Transcendence of Sturmian Numbers over an Algebraic Base

F Luca, J. Ouaknine, and J. Worrell

One World Numeration Seminar, 2023

Normal Numbers are Normal

E. Borel. Les probabilités dénombrables et leurs applications arithmétiques. *Rend. Circ. Mat. Palermo*, 27 (1909).

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Theorem (Borel 1909)

Almost every number in $[0, 1]$ *is normal.*

Specific Cases

- Champernowne (1933): explicit base-10 normal number

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- Borel (1950): does decimal expansion of $\sqrt{2}$ have infinitely many 5's? Is it normal?

$$\sqrt{2} = 1.4142135623730950488016887242 0$$

Conjecture (Borel 1950)

Let x be a real irrational algebraic number and $b \geq 2$ a positive integer. Then x is normal in base b.

Conjectures with a Computational Aspect

If the base-b expansion of a real irrational number x is "simple" then x is transcendental.

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Cobham's Second Conjecture (1968)

The base-b expansion of an algebraic number cannot be generated by a morphism of exponential growth (equivalently, by a **tag machine** with exponential dilation factor > 1).

Tag Machines (Cobham 1968)

- A finite work-tape alphabet,
- B finite output-tape alphabet,
- Start symbol $a \in A$,
- $\sigma : A^* \to A^*$ morphism, prolongable on a,
- $\varphi : A^* \to B^*$ letter-to-letter morphism.

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$$\begin{array}{llllll} & & \downarrow & & & \downarrow \\ \text{working} & a & b & a & a & c \\ \text{output} & x & y & & & \end{array}$$

Example: The Fibonacci Word

The sequence of finite binary words

$$F_0 = 0, F_1 = 01, F_2 = 010, F_2 = 01001, \ldots$$

satisfying recurrence

$$F_n = F_{n-1}F_{n-2} \quad (n \geq 2)$$

converges to infinite Fibonacci word

$$F_\infty = 010010100100101001010010010100\ldots$$

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- Fibonacci word is **morphic**: $F_\infty = \lim_{n \to \infty} \sigma^n(0)$, where $\sigma : \{0,1\}^* \to \{0,1\}^*$ is given by $\sigma(0) = 01$ and $\sigma(1) = 0$.

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- Incidence matrix

$$M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

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Theorem (Danilov 1972)

Let $\boldsymbol{u}$ *be the Fibonacci word. Then for all integers* $b \geq 2$ *the word*

$$S_b(\boldsymbol{u}) := \sum_{n=0}^{\infty} \frac{u_n}{b^n}$$

is transcendental.

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$$x_n := \begin{cases} 1 & \text{if } R_\theta^n(x) \in [0, \theta) \\ 0 & \text{otherwise} \end{cases}$$

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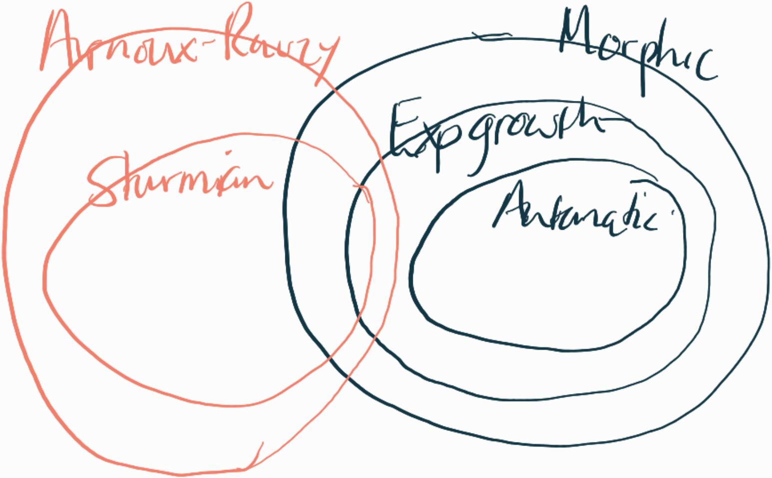
- Sequence is Sturmian of **slope** θ iff it is coding of some x

Taxonomy of Simple Words



Transcendence of Sturmian Words

Theorem (Ferenczi and Mauduit 1997)

Let $b \geq 2$ be an integer. Given $\boldsymbol{u} \in \{0, 1, \ldots, b-1\}^\omega$, suppose that there exist $\varepsilon > 0$ and infinite sequences $(U_n)_{n=0}^{\infty}$ and $(V_n)_{n=0}^{\infty}$ of finite words such that:

- $\lim_n |V_n| = \infty$
- $\sup_n \frac{|U_n|}{|V_n|} < \infty$
- $U_n V_n^{2+\varepsilon}$ *is a prefix of* $\boldsymbol{u}$

Then $S_b(\boldsymbol{u})$ is transcendental.

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Corollary (Ferenzci and Mauduit 1997)

Let $b \geq 2$ be an integer. If $\boldsymbol{u} \in \{0, 1\}^\omega$ is Sturmian then $S_b(\boldsymbol{u})$ is transcendental.

Transcendence of Automatic Numbers

Theorem (Adamczewski, Bugeaud, Luca 2004)

Let $b \geq 2$ be an integer. Given $\boldsymbol{u} \in \{0, 1, \ldots, b-1\}^\omega$, suppose that there exist $\varepsilon > 0$ and infinite sequences $(U_n)_{n=0}^\infty$ and $(V_n)_{n=0}^\infty$ of finite words such that:

- $\lim_n |V_n| = \infty$
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Then $S_b(\boldsymbol{u})$ is either rational or transcendental.

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Corollary

Let $b \geq 2$ *be an integer. If* $\boldsymbol{u} \in \{0, 1\}^\omega$ *is automatic then* $S_b(\boldsymbol{u})$ *either rational or transcendental.*

Diophantine Exponent

Definition (Adamczewski and Bugeaud 2007)

The **Diophantine exponent** of $\boldsymbol{u}$ is the supremum of all real ρ such that $\boldsymbol{u}$ has arbitrarily long prefixes of the form UV^α, for $\alpha \geq 1$, satisfying

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Theorem (Adamczewski-Bugeaud-Luca Reformulated)

For an integer $b \geq 2$ and sequence $\boldsymbol{u} \in \{0, \ldots, b-1\}$, if $\mathrm{Dio}(\boldsymbol{u}) > 1$ then $S_b(\boldsymbol{u})$ is either rational or transcendental.

Combinatorial Part

- [Ferenczi and Mauduit 1997] show that Sturmian words have Diophantine exponent > 1.

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- [Adamczewski, Cassaigne, Le Gonidec 2020] shows that words generated by morphims of exponential growth have Diophantine exponent > 1.

Number-Theoretic Part

Theorem (Schlickewei 75)

Let $m \geq 2$ be an integer, ε a positive real, and S a finite set of prime numbers. Let $L_1, \ldots, L_m$ be linearly independent linear forms with real algebraic coefficients. Then the set of solutions $\boldsymbol{x} \in \mathbb{Z}^m$ of the inequality

$$\left(\prod_{i=1}^{m} \prod_{p \in S} |x_i|_p\right) \cdot \prod_{i=1}^{m} |L_i(\boldsymbol{x})| \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

are contained in finitely many proper linear subspaces of $\mathbb{Q}^m$.

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4. Weaker condition $\mathrm{Dio}(\boldsymbol{u}) > 1$ yields infinite sequence of points in $\mathbb{Z}^3$ on which linear form $L(x_1, x_2, x_3) = \alpha x_1 - \alpha x_2 - x_3$ is "small"

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5. Apply Subspace Theorem to conclude that α is **rational**

Transcendence Results over an Algebraic Base

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Theorem (Adamczewski and Bugeaud 2007a)

Let β be a Pisot or a Salem number and let $\boldsymbol{u}$ be a bounded sequence of integers. Then $S_\beta(\boldsymbol{u})$ either lies in $\mathbb{Q}(\beta)$ or is transcendental.

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Theorem (Adamczewski and Bugeaud 2007b)

Let β be an algebraic integer with $|\beta| > 1$. Let $\boldsymbol{u}$ be a bounded sequence of rational integers. Assume that $\mathrm{Dio}(\boldsymbol{u}) > \frac{\log M(\beta)}{\log |\beta|}$. Then $S_\beta(\boldsymbol{u})$ either lies in $\mathbb{Q}(\beta)$ or is transcendental.

Our Main Result

Theorem (Luca, Ouaknine, W. 2022)

Let β be algebraic with $|\beta| > 1$. Let $\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(k)}$ be Sturmian sequences, all having the same slope and such that no sequence is a tail of another. Then $\{1, S_\beta(\boldsymbol{u}^{(1)}), \ldots, S_\beta(\boldsymbol{u}^{(k)})\}$ is linearly independent over $\overline{\mathbb{Q}}$.

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Corollary

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Theorem (Bugeaud, Kim, Laurent, and Nogueira 2021)

Let $\beta \geq 2$ be integer and $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ Sturmian sequences of the same slope, neither a tail of the other. Then $\{1, S_\beta(\boldsymbol{u}^{(1)}), S_\beta(\boldsymbol{u}^{(2)})\}$ is linearly independent over $\overline{\mathbb{Q}}$.

Diophantine Approximation Modulo Errors

Let $(r_n)_{n=0}^{\infty}$ be Fibonacci sequence and write $F_{\infty}^{(n)}$ for tail of Fibonacci word after dropping first r_n letters.

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- Gaps between these pairs expand with n
- For all n we have $s_n \geq 5r_n$

Stuttering Sequences

Sequence $\boldsymbol{u}$ is **stuttering** if for all $\rho > 0$ there exist sequences $\langle r_n \rangle_{n=0}^{\infty}$ and $\langle s_n \rangle_{n=0}^{\infty}$ of positive integers and $d \geq 2$ such that:

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S3 we have $i_d(n) - i_1(n) = \omega(\log r_n)$ and $i_{j+1}(n) - i_j(n) = \omega(1)$ for all $j \in \{1, \dots, d-1\}$;

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- S2 the strings $u_0 \dots u_{s_n}$ and $u_{r_n} \dots u_{r_n+s_n}$ differ at d pairs with respective positions $i_1(n) < \dots < i_d(n)$;
- S3 we have $i_d(n) - i_1(n) = \omega(\log r_n)$ and $i_{j+1}(n) - i_j(n) = \omega(1)$ for all $j \in \{1, \dots, d-1\}$;
- S4 for all $n \in \mathbb{N}$ and $j \in \{1, 2 \dots, d\}$ we have $u_{i_j(n)} = u_{i_j(n)+r_n+1}$ and $u_{i_j(n)+1} = u_{i_j(n)+r_n}$.

Key Number-Theoretic and Combinatorial Ingredients

Theorem

Let A be a finite set of algebraic numbers and suppose that $\boldsymbol{u} \in A^\omega$ is a stuttering sequence. Then for any algebraic number β with $|\beta| > 1$ the sum $S_\beta(\boldsymbol{u}) = \sum_{n=0}^{\infty} \frac{u_n}{\beta^n}$ is transcendental.

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- Subspace Theorem with a linear form “adapted to errors”

Theorem

Let $\boldsymbol{u}^{(1)}, \ldots, \boldsymbol{u}^{(k)}$ be Sturmian sequences all having the same slope and such that no sequence is a tail of another. Given $c_1, \ldots, c_k \in \mathbb{C}$, define $u_n := \sum_{i=1}^{k} c_i u_n^{(i)}$ for all $n \in \mathbb{N}$. Then $\boldsymbol{u} = \langle u_n \rangle_{n=0}^{\infty}$ is stuttering.

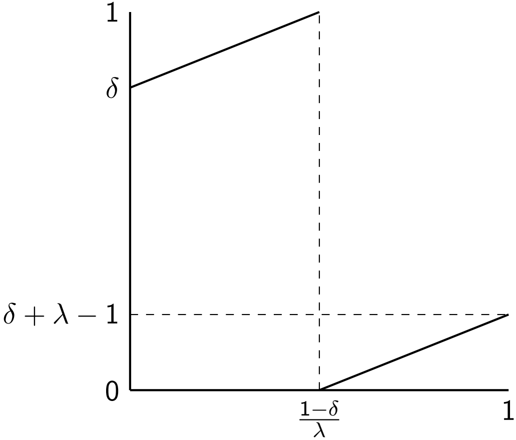
Application to Dynamical Systems

"Are all irrational elements of the Cantor ternary set transcendental?"

K. Mahler, Some suggestions for further research, *Bull. Austral. Math. Soc.* 29 (1984).

Contracted Rotations

Given $0 < \lambda, \delta < 1$ such that $\lambda + \delta > 1$, map $f : I \to I$ given by $f(x) := \{\lambda x + \delta\}$ is a **contracted rotation** with **slope** λ and **offset** δ.



Cantor Sets from Rotations

Rotation Number

Consider the limit set $C := \bigcap_{n=0}^{\infty} f^n(I)$. Then f has a **rotation number** θ such that restriction of f to C is conjugate to the rotation map R_θ and $\overline{C}$ is a Cantor set[a] if θ is irrational.

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Theorem (Luca, Ouaknine, W., 2023)

If f has algebraic slope and irrational rotation number then every element of the Cantor set $\overline{C}$ other than 0 and 1 is transcendental.

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- Generalises result of Bugeaud, Kim, Laurent, Nogueira, which had $\lambda^{-1} \in \mathbb{Z}$.

MSc Thesis of Pavol Kebis

Let $\Sigma = \{0, \ldots, k-1\}$ for some $k \geq 2$. A sequence $\boldsymbol{u} \in \Sigma^\omega$ is **Arnoux-Rauzy** if

- it is uniformly recurrent
- it has subword complexity $p(n) = (k-1)n + 1$
- for each n there is one left-special and one right-special factor of length n.

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Example

The **Tribonacci word** is the limit of the infinite sequence defined by recurrence

$$T_n = T_{n-1}T_{n-2}T_{n-3} \quad T_0 = 0,\, T_1 = 01,\, T_2 = 0102$$

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Also generated by the morphism $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$.

LTI Reachability

Consider LTI system in $\mathbb{R}^2$ with

- Control polyhedron: $U := [0, 1] \times \{0\}$
- Transition matrix $A := \frac{1}{b} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

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Determine whether $\sum_{n=0}^{\infty} u_n \frac{\cos(n\theta)}{b^n} \geq c$, where $u_n = 1$ if $\cos(n\theta) \geq 0$ and $u_n = 0$ otherwise.