From the Thue-Morse sequence to the apwenian sequences

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1. **Thue-Morse sequence**
   - Zero-one Thue-Morse sequence
   - Thue-Morse constant
   - Plus minus one Thue-Morse sequence

2. **Apwenian sequences**
   - Results
   - Examples and remarks

Thue-Morse sequence on $\{0, 1\}$

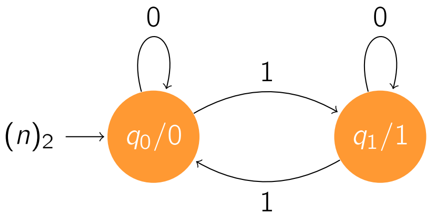
Let $t_0 = 0$ and $t_{2n} = t_n$ and $t_{2n+1} = 1 - t_n$ for all $n \geq 0$. Then

$$\mathbf{t} = (t_0, t_1, t_2, \dots) = (0, 1, 1, 0, 1, 0, 0, 1, \dots)$$

is the famous Thue-Morse sequence.

It is also

- a fix point of the substitution $0 \mapsto 01$ and $1 \mapsto 10$, and
- a 2-automatic sequence.



The Thue-Morse constant

The **generating function** of **t** is

$$f_0(z) = t_0 + t_1 z + t_2 z^2 + \cdots .$$

In base-2,

$$\tau_{TM} := \frac{1}{2} f_0(1/2) = \sum_{n \geq 0} \frac{t_n}{2^{n+1}} = 0.01101001 \cdots \in \mathbb{Q}^c$$

is the Thue-Morse constant.

Questions related to τ_{TM}:

- Is τ_{TM} an algebraic number?
- How well can rational numbers approximate τ_{TM}?

Let $\xi \in \mathbb{Q}^c$, the **irrationality exponent** of ξ, denoted by $\mu(\xi)$, is

$$\mu(\xi) = \sup\left\{\mu > 0 : \exists \text{ i.m. } \frac{p}{q} \in \mathbb{Q} \text{ s.t. } \left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}\right\}.$$

Some arithmetic properties of τ_{TM}:

- (Mahler 1929') τ_{TM} is transcendental;
- (Adamczewski & Cassaigne 06', A. & Rivoal 09') $\mu(\tau_{TM}) \leq 5$ (06'), 4 (09');
- (Bugeaud 11') $\mu(\tau_{TM}) = 2$;
- (Bugeaud & Queffélec 13') i.m. partial quotients of $\tau_{TM} = 4$ or 5 and i.m. ≥ 50;
- (Badziahin & Zorin 15') τ_{TM} is not badly approximable;
    - τ_{TM} is $\frac{C}{q^2 \log\log q}$-well approximable for some $C > 0$.
- …

Thue-Morse sequence on $\{-1, +1\}$

- Recall that $f_0(z) = \sum_{n \geq 0} t_n z^n$. Applying the recurrence relations $t_{2n} = t_n$ and $t_{2n+1} = 1 - t_n$, one has

$$f_0(z) = (1 - z) \cdot f_0(z^2) + \frac{z}{1 - z^2}. \tag{1}$$

- Let

$$f_1(z) = \frac{1}{1 - z} - 2 f_0(z) = \sum_{n \geq 0} (1 - 2t_n) z^n.$$

- Eq. (1) implies

$$\begin{aligned} f_1(z) &= (1 - z) \cdot f_1(z^2) \\ &= (1 - z) \cdot (1 - z^2) \cdot f_1(z^4) = \cdots = \prod_{n=0}^{\infty} (1 - z^{2^n}). \end{aligned}$$

- For $n \geq 0$, let $t'_n := 1 - 2t_n$. The **Thue-Morse sequence on** $\{-1, +1\}$ is

$$\mathbf{t}' = (t'_n)_{n \geq 0},$$

which satisfies $t'_0 = 1$, $t'_{2n} = t'_n$ and $t'_{2n+1} = -t'_n$ for all $n \geq 0$.

- Recall that

$$f_1(z) = \sum_{n \geq 0} t'_n z^n = (1 - z^2) \cdot f_1(z^2) = \prod_{n=0}^{\infty} (1 - z^{2^n}).$$

- This inspires us to study the ± 1 sequences with generating functions given by

$$g(z) = (v_0 + v_1 z + \cdots + v_{p-1} z^{p-1}) g(z^p) =: P(z) g(z^p)$$

where $p \geq 2$ and $v_0, \ldots, v_{p-1} \in \{-1, +1\}$.

Diophantine property of $g(1/b)$

- Let $\mathbb{F}$ be a field and $f(x) = \sum_{i=0}^{\infty} u_i x^i$ is a formal power series over $\mathbb{F}$.
- For each $n \geq 1$ and $p \geq 0$, the Hankel determinant of the series f (or of the sequence $(u_n)_{n\geq 0}$) is defined by

$$H_n^p(f) = \begin{vmatrix} u_p & u_{p+1} & \cdots & u_{p+n-1} \\ u_{p+1} & u_{p+2} & \cdots & u_{p+n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{p+n-1} & u_{p+n} & \cdots & u_{p+2n-2} \end{vmatrix} \in \mathbb{F}.$$

- (Brezinski 1980) If $H_k(f) := H_n^0(f) \neq 0$, then there exist $P(x), Q(x) \in \mathbb{Q}[x]$ with $\deg P(x) \leq k-1$ and $\deg Q(x) \leq k$ such that

$$f(z) - \frac{P(z)}{Q(z)} = \frac{H_{k+1}(f)}{H_k(f)} z^{2k} + \mathcal{O}(z^{2k+1}).$$

- $\frac{P(z)}{Q(z)}$ is the $(k-1, k)$-order Padé approximation of f. It is also a convergent of f.

Let $b \geq 2$ be an integer such that $P(\frac{1}{b^{p^m}}) \neq 0$ for all $m \geq 0$.

(Buguead, Han, Wen and Yao 2016) If $\exists (n_i)_{i\geq 0} \uparrow$ such that

$$H_{n_i}(g) \neq 0 \quad \forall i \geq 0 \quad \text{and} \quad \limsup_{i\to\infty} \frac{n_{i+1}}{n_i} = 1,$$

then $g(1/b)$ is transcendental and $\mu(g(1/b)) = 2$.

(Badziahin 2019) If $g(1/z) \notin \mathbb{Q}(z)$, then

$$\mu(g(1/b)) = 1 + \limsup_{k\to\infty} \frac{d_{k+1}}{d_k}$$

where d_k is the degree of the denominator of the (co-prime) nth convergent of $g(1/z)$.

- Need either the Hankel determinants or the continued fraction of g.

Hankel determinants of Thue-Morse sequence

Recall that $\mathbf{t}$ and $\mathbf{t}'$ are the Thue-Morse sequences on $\{0, 1\}$ and $\{-1, +1\}$ respectively.

- **(Allouche, Peyrière, Wen and Wen 98')** For all $n \geq 1$,

$$\frac{H_n(\mathbf{t}')}{2^{n-1}} \equiv 1 \pmod 2$$

  and

$$H_n(\mathbf{t}) \equiv 1 \pmod 2.$$

- Using BHWY's (or Badziahin's) result, the non-vanishing of Hankel determinants of f_1 yields for all integer $b \geq 2$,

$$f_1(1/b) \text{ is transcendental and } \mu(f_1(1/b)) = 2.$$

- How about $g(1/b)$?

1 Thue-Morse sequence
- Zero-one Thue-Morse sequence
- Thue-Morse constant
- Plus minus one Thue-Morse sequence

2 Apwenian sequences
- Results
- Examples and remarks

Apwenian sequences

Fu and Han (2016) introduced the apwenian sequences in honour of APWW.

Definition (Apwenian sequences)

A sequence $\mathbf{d} \in \{-1, 1\}^\infty$ is ± 1 **apwenian** if

$$\forall n \geq 1, \quad \frac{H_n(\mathbf{d})}{2^{n-1}} \equiv 1 \pmod 2.$$

We restrict our interest on the ± 1 sequences $\mathbf{d}$ whose generating function satisfies

$$g(z) = \sum_{n \geq 0} d_n z^n = (v_0 + v_1 z + \cdots + v_{p-1} z^{p-1}) g(z^p)$$

where $v_i \in \{-1, 1\}$.

- Are there any other apwenian sequences?
- Moreover, how many apwenian sequences are there?

Are there any other apwenian sequences?

Apwenian sequences occur in pairs: **d** and $-$**d**.

For $p = 2, 3, 5, 7, 11, 13, 17$, **Fu and Han** (2016) showed that apwenian sequences are **quite rare**.

Han and Fu's computer assistant proof gives

p	2	3	5	7	11	13	17	19
Num	2	2	2	0	2	2	4	2?

Results

When p **is odd**, the mapping $j \mapsto 2j \pmod p$ induces the permutation

$$\tau : \begin{pmatrix} 1 & 2 & \cdots & \frac{p-1}{2} & \frac{p+1}{2} & \frac{p+3}{2} & \cdots & p-1 \\ 2 & 4 & \cdots & p-1 & 1 & 3 & \cdots & p-2 \end{pmatrix}.$$

Theorem (Guo, Han and W. 2021)

If p is even, then

$$\mathbf{d} \text{ is apwenian} \iff \mathbf{d} = \mathbf{t}' \text{ or } -\mathbf{t}'.$$

If p is **odd**, *then there are*

- **no** *apwenian sequences if the cycle decomposition of τ has a cycle of odd length;*
- 2^k *apwenian sequences if the cycle decomposition of τ has only k cycles of even lengths and no cycles of odd length.*

Proposition (GHW 2021)

Let $p \geq 3$ be **odd** and

$$\mu(p) := \min\{1 \leq j \leq p-1 \mid p|(2^j-1)\}.$$

Then there are k cycles in the cycle decomposition of τ where

$$k = \frac{1}{\mu(p)} \sum_{j=1}^{\mu(p)-1} \gcd(2^j-1, p) - 1.$$

p	cycle decomposition of τ	k	Num
3	$(1,2)$	1	2
5	$(1,2,4,3)$	1	2
7	$(1,2,4)(3,6,5)$	2	0
9	$(1,2,4,8,7,5)(3,6)$	2	4
11	$(1,2,4,8,5,10,9,7,3,6)$	1	2
13	$(1,2,4,8,3,6,12,11,9,5,10,7)$	1	2
15	$(1,2,4,8)(3,6,12,9)(5,10)(7,14,13,11)$	4	16
17	$(1,2,4,8,16,15,13,9)(3,6,12,7,14,11,5,10)$	2	4
19	$(1,2,4,8,16,13,7,14,9,18,17,15,11,3,6,12,5,10)$	1	2
21	$\cdots$	5	0
23	$\cdots$	2	0
25	$\cdots$	2	4
27	$\cdots$	3	8
29	$\cdots$	1	2

0-1 awpenian sequences

If $\mathbf{d} \in \{-1, 1\}^\infty$ satisfies $d_0 = 1$ and

$$g(z) = \sum_{n \geq 0} d_n z^n = (v_0 + v_1 z + \cdots + v_{p-1} z^{p-1}) g(z^p),$$

then $\mathbf{d}$ is the fixed point of the morphism

$$1 \mapsto v_0 v_1 \cdots v_{p-1}, \quad -1 \mapsto \bar{v}_0 \bar{v}_1 \cdots \bar{v}_{p-1}$$

where $\bar{x} = -x$ for $x \in \{-1, 1\}$.

We restrict on the 0-1 sequences $\mathbf{c} = \sigma(1)^\infty \in \{0, 1\}^\infty$ where

$$\sigma : 1 \mapsto 1 a_2 \cdots a_p, \quad 0 \mapsto b_1 b_2 \cdots b_p.$$

Definition (0-1 awpenian sequences)

A sequence $\mathbf{c} \in \{0,1\}^{\infty}$ is 0-1 **apwenian** if

$$\forall n \geq 1, \quad H_n(\mathbf{c}) \equiv 1 \pmod 2.$$

Example

Let σ be the morphism

$$1 \mapsto 10,\ 0 \mapsto 11.$$

The **period doubling** sequence $\mathbf{p} = \sigma^{\infty}(1)$ **is apwenian**.

- How many 0-1 sequences are apwenian?

How many 0-1 sequences are apwenian?

The only 0-1 apwenian sequence that is a fixed point of substitution of constant length is the period doubling sequence.

Theorem (Guo, Han and W. 2021)

Let $\mathbf{c} = \sigma(1)^\infty \in \{0,1\}^\infty$ *where*

$$\sigma : 1 \mapsto 1a_2 \cdots a_p, \quad 0 \mapsto b_1 b_2 \cdots b_p.$$

Then $\mathbf{c}$ *is apwenian* $\iff$ $\mathbf{c} = \mathbf{p}$.

If we allow projection, then there are more apwenian sequences. For example, consider the projection $\iota : 1 \mapsto 11,\ 0 \mapsto 00$. Then $\iota(\mathbf{p})$ is also apwenian.

Criterion for apwenian sequences

Theorem (0-1 criterion, Guo, Han and W. 2021)

$\mathbf{c} \in \{0,1\}^\infty$ *is apwenian if and only if*

$$c_0 = 1 \text{ and } c_n \equiv c_{2n+1} + c_{2n+2} \pmod 2, \forall n \geq 0. \tag{2}$$

Theorem (± 1 criterion, Guo, Hand and W. 2021)

$\mathbf{d} \in \{+1,-1\}^\infty$ *is apwenian if and only if*

$$\frac{d_n + d_{n+1}}{2} - \frac{d_{2n+1} + d_{2n+2}}{2} \equiv 1 \pmod 2, \forall n \geq 0.$$

An application

Let $p \geq 3$ be an odd number and $P(z) = v_0 + v_1 z + \cdots + v_{p-1} z^{p-1}$ with $v_0 = 1$ and $v_i \in \{-1, 1\}$ for $i = 1, \cdots, p-1$. Recall that

$$g(z) = P(z) g(z^p) = \prod_{i=0}^{\infty} P(z^{p^i}).$$

For $m \geq p$, define $v_m := v_i$ with $m \equiv i \pmod{p}$ and $0 \leq i < p$.

Theorem (Guo, Han and W. 2021)

Assume that $b \in \mathbb{Z}_{\geq 2}$ such that $P(\frac{1}{b^{p^i}}) \neq 0$ for all $i \geq 0$. If

$$\frac{v_j + v_{j+1} + v_{2j+1} + v_{2j+2}}{2} \equiv 1 \pmod{2}, 0 \leq j \leq p-2,$$

then the real number $g(1/b)$ is transcendental and its irrationality exponent equals 2.

General cases

Let $\mathbf{d} \in \{-1, 1\}^\infty$ given by

$$\sigma : 1 \mapsto v_0 v_1 \cdots v_{p-1}, \quad -1 \mapsto w_0 w_1 \cdots w_{p-1}$$

where v_i, $w_i \in \{-1, 1\}$.

In the previous discussion, we know about the case

$$w_i = -v_i$$

for all $0 \leq j \leq p - 1$.

Write

$$A := \{j \mid w_j = v_j, 0 \le j \le p-1\}.$$

- $\sharp A = 0$ is the previous case.
- If $\sharp A = p$, then $\mathbf{d}$ is periodic which can not be apwenian.

The following is a criterion which applies to σ.

Theorem (Guo, Han and W. 2021)

If $\sharp A > 0$, then $\mathbf{d}$ is not apwenian.

That is to say, if **d** is a pure substitution **of constant length** sequence, then **d** could be apwenian only if the substitution is of the form

$$1 \mapsto v_0 v_1 \cdots v_{p-1}, \quad -1 \mapsto \bar{v}_0 \bar{v}_1 \cdots \bar{v}_{p-1}$$

or its generating function is of the form

$$g(z) = \prod_{n \geq 0} (v_0 + v_1 z^{p^n} + \cdots + v_{p-1} z^{p^n(p-1)}).$$

Conjecture (GHW)

If **d** is a pure substitution (of non-constant length) sequence, then **d** is not apwenian.

- GHW (2021) verified that all Sturmian sequences are not apwenian.

Examples

Example

Apply the projection $1 \mapsto 11$, $-1 \mapsto -1-1$ to the Thue-Morse sequence $\mathbf{t}'$. Then by our criterion, the resulting sequence is awpenian. However, it is not a pure substitution sequence.

- In general, we are still not clear about when substitution sequences (with projection) is apwenian or not. For concrete examples, our criterion could be applied.

Example

Assume that $\mathbf{c} \in \{0,1\}^\infty$ satisfies

$$c_0 = 1, c_n \equiv c_{2n+1} + c_{2n+2} \pmod 2 \quad (\forall n \geq 0)$$

and $(c_{2n+1})_{n\geq 0}$ is the Fibonacci sequence given by $1 \mapsto 10$, $0 \mapsto 1$. Then by our criterion, $\mathbf{c}$ is apwenian.

- One can also pick a 0-1 sequence with high complexity to be the odd sub-sequence of $\mathbf{c}$. So that $\mathbf{c}$ is not a substitution sequence but still apwenian.
- **(Allouche, Han and Niederreiter 2020)** $\mathbf{c}$ is 2-automatic if and only if $(c_{2n+1})_{n\geq 0}$ is 2-automatic.

Connection with PLCP sequences

Let $\mathbf{s} = (s_m)_{m\geq 1}$ be a sequence of elements in a field $\mathbb{F}$.

- $\mathbf{s}$ is called a k-th order *shift-register sequence* if there exist constants $a_0, a_1, \ldots, a_{k-1}$ in $\mathbb{F}$ such that, for all $i \geq 1$,

$$s_{i+k} + a_{k-1}s_{i+k-1} + \cdots + a_1 s_{i+1} + a_0 s_i = 0.$$

- The n-th *linear complexity* $L(n)$ of $\mathbf{s}$ is defined as the least k such that $s_1, s_2, \ldots, s_n$ are the first n terms of a k-th order shift-register sequence.
- $\mathbf{s}$ is said to have a **perfect linear complexity profile** (PLCP) if for all $n \geq 1$ one has $L(n) = \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$.

Let $\mathbf{s} = (s_n)_{n\geq 1}$ be a sequence in $\mathbb{F}_2$ with $s_1 = 1$. Let

$$f(t) = s_1 t + s_2 t^2 + s_3 t^3 \cdots \in \mathbb{F}_2[[t]].$$

Let $\mathbf{c} = (c_n)_{n\geq 0}$ be the sequence defined by $c_n = s_{n+1}$ for all $n \geq 0$.

Allouche, Han and Niederreiter (2020) found that

- $\mathbf{s}$ has a PLCP $\iff$ $\mathbf{c}$ is apwenian.

They also address the question of which 0-1 apwenian sequences are automatic.

- If $\mathbf{s}$ has a PLCP, then $f(t)$ is algebraic over $\mathbb{F}_2(t)$ if and only if it can be written as $f(t) = v^2 + tu^2$, with u any series in $1 + t\mathbb{F}_2[[t]]$ algebraic over $\mathbb{F}_2(t)$ and v the root of $v^2 + v = 1 + u + tu^2$ lying in $t\mathbb{F}_2[[t]]$.

Thank you!