On the order of magnitude of Sudler products

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Introduction

- Given a real number α and an integer $N \geq 1$, the *Sudler product* at stage N with parameter α is defined as

$$P_N(\alpha) = \prod_{r=1}^{N} 2|\sin \pi r \alpha|.$$

- Sudler products have been studied in various contexts, as they appear to have connections with many different areas of research (dynamical systems, KAM Theory, q-series, partition theory).

Introduction

- When $\alpha = m/n$ is a rational number we trivially have

$$P_N(\alpha) = 0 \text{ for all } N \geqslant n.$$

- We note that for $\alpha = 1/n$ and $N = n - 1$ we have the important trigonometric identity

$$\prod_{r=1}^{n-1} 2\sin(\pi r/n) = n.$$

- Since the periodicity of the sine function implies that $P_N(\alpha) = P_N(\{\alpha\})$, in order to study the asymptotic behaviour of products $P_N(\alpha)$ as $N \to \infty$ we can restrict to irrational numbers $0 \leqslant \alpha < 1$.

The first known appearance of Sudler products is in a paper of Erdős and Szekeres. There it is proved that

Theorem (Erdős, Szekeres)

$$\liminf_{N\to\infty} P_N(\alpha) = 0 \text{ and } \limsup_{N\to\infty} P_N(\alpha) = \infty \quad \text{for almost all } \alpha.$$

Furthermore it was conjectured that

Conjecture

$$\liminf_{N\to\infty} P_N(\alpha) = 0 \quad \textit{for all } \alpha.$$

- Erdős & Szekeres also claimed without proof that the limit

$$E := \lim_{N\to\infty} \|P_N\|_\infty^{1/N}$$

exists and $1 < E < 2$. (Here $\|\cdot\|_\infty$ denotes the supremum norm on the interval $[0,1]$.)

- A formal proof of this claim was given by Sudler, who also gave a precise formula for the limit E and provided asymptotic estimates for the points $\alpha_N \in (0,1)$ for which $\|P_N\|_\infty = P_N(\alpha_N)$.

- Although the exponential growth of $\|P_N\|_\infty$ could lead one to believe that the sequence $(P_N(\alpha))_{N=1}^{\infty}$ also exhibits the same behaviour for most values of α (from the metrical point of view), it has been shown that this is not the case.

Theorem (Lubinsky and Saff)

We have

$$\lim_{N\to\infty} P_N(\alpha)^{1/N} = 1$$

for almost all α.

Introduction

- Lubinsky proved several results which explicitly exhibit the underlying relation between the asymptotic order of magnitude of $P_N(\alpha)$ and the Diophantine properties of α as encoded in its continued fraction expansion $\alpha = [a_1, a_2, \ldots]$.

Theorem (Lubinsky)

For any $\varepsilon > 0$ we have

$$\log P_N(\alpha) \ll \log N (\log \log N)^{1+\varepsilon}, \quad N \to \infty \qquad \textit{for almost all } \alpha.$$

- On the other hand:

Theorem (Lubinsky)

We have

$$\limsup_{N\to\infty} \frac{\log P_N(\alpha)}{\log N} \geqslant 1 \qquad \text{for all irrational } \alpha.$$

- This means heuristically that the higher order of magnitude of $P_N(\alpha)$ is at least linear.

Introduction

- Since then it has been conjectured that equality is true in the aforementioned theorem, and additionally that an even stronger statement holds:

Conjecture

For all irrational α,

$$\limsup_{N\to\infty} \frac{P_N(\alpha)}{N} < \infty.$$

- The answer to this will be given later!!!

- Furthermore, Lubinsky showed that

$$\liminf_{N\to\infty} P_N(\alpha) = 0$$

for any irrational α with unbounded partial quotients in its continued fraction expansion.

- For irrational numbers α with bounded partial quotients (such numbers are called *badly approximable*) he showed that

$$\log P_N(\alpha) \ll \log N, \quad N \to \infty.$$

Introduction

- In addition to all previous results stated explicitly, Lubinsky mentions a statement as a byproduct of the proof that $\liminf P_N(\alpha) = 0$ for irrationals with unbounded partial quotients.
- The relation
$$\liminf_{N \to \infty} P_N(\alpha) = 0$$
is actually also true for α with bounded partial quotients, provided that the partial quotients are infinitely often large enough.

- More recently, Mestel and Verschueren examined the behaviour of the sequence of Sudler products evaluated on the golden ratio $\phi = \frac{\sqrt{5}-1}{2}$.
- They established the convergence of the subsequence $P_{F_n}(\phi)$ to some positive and finite limit.

Theorem (Mestel, Verschueren)

For the sequence $P_{F_n}(\phi)$, there exists a constant $C_1 > 0$ such that

$$C_1 = \lim_{n\to\infty} P_{F_n}(\phi).$$

Moreover, for the same constant C_1 we have

$$\lim_{n\to\infty} \frac{P_{F_n-1}(\phi)}{F_n} = \frac{C_1\sqrt{5}}{2\pi}.$$

- The methods developed by Mestel & Verschueren form the basis for many subsequent works .
- Expanding these techniques, Grepstad and Neumüller showed the convergence of specific subsequences of $P_N(\alpha)$ when α is a *quadratic irrational*.
- Grepstad, Kaltenböck and Neumüller proved that for the specific case of the golden ratio we actually have:

Theorem (Grepstad, Kaltenböck and Neumüller)

$$\liminf_{N\to\infty} P_N(\phi) > 0.$$

- This gave a negative answer to the conjecture of Erdős and Szekeres!

- This last result is particularly striking, since it shows that whether α satisfies

$$\liminf_{N\to\infty} P_N(\alpha) = 0$$

or not depends on the actual size of the partial quotients of α, and not only on whether or not they are bounded.

- Sketch of the proof for the fact that $\liminf_{N\to\infty} P_N(\phi) > 0$:

Proposition (Zeckendorff representation of integers)

Let $N \geqslant 1$ be a positive integer. Then N can be written in a unique manner in the form

$$N = F_{n_k} + F_{n_{k-1}} + \ldots + F_{n_1},$$

where the integers $1 \leqslant n_1 \leqslant \ldots \leqslant n_k$ are such that $n_{i+1} - n_i \geqslant 2$ for all $i = 1, 2, \ldots, k-1$.

- Given $N \geqslant 1$, Grepstadt, Kaltenböck & Neumüller expand N in its Zeckendorff representation and then write

$$P_N(\phi) = \left(\prod_{r=1}^{F_{n_k}} 2|\sin \pi r\phi|\right) \left(\prod_{r=F_{n_k}+1}^{F_{n_k}+F_{n_{k-1}}} 2|\sin \pi r\phi|\right) \cdots \left(\prod_{r=F_{n_k}+\cdots+F_{n_2}+1}^{F_{n_k}+\cdots+F_{n_1}} 2|\sin \pi r\phi|\right).$$

- The first product on the right-hand side is equal to $P_{F_{n_k}}(\phi)$. By Mestel and Verschueren's result it converges to C_1.
- The second product can be rewritten as

$$\prod_{r=F_{n_k}+1}^{F_{n_k}+F_{n_k-1}} 2|\sin \pi r\phi| = \prod_{r=1}^{F_{n_{k-1}}} 2|\sin \pi(r\phi + F_{n_k}\phi)|,$$

so it is of the form $\prod_{r=1}^{F_n} 2|\sin \pi(r\phi + \varepsilon)|$.

- Similarly for the remaining products.
- The result follows provided that products of the form $\prod_{r=1}^{F_n} 2|\sin \pi(r\phi + \varepsilon)|$ are > 1 within some certain range of the perturbation argument ε.

Following the approach of Grepstad, Kaltenböck and Neumüller, we define perturbed Sudler products of the form

$$P_{F_n}(\phi, \varepsilon) = \prod_{r=1}^{F_n} 2 \left| \sin \pi \left(r\phi + (-1)^{n+1} \frac{\varepsilon}{F_n} \right) \right| .$$

- For the sake of convenience we use the notation

$$u(r) = 2\sqrt{5}\left(r - \frac{1}{\sqrt{5}}\left(\{r\phi\} - \frac{1}{2}\right)\right), \quad r = 1, 2, \ldots$$

Theorem (Aistleitner, Technau, Z.)

For every $\varepsilon \in \mathbb{R}$, the limit $\lim_{n \to \infty} P_{F_n}(\phi, \varepsilon)$ exists and is equal to

$$G(\varepsilon) = K|\varepsilon\sqrt{5} + 1| \cdot \prod_{r=1}^{\infty} \left| 1 - \frac{(2\varepsilon\sqrt{5} + 1)^2}{u(r)^2} \right|,$$

where $K > 0$ is some absolute constant.

- Furthermore, we are able to determine the precise value of the constant C_1 in Mestel & Verschueren's Theorem.

Theorem (Aistleitner, Technau, Z.)

Let $C_1 = \lim_{n \to \infty} P_{F_n}(\phi)$ *. Then*

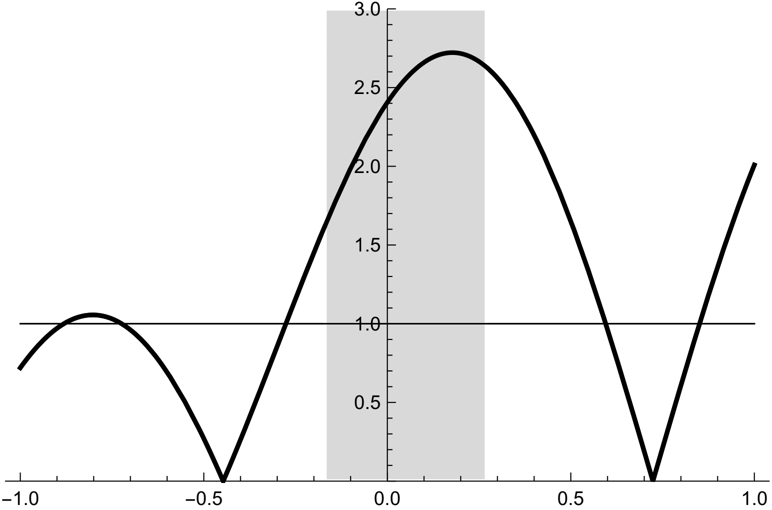
$$C_1 = (1+\phi) \cdot \prod_{r=1}^{\infty} \left(1 - \frac{1}{u(r)^2}\right) \left(1 - \frac{(1+2\phi)^2}{u(r)^2}\right)^{-1}.$$

Since $G(0) = C_1$, we can calculate the constant K in the Theorem and obtain an explicit formula for the function G.

Corollary

The limiting function G satisfies

$$G(\varepsilon) = (1+\phi)|\varepsilon\sqrt{5}+1| \cdot \prod_{r=1}^{\infty} \left|1 - \frac{(2\varepsilon\sqrt{5}+1)^2}{u(r)^2}\right| \left|1 - \frac{(2\phi+1)^2}{u(r)^2}\right|^{-1}.$$



3.0
2.5
2.0
1.5
1.0
0.5
−1.0
−0.5
0.0
0.5
1.0

- This enables us to show the following fact.

Theorem (Aistleitner, Technau, Z.)

We have

$$\limsup_{N\to\infty} \frac{P_N(\phi)}{N} < \infty.$$

- The proof of this theorem is a natural dual of the proof of $\liminf P_N(\phi) > 0$, sketched above. When n is such that $F_{n-1} \leqslant N+1 < F_n$ we write

$$\begin{aligned} P_N(\phi) &= P_{F_n - 1}(\phi) / \prod_{r=N+1}^{F_n - 1} 2|\sin \pi r \phi| \\ &= P_{F_n - 1}(\phi) / \prod_{r=1}^{F_n - N - 1} 2|\sin \pi (r - F_n)\phi|, \end{aligned}$$

  where $\|F_n \phi\|$ is "small".
- The motivation for this decomposition is that since the quotients $P_{F_n - 1}(\phi)/F_n$ converge and since $F_{n-1} \leqslant N+1 < F_n$ we can show that $P_{F_n - 1}(\phi)/N$ will be bounded from above.
- Thus establishing an upper bound for $P_N(\phi)/N$ reduces to bounding a perturbed Sudler product from below. This can be done by showing that $G(\varepsilon) > 1$ in an appropriate range of values of ε.

- Is it true that

$$\limsup_{N\to\infty} \frac{P_N(\alpha)}{N} < \infty$$

for all irrationals α?

- We have already seen that the asymptotic behaviour of the Sudler product might depend on the actual size of the partial quotients.

For any positive integer $b \geq 1$, let

$$\beta = \beta(b) = [b, b, b, \ldots]$$

be the quadratic irrational with all partial quotients in its continued fraction expansion equal to b.

- The theorem of Mestel and Verschueren can be generalised for the quadratic irrational β as follows.

Theorem

Let $\beta = [b, b, \ldots]$ and $(q_n)_{n=1}^{\infty}$ be the sequence of denominators associated with its continued fraction expansion. There exists a constant $C_b > 0$ such that

$$C_b = \lim_{n\to\infty} P_{q_n}(\beta).$$

Moreover, for the constant C_b we have

$$\lim_{n\to\infty} \frac{P_{q_n-1}(\beta)}{q_n} = \frac{C_b\sqrt{b^2+4}}{2\pi}.$$

- The first relation is a special case of the result by Grepstad and Neumüller for arbitrary quadratic irrationals. The second limit follows then if we adapt an argument by Mestel and Verschueren.

We define

$$P_{q_n}(\beta,\varepsilon) = \prod_{r=1}^{q_n} 2\left|\sin\pi\left(r\beta + (-1)^n\frac{\varepsilon}{q_n}\right)\right|.$$

Theorem (Aistleitner, Technau, Z.)

For every $\varepsilon \in \mathbb{R}$, the limit $\lim_{n\to\infty} P_{q_n}(\beta,\varepsilon)$ exists and is equal to

$$G_\beta(\varepsilon) = K_b \cdot |\varepsilon\sqrt{b^2+4}+1| \cdot \prod_{r=1}^{\infty}\left|1 - \frac{(2\varepsilon\sqrt{b^2+4}+1)^2}{u_b(r)^2}\right|,$$

where $K_b > 0$ is some absolute constant.

(Here $(q_n)_{n=1}^{\infty}$ denotes the)

Theorem (Aistleitner, Technau, Z.)

Let $C_b = \lim_{n\to\infty} P_{q_n}(\beta)$. *Then*

$$C_b^b = \frac{1}{\beta(\beta+1)\cdots(\beta+b-1)} \cdot \prod_{r=1}^{\infty}\prod_{j=1}^{b}\left(1-\frac{1}{u_b(r)^2}\right)\left(1-\frac{(2b-2j+2\beta+1)^2}{u_b(r)^2}\right)^{-1}.$$

Corollary

The limiting function $G_\beta(\varepsilon)$ *satisfies*

$$G_\beta(\varepsilon)^b = \frac{|\varepsilon\sqrt{b^2+4}+1|^b}{\beta\cdots(\beta+b-1)} \cdot \prod_{r=1}^{\infty}\prod_{j=1}^{b}\frac{\left|1-\frac{(2\varepsilon\sqrt{b^2+4}+1)^2}{u_b(r)^2}\right|}{\left|1-\frac{(2b-2j+2\beta+1)^2}{u_b(r)^2}\right|}.$$

The value of the constant C_b may not be of interest on its own, but it turns out to provide information on the order of magnitude of $P_N(\beta)$.

Proposition (Aistleitner, Technau, Z.)

If for the irrational $b = [0; b, b, b, \ldots]$ we have $C_b < 1$ then

$$\liminf_{N\to\infty} P_N(\beta) = 0 \text{ and } \limsup_{N\to\infty} \frac{P_N(\beta)}{N} = \infty.$$

The following theorem provides a necessary and sufficient condition so that $P_N(\beta)$ satisfies the linear growth condition we have already proved for the specific case $\beta = \phi$.

Theorem

Let $\beta = [b, b, b, \dots]$, where $b \in \mathbb{N}$ is a positive integer. Then the following holds.

(i) *If $b \leqslant 5$, then* $\liminf_{N\to\infty} P_N(\beta) > 0$ *and* $\limsup_{N\to\infty} \frac{P_N(\beta)}{N} < \infty$.

(ii) *If $b \geqslant 6$, then* $\liminf_{N\to\infty} P_N(\beta) = 0$ *and* $\limsup_{N\to\infty} \frac{P_N(\beta)}{N} = \infty$.

We have determined the behaviour of the Sudler product when the argument is a quadratic irrational of the form $\beta = [b, b, b, \ldots]$, but what happens when the argument is an arbitrary quadratic irrational?

Recall that an arbitrary quadratic irrational has a periodic continued fraction expansion, i.e. of the form

$$\alpha = [0; a_1, \ldots, a_k, \overline{a_{k+1}, a_{k+2}, \ldots, a_{k+\ell}}].$$

We first examine the purely periodic quadratic irrationals, i.e. those with continued fraction expansions of the form $\alpha = [0; \overline{a_1, a_2, \ldots, a_\ell}]$. The analogue of the theorem of Mestel and Verschueren is the following.

Theorem (Grepstad, Neumüller)

Let $\alpha = [0; \overline{a_1, a_2, \ldots, a_\ell}]$ be a purely periodic quadratic irrational, where $\ell \geqslant 1$ and $a_1, \ldots, a_\ell \in \mathbb{N}$, and let $(q_n)_{n=1}^{\infty}$ be the sequence of denominators of convergents of α. Then there exist constants $C_1, C_2, \ldots, C_\ell > 0$ such that

$$\lim_{m \to \infty} P_{q_{m\ell+k}}(\alpha) = C_k, \qquad k = 1, 2, \ldots, \ell.$$

The study of $P_N(\alpha)$ again passes through the study of perturbed products

$$P_{q_n}(\alpha, \varepsilon) := \prod_{r=1}^{q_n} 2\Big|\sin\pi\Big(r\alpha + (-1)^{n+1}\frac{\varepsilon}{q_n}\Big)\Big|.$$

For these products, we are able to show the following convergence result along subsequences.

Theorem (Grepstad, Neumüller, Z.)

Let $\alpha = [0; \overline{a_1, \dots, a_\ell}]$ where $\ell, a_1, \dots, a_\ell \in \mathbb{N}$ and $P_{q_n}(\alpha, \varepsilon)$ is the sequence of perturbed Sudler products. Then for each $k = 1, \dots, \ell$ the subsequence $P_{q_{m\ell+k}}(\alpha, \varepsilon)$ converges locally uniformly to a function $G_k(\alpha, \varepsilon)$. The limit function satisfies

$$G_k(\alpha,\varepsilon) = \left|1 + \frac{\varepsilon}{|c_k e_k|}\right| \left(1 + \frac{1}{|b|^2}\right)^{\frac{1}{c-2}} \frac{1}{(c!)^{1/(c-2)}} \times$$

$$\times \prod_{t=1}^{\infty} \left| \left(1 - \frac{\left(1 + \frac{2\varepsilon}{|e_k c_k|}\right)^2}{u_k(t)^2}\right) \left(1 - \frac{\left(1 + \frac{2}{|b|^2}\right)^2}{u_k(t)^2}\right)^{\frac{1}{c-2}} \right|$$

$$\prod_{s=1}^{c-1} \left|1 - \frac{(1+2s)^2}{u_k(t)^2}\right|^{-\frac{1}{c-2}}$$

when ℓ is even,

Theorem (Grepstad, Neumüller, Z.)

and

$$G_k(\alpha,\varepsilon)=\frac{\left|1+\frac{\varepsilon}{|c_k e_k|}\right|}{\prod_{s=1}^{c}|s-a|^{\frac{1}{c}}}\times\prod_{t=1}^{\infty}\left|1-\frac{\left(1+\frac{2\varepsilon}{|e_k c_k|}\right)^2}{u_k(t)^2}\right|$$

$$\prod_{s=0}^{c-1}\left|1-\frac{\left(1+2(s-\frac{1}{|b|})\right)^2}{u_k(t)^2}\right|^{-\frac{1}{c}}$$

when ℓ is odd.

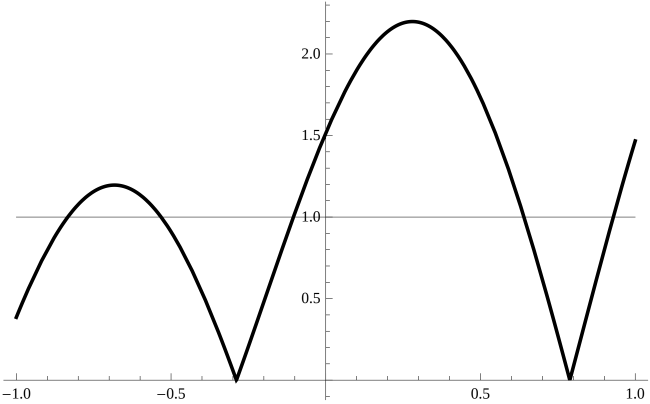
As in the case of period$-\,-1$ irrationals, we have the following phenomenon.

Theorem (Grepstad, Neumüler, Z.)

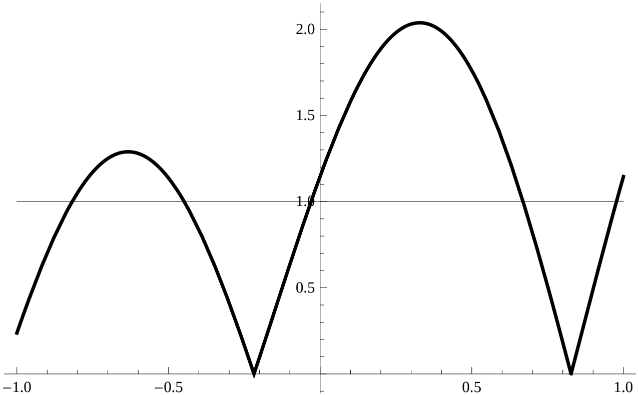
Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ *and* $(C_k)_{k=1}^{\ell}$ *be the limits of the* k *subsequences of the perturbed Sudler product. If* $C_{k_0} < 1$ *for some index* $1 \leqslant k_0 \leqslant \ell$ *then*

$$\liminf_{N\to\infty} P_N(\alpha) = 0 \qquad \text{and} \qquad \limsup_{N\to\infty} \frac{P_N(\alpha)}{N} = \infty. \qquad (0.1)$$

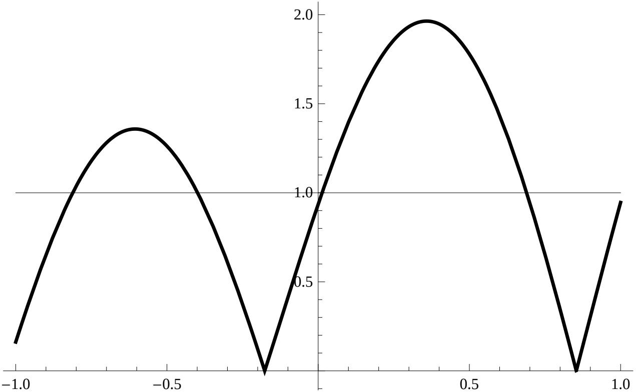
This Theorem tells us that as long as *one* of the constants C_k $(1 \leq k \leq \ell)$ is less than 1, the Sudler product corresponding to the irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ satisfies (0.1). This raises the question of which out of the ℓ constants C_k associated with α is expected to be minimal.



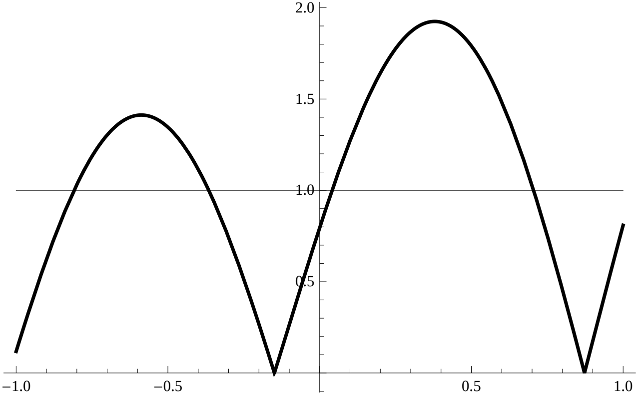
Plot of the limit function for $\alpha = [0; \overline{1, 2}]$



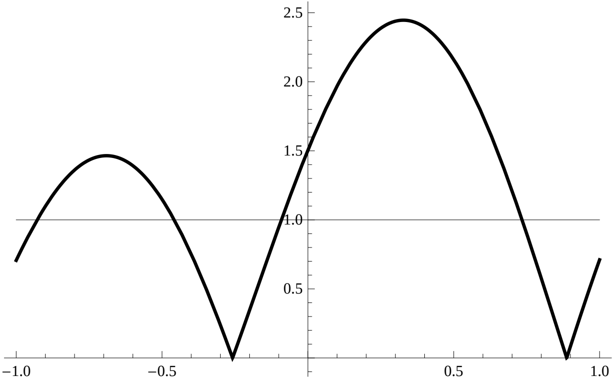
Plot of the limit function for $\alpha = [0; \overline{1, 3}]$



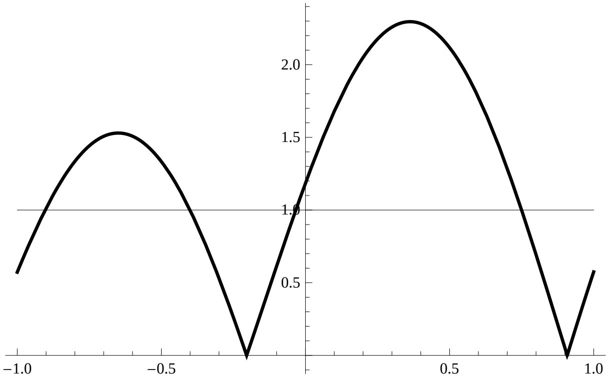
Plot of the limit function for $\alpha = [0; \overline{1, 4}]$



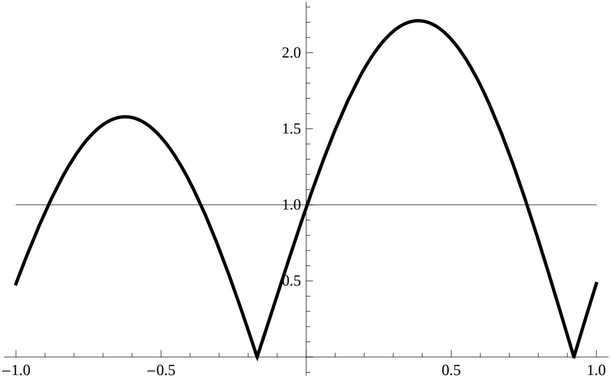
Plot of the limit function for $\alpha = [0; \overline{1, 5}]$



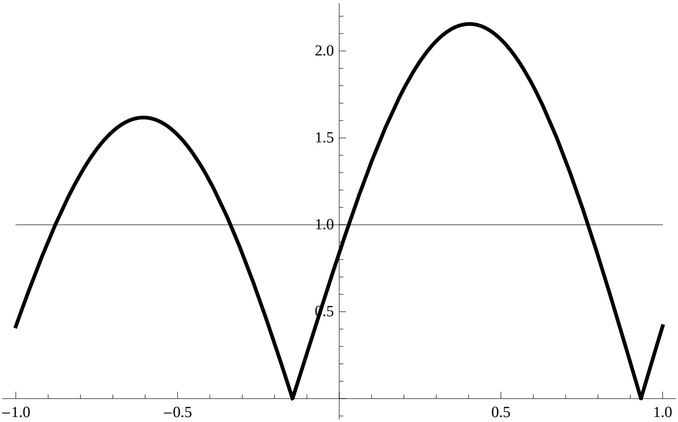
Plot of the limit function for $\alpha = [0; \overline{2, 3}]$



Plot of the limit function for $\alpha = [0; \overline{2, 4}]$



Plot of the limit function for $\alpha = [0; \overline{2,5}]$



Plot of the limit function for $\alpha = [0; \overline{2, 6}]$

The following questions come up naturally.

- Is the phenomenon implied by the graphs indeed true, i.e. for any index $1 \leqslant k \leqslant \ell$, is C_k decreasing as a function of the digit a_k?

- Suppose we fix some period length $\ell \geqslant 2$. Does there exist an integer $K = K_\ell \geq 1$ such that for any irrational $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ with $\max_{1 \leqslant i \leqslant \ell} a_i \geqslant K$ the Sudler product $P_N(\alpha)$ satisfies $\liminf_{N\to\infty} P_N(\alpha) = 0$ and $\limsup\limits_{N\to\infty} \frac{P_N(\alpha)}{N} = \infty$?

- If such an integer exists, can it be chosen independently of the period length ℓ?

Theorem (Grepstad, Neumüller, Z.)

Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ *with period length* $\ell \geqslant 2$*, and say* $a_k = \max_j a_j$*. Then*

$$\liminf_{N \to \infty} P_N(\alpha) = 0 \qquad \textit{and} \qquad \limsup_{N \to \infty} \frac{P_N(\alpha)}{N} = \infty,$$

whenever $a_k \geqslant 23$*.*

Some even more recent advances:
Aistleitner and Borda have shown a general dichotomy for the asymptotic behaviour of Sudler products.

Theorem (Aistleitner, Borda)

Let $\alpha \in (0, 1)$ be an arbitrary irrational. We have

$$\liminf_{N\to\infty} P_N(\alpha) > 0 \quad \Leftrightarrow \quad \limsup_{N\to\infty} \frac{P_N(\alpha)}{N} < \infty.$$

In a paper to appear, Hauke has improved the threshold value for arbitrary quadratic irrationals.

Theorem (Hauke)

Let $\alpha = [0; \overline{a_1, \ldots, a_\ell}]$ *with period length* $\ell \geqslant 2$*, and say* $a_k = \max_j a_j$*. If* $a_k \geqslant 8$*, then*

$$\liminf_{N\to\infty} P_N(\alpha) = 0 \qquad \textit{and} \qquad \limsup_{N\to\infty} \frac{P_N(\alpha)}{N} = \infty.$$

End

Thank you for your attention!