda < \gamma$. Consider

$$\Lambda(x) \cap [0, \lambda] \xrightarrow{\Psi_x} \Omega \xrightarrow{\pi_\lambda} K_\lambda.$$

Take $\lambda_1, \lambda_2 \in \Lambda(x) \cap [0, \lambda]$ with $\lambda_1 < \lambda_2$. We can show that

$$\pi_\lambda\big(\Psi_x(\lambda_1)\big) - \pi_\lambda\big(\Psi_x(\lambda_2)\big) \ge \frac{\alpha x(1 - \lambda - \lambda^\alpha)}{\lambda}(\lambda_2 - \lambda_1).$$

Then we have $\dim_H \big(\Lambda(x) \cap [0, \lambda]\big) \le \dim_H \pi_\lambda \circ \Psi_x\big(\Lambda(x) \cap [0, \lambda]\big) \le \dim_H K_\lambda$.

## Proposition

*For $0 < x < \gamma^{\alpha-1}$ and $\lambda \in \Lambda(x) \setminus \{\gamma\}$, if the sequence $\Psi_x(\lambda)$ does not end with $0^\infty$, then for any $\delta > 0$ we have*

$$\dim_H \left(\Lambda(x) \cap [\lambda, \lambda+\delta]\right) \geq \frac{\log \gamma}{\log \lambda}. \tag{3.3}$$

## Proof

Write $\Psi_x(\lambda) = d_1 d_2 d_3 \cdots$. Note that $\Psi_x(\gamma) \prec \Psi_x(\lambda)$. Choose a $n_0 \geq 1$ such that $\Psi_x(\gamma) \prec d_1 d_2 \cdots d_{n_0} 0^\infty$. Then choose an infinite sequence $n_0 < n_1 < n_2 < \cdots$ such that $d_{n_k} = 1$ for $k \geq 1$.
For $k \geq 1$, we define

$$\Omega_{\lambda,k} = \left\{ d_1 \cdots d_{n_k - 1} 0 i_1 i_2 i_3 \cdots : \ i_1 i_2 i_3 \cdots \text{ does not contain } 0^k \right\}.$$

## Proof

Recall that

$$\Psi_x\big(\Lambda(x)\cap[\lambda,\gamma]\big)=\{i_1i_2i_3\cdots\in\Omega:\Psi_x(\gamma)\preceq i_1i_2i_3\cdots\preceq\Psi_x(\lambda)\}\,.$$

Note that $\Omega_{\lambda,k}\subseteq\Psi_x\big(\Lambda(x)\cap[\lambda,\gamma]\big)$ for $k\geq 1$. By the continuity of $\Psi_x$, $\Omega_{\lambda,k}\subseteq\Psi_x(\Lambda(x)\cap[\lambda,\lambda+\delta])$ for sufficiently large $k$.

$$\Psi_x^{-1}\big(\Omega_{\lambda,k}\big)\xrightarrow{\Psi_x}\Omega_{\lambda,k}\xrightarrow{\pi_\lambda}\pi_\lambda(\Omega_{\lambda,k}).$$

Take $\lambda_1,\lambda_2\in\Psi_x^{-1}(\Omega_{\lambda,k})$ with $\lambda_1<\lambda_2$. We can show that

$$\pi_\lambda(\Psi_x(\lambda_1))-\pi_\lambda(\Psi_x(\lambda_2))\leq\frac{C_\alpha(\lambda_2-\lambda_1)}{\lambda^{k+\alpha}(1-\lambda^\alpha)}.$$

It follows that $\dim_H\big(\Lambda(x)\cap[\lambda,\lambda+\delta]\big)\geq\dim_H\pi_\lambda(\Omega_{\lambda,k})$. Letting $k\to\infty$, we obtain the desired result.

## Proposition

*For* $0 < x < \gamma^{\alpha-1}$ *and* $\lambda \in \Lambda(x)$, *if the sequence* $\Psi_x(\lambda)$ *does not end with* $1^\infty$, *then for any* $\delta > 0$ *we have*

$$\dim_H \left(\Lambda(x) \cap [\lambda - \delta, \lambda]\right) \geq \frac{\log \gamma}{\log \lambda}. \tag{3.4}$$

Now we can conclude the second main result.

## Theorem (main results-2)

*For* $0 < x < \gamma^{\alpha-1}$ *and* $\lambda \in \Lambda(x)$, *we have*

$$\lim_{\delta \to 0^+} \dim_H \left(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)\right) = \frac{\log \gamma}{\log \lambda} = \dim_H K_\lambda.$$

# Thickness of Cantor set

Let $F$ be a Cantor set in $\mathbb{R}$ and let $F_0$ be the convex hull of $F$.

- Write $F_0 \setminus F = \bigcup_{j=1}^{\infty} V_j$ as the union of countable disjoint open intervals.
- The sequence $\mathscr{V} = \{V_j\}_{j=1}^{\infty}$ is called a *defining sequence* for $F$. If moreover $|V_1| \geq |V_2| \geq |V_3| \geq \cdots$, we call $\mathscr{V}$ an *ordered defining sequence* for $F$, where $|E|$ denotes the diameter of the set $E$.
- Let $F_j = F_0 \setminus \bigcup_{k=1}^{j} V_k$, which is a finite union of closed intervals.
- The open interval $V_j$ is contained in some connected component of $F_{j-1}$, denoted by $F_{j-1}^*$. We can write

$$F_{j-1}^* = L_{\mathscr{V}}(V_j) \cup V_j \cup R_{\mathscr{V}}(V_j),$$

  where $L_{\mathscr{V}}(V_j)$ and $R_{\mathscr{V}}(V_j)$ are two closed intervals.

We define

$$\tau_{\mathscr{V}}(F) = \inf\left\{\frac{|L_{\mathscr{V}}(V_j)|}{|V_j|}, \frac{|R_{\mathscr{V}}(V_j)|}{|V_j|} : j \geq 1\right\}.$$

The *thickness* of $F$ is defined by

$$\tau(F) = \sup_{\mathscr{V}} \tau_{\mathscr{V}}(F)$$

where the supremum is taken over all defining sequences $\mathscr{V} = \{V_j\}_{j=1}^{\infty}$ for $F$ [Newhouse-1979]. It is shown that $\tau(F) = \tau_{\mathscr{V}}(F)$ for every ordered defining sequence $\mathscr{V}$ for $F$ [Williams-1991].

**Lemma (Palis-Takens-1993, p. 77)**

*If $F$ is a Cantor set in $\mathbb{R}$, then we have*

$$\dim_H F \geq \frac{\log 2}{\log\left(2 + \frac{1}{\tau(F)}\right)}.$$

We say that two Cantor sets in $\mathbb{R}$ are *interleaved* if each set intersects the interior of the convex hull of the other set.

**Theorem (Hunt-Kan-Yorke, 1993)**

*There exists a function $\varphi : (0,\infty) \to (0,\infty)$ such that for all interleaved Cantor sets $E$ and $F$ with $\tau(E) \geq \varphi(t)$ and $\tau(F) \geq \varphi(t)$ there exists a Cantor subset $K \subseteq E \cap F$ with $\tau(K) \geq t$.*

If $\xi = \min E = \min F$ (or $\xi = \max E = \max F$), from the proof of the above theorem, then the resulting Cantor set $K$ also contains $\xi$.
Let $\varphi(t)$ be as that in the above theorem.

**Theorem**

*If $E$ and $F$ are two Cantor sets in $\mathbb{R}$ with $\tau(E) \geq \varphi(t)$, $\tau(F) \geq \varphi(t)$, and $\xi = \max E = \max F$, then there exists a Cantor subset $K \subseteq E \cap F$ such that*

$$\tau(K) \geq t \text{ and } \xi \in K.$$

# Construction of large thickness subsets of $\Lambda(x)$

Fix $0 < x < \gamma^{\alpha-1}$, and write $\Psi_x(\gamma) = d_1 d_2 d_3 \cdots$.

- Let $\{n_1 < n_2 < \cdots < n_k < \cdots\} = \{n \geq 1 : d_n = 0\}$.
- For $k \geq 1$, we define

$$\Omega_{x,k} = \{i_1 i_2 \cdots \in \Omega : d_1 \cdots d_{n_k-1} 10^\infty \preceq i_1 i_2 \cdots \preceq d_1 \cdots d_{n_k-1} 1^\infty\}.$$

- Let $F_k = \Psi_x^{-1}(\Omega_{x,k})$. Then we have $F_k \subseteq \Lambda(x)$ since $\Omega_{x,k} \subseteq \Omega(x)$.
- Let $\eta_k = \Psi_x^{-1}(d_1 d_2 \cdots d_{n_k-1} 1^\infty)$, $\theta_k = \Psi_x^{-1}(d_1 d_2 \cdots d_{n_k-1} 10^\infty)$.

We define $F_{k,0} = [\eta_k, \theta_k]$, and for $\ell \geq 1$,

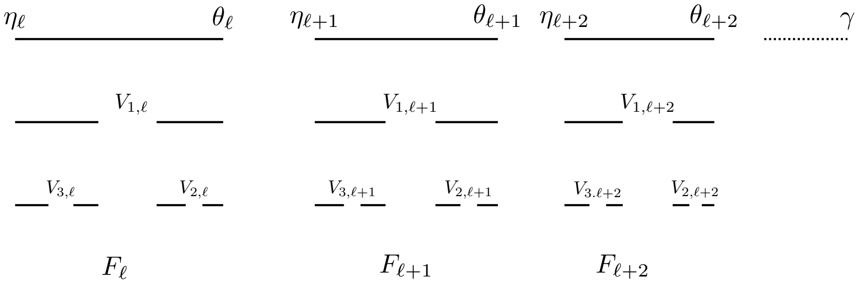
$$F_{k,\ell} = \bigcup_{i_1 \cdots i_\ell \in \{0,1\}^\ell} \left[\Psi_x^{-1}(d_1 \cdots d_{n_k-1} 1 i_1 \cdots i_\ell 1^\infty), \Psi_x^{-1}(d_1 \cdots d_{n_k-1} 1 i_1 \cdots i_\ell 0^\infty)\right].$$

- $F_k = \bigcap_{\ell=0}^{\infty} F_{k,\ell}$.
-
$$[\eta_k, \theta_k] \setminus F_k = \bigcup_{\omega \in \Omega^*} V_{w,k},$$
  where $\Omega^* = \bigcup_{n=0}^{\infty} \{0,1\}^n$ is the collection of all finite 0-1 words and
$$V_{\omega,k} = \left(\Psi_x^{-1}(d_1 d_2 \cdots d_{n_k-1} 1\ \omega\ 10^\infty), \Psi_x^{-1}(d_1 d_2 \cdots d_{n_k-1} 1\ \omega\ 01^\infty)\right).$$
- We enumerate $\{V_{w,k} : \omega \in \Omega^*\}$ as $\mathscr{V}_k = \{V_{j,k}\}_{j=1}^{\infty}$ by
$$V_{\vartheta,k}, V_{0,k}, V_{1,k}, V_{00,k}, V_{01,k}, V_{10,k}, V_{11,k}, V_{000,k}, V_{001,k}, V_{010,k}, V_{011,k}, \cdots,$$

For $\ell \geq 1$, we define
$$C_\ell = \{\gamma\} \cup \bigcup_{k=\ell}^{\infty} F_k = \Lambda(x) \cap [\eta_\ell, \gamma].$$

Note that

$$[\eta_\ell, \gamma] \setminus C_\ell = \bigcup_{k=\ell}^{\infty} (\theta_k, \eta_{k+1}) \ \cup \ \bigcup_{k=\ell}^{\infty} \bigcup_{j=1}^{\infty} V_{j,k}.$$



We enumerate $\{(\theta_k, \eta_{k+1}), V_{j,k} : k \geq \ell, j = 1, 2, \cdots\}$ as the following

$$\begin{array}{ccccc}
(\theta_\ell, \eta_{\ell+1}) & (\theta_{\ell+1}, \eta_{\ell+2}) & (\theta_{\ell+2}, \eta_{\ell+3}) & (\theta_{\ell+3}, \eta_{\ell+4}) & \cdots \\
V_{1,\ell} & V_{2,\ell} & V_{3,\ell} & V_{4,\ell} & \cdots \\
V_{1,\ell+1} & V_{2,\ell+1} & V_{3,\ell+1} & V_{4,\ell+1} & \cdots \\
V_{1,\ell+2} & V_{2,\ell+2} & V_{3,\ell+2} & V_{4,\ell+2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

which gives us a defining sequence $\mathscr{V}$ of $C_\ell$.

**Proposition**

*We have*

$$\lim_{\ell \to \infty} \tau_{\mathscr{V}}(C_\ell) = \infty,$$

*which implies*

$$\lim_{\ell \to \infty} \tau(C_\ell) = \infty.$$

Now we can prove the third main result.

**Theorem (main results-3)**

*For* $x_1, x_2, \cdots, x_\ell \in \left(0, \gamma^{\alpha-1}\right)$, *we have*

$$\dim_H \left( \bigcap_{i=1}^{\ell} \Lambda(x_i) \right) = 1.$$

# The two-parameters case

Consider the inhomogeneous IFS

$$\Psi_{a,b} := \{f_0(x) = ax,\ f_1(x) = b(x+1)\},\ a > 0, b > 0, a + b \leq 1.$$

Let $E_{a,b}$ be the invariant set of the IFS $\Psi_{a,b}$. For $x > 0$, we define

$$\Upsilon(x) = \{(a,b) : x \in E_{a,b}, a > 0, b > 0, a + b \leq 1\}. \tag{3.5}$$

**Theorem**

*(i)* $\Upsilon(x)$ *is a Borel subset in* $\mathbb{R}^2$*;*
*(ii)* $\Upsilon(x)$ *is a Lebesgue null set with full Hausdorff dimension in* $\mathbb{R}^2$*;*
*(iii) The intersection of* $\Upsilon(x_1), \Upsilon(x_2), \cdots, \Upsilon(x_\ell)$ *still has full Hausdorff dimension in* $\mathbb{R}^2$ *for any finitely many positive real number* $x_1, x_2, \cdots, x_\ell$*.*

B. R. Hunt, I. Kan, J. A. Yorke,
When Cantor sets intersect thickly.
*Trans. Amer. Math. Soc.* 339 (1993), no. 2, 869-888.

S. E. Newhouse,
The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms.
*Publ. Math.* No. 50 (1979), 349-399.

J. Palis, F. Takens,
*Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcation.*
Cambridge University Press, 1993.

R. F. Williams,
How big is the intersection of two thick Cantor sets?
*Continuum theory and dynamical systems*, 163-175, Contemp. Math., 117, Amer. Math. Soc., Providence, RI, 1991.

**Thank you!**