On normal numbers in fractals

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Normal numbers

- $p \in \mathbb{N}, p \geq 2$.
- $x \in [0, 1)$ is p-normal if $\{p^n x \mod 1\}_{n \geq 1}$ is uniformly distributed.
- $\{p^n x \mod 1\}_{n \geq 1} = \{T^n(x)\}_{n \geq 1}$, orbit of $T : x \to px \mod 1$.
- normal number: É. Borel (1909)
- Borel: $\mathcal{L}$-a.e. x is p-normal, for $p \geq 2$, Borel–Cantelli lemma.
- Concrete normal numbers:
  Champernowne (1933): 0.1234567891011121314151617181920 ...
  Copeland-Erdös (1946): 0.2357111317192329313741434753596 1 ...

Normal numbers in fractals

- Cassels (1956), W. Schmidt (1960): there are plenty of 2-normal numbers in the one third Cantor set $C_{1/3}$.
- $C_{1/3}$: Cantor-Lebesgue measure $\mu_{1/3}$, $\mu_{1/3}$-a.e. point is 2-normal.
- Points in $C_{1/3}$ are not 3-normal, but typically they are 2-normal.
- Motivation: Steinhaus question: normal in infinitely many bases $\Rightarrow$ normal in all bases?
- Cassels, Schmidt: $\{2\text{-normal numbers}\} \neq \{3\text{-normal numbers}\}$.

Classical method: Fourier analysis

- if $\widehat{\mu_{1/3}}(2^n)$ decay fast enough $+$ L^2 method $+$ Weyl's equidistribution criterion.
- Many generalizations: Brown, Pearce, Pollington, Moran, ...
- Riesz products (easy to calculate the Fourier transforms), also higher dimensional generalization.

Dynamical motivation: Furstenberg's $\times 2, \times 3$ conjecture about measures

- Furstenberg (1960's): $\mu = \times 2, \times 3$-invariant, ergodic, continuous, is $\mu = \mathcal{L}_{\mathbb{T}}$?
- $\rightarrow$ very rigid structure for measures that are jointly $\times 2, \times 3$-invariant.
- more generally $p, q \geq 2$ with $\log p / \log q \notin \mathbb{Q}$.
- Rudolph (1990), Johnson: yes, if μ has positive entropy.
- Rigidity phenomena: many profound generalizations and developments.
- Benjy Weiss: $\mu = \times 3$-invariant, ergodic, continuous, is μ-a.e. x 2-normal?
- Lyons : yes if μ is K-mixing.

Dynamical motivation: Furstenberg's $\times 2, \times 3$ conjecture about measures

- Host (1995): if $(p, q) = 1$, $\mu = \times p$-invariant, ergodic, positive entropy, then μ-a.e. point is q-normal.
- Host method: Fourier analysis.
- How about $\log p / \log q \notin \mathbb{Q}$?
- Lindenstrauss, Meiri, Peres, ... (aim to show $\log p / \log q \notin \mathbb{Q}$ is enough)
- Hochman, Shmerkin (2015): optimal condition: $\log p / \log q \notin \mathbb{Q}$, also for pairs of Pisot bases.

Another way of generating normal numbers in fractals: deforming the fractals (rescaling and translating)

- Dayan, Ganguly, Barak Weiss (2024): let $\lambda, t \in \mathbb{R}, \lambda \notin \mathbb{Q}$, $\mu = \times 3$-invariant Bernoulli measure, then for μ-a.e. x, $(\lambda x + t)$ is 3-normal.
- DGW methods: random walk method, $\rightarrow$ Simmons-Weiss, $\rightarrow$ Benoist-Quint.
- The results of DGW are more general.
- Question: how about $\mu = \times 3$-invariant, ergodic but not Bernoulli ?
- We introduce a new method to deal with this problem.

Results

let $\lambda, t \in \mathbb{R}, \lambda \notin \mathbb{Q}$, $\mu = \times 3$-invariant, ergodic with positive entropy, then for μ-a.e. x, $(\lambda x + t)$ is 3-normal.

In our proof we make use of recent advances on dimension of self-similar measures with overlaps (Hochman, ..., Jordan-Rapaport, ...)

Thank you!