Random β-transformation on fat Sierpiński gasket

Tingyu Zhang
tingyuzhangecnu@163.com

Department of Mathematics
East China Normal University, Shanghai, China

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Contents

β-transformations

- Let $\beta > 1$.
- The greedy transformation T_β [Rényi1957] on $[0,1)$:

$$T_\beta x = \beta x - \lfloor \beta x \rfloor = \beta x \pmod 1.$$

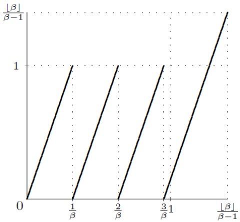
- T_β-invariant Parry measure;[Gelfond1959, Parry1960]
- unique measure that is equivalent to Lebesgue measure;[Rényi1957,Parry1960]
- unique T_β-invariant measure of maximal entropy.[Hofbauer1980]

β-transformations

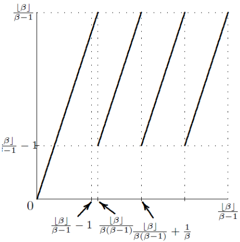
- The greedy transformation T_β on $[0, \frac{\lfloor\beta\rfloor}{\beta-1})$:

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & \text{if } 0 < x < 1, \\ \beta x - \lfloor\beta\rfloor, & \text{if } 1 \le x < \frac{\lfloor\beta\rfloor}{\beta-1}. \end{cases}$$

- The lazy transformation L_β on $[0, \frac{\lfloor\beta\rfloor}{\beta-1})$:

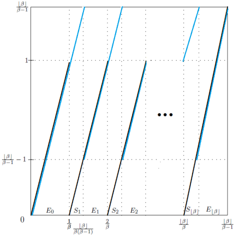


(a) the greedy map T_β



(b) the lazy map L_β

β-transformations



- $(\{0,1\}^{\mathbb{N}} \times [0, \frac{[\beta]}{\beta-1}], K_\beta)$ [Dajani & Kraaikamp, 2003]
-

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - l), & \text{if } x \in E_l, l = 0, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - l), & \text{if } \omega_1 = 1, \text{and } x \in S_l, l = 1, \dots, \lfloor\beta\rfloor, \\ (\sigma(\omega), \beta x - l + 1), & \text{if } \omega_1 = 0, \text{and } x \in S_l, l = 1, \dots, \lfloor\beta\rfloor. \end{cases}$$

β-transformations

- $(\{0,1\}^{\mathbb{N}} \times [0, \frac{\lfloor\beta\rfloor}{\beta-1}], K_\beta)$[Dajani & Kraaikamp, 2003]
  - unique K_β-invariant measure of maximal entropy;[Dajani & de Vries, 2005]
  - unique K_β-invariant probability measure, absolutely continuous with respect to $m_p \otimes \lambda_1$; the measure of maximal entropy and $m_p \otimes \lambda_1$ are mutually singular.[Dajani & de Vries, 2007]
- Question: β-transformations on $\mathbb{R}^2$?

Sierpinski gasket

- Let $\beta > 1$. Consider the *iterated function system*(IFS):

$$f_{\vec{q}_0}(\vec{z}) = \frac{\vec{z}+\vec{q}_0}{\beta}, f_{\vec{q}_1}(\vec{z}) = \frac{\vec{z}+\vec{q}_1}{\beta}, f_{\vec{q}_2}(\vec{z}) = \frac{\vec{z}+\vec{q}_2}{\beta},$$

where $\vec{q}_0, \vec{q}_1, \vec{q}_2$ are $(0,0), (1,0), (0,1)$, respectively. $\vec{q}_0 < \vec{q}_1 < \vec{q}_2$.

- There exists a unique non-empty compact set $S_\beta \subset \mathbb{R}^2$ such that

$$S_\beta = f_{\vec{q}_0}(S_\beta) \cup f_{\vec{q}_1}(S_\beta) \cup f_{\vec{q}_2}(S_\beta).$$

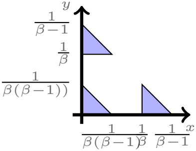
- For $\vec{z} \in S_\beta$, there exists a sequence $(a_i)_{i=1}^\infty \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}}$ such that

$$\vec{z} = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(\vec{q}_0) = \sum_{i=1}^\infty \frac{a_i}{\beta^i}.$$

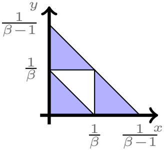
We call $(a_i)_{i=1}^\infty$ a *coding* of $\vec{z}$ and $\sum_{i=1}^\infty a_i\beta^{-i}$ a *representation* of $\vec{z}$ in base β.

For different β

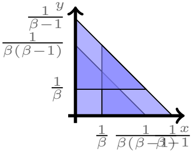
Δ : the convex hull of S_β;
a triangle with vertices at $(0,0), (\frac{1}{\beta-1}, 0)$ and $(0, \frac{1}{\beta-1})$.



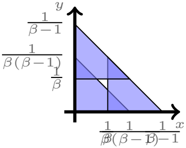
(c) $f_{\vec{q}_i}(\Delta)$ for $\beta > 2$



(d) $f_{\vec{q}_i}(\Delta)$ for $\beta = 2$



(e) $f_{\vec{q}_i}(\Delta)$ for $1 < \beta \leq \frac{3}{2}$



(f) $f_{\vec{q}_i}(\Delta)$ for $\frac{3}{2} < \beta \leq 2$

For $\beta > 2$

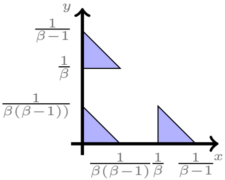


Figure: $f_{\vec{q}_i}(\Delta)$ for $\beta > 2$

Since $S_\beta = f_{\vec{q}_0}(S_\beta) \cup f_{\vec{q}_1}(S_\beta) \cup f_{\vec{q}_2}(S_\beta)$ and $f_{\vec{q}_i}(S_\beta)$ are disjoint, then we define

$$T(\vec{z}) = \begin{cases} f_{\vec{q}_0}^{-1}\vec{z}, & \text{if } \vec{z} \in f_{\vec{q}_0}(S_\beta), \\ f_{\vec{q}_1}^{-1}\vec{z}, & \text{if } \vec{z} \in f_{\vec{q}_1}(S_\beta), \\ f_{\vec{q}_2}^{-1}\vec{z}, & \text{if } \vec{z} \in f_{\vec{q}_2}(S_\beta). \end{cases}$$

Question: how is the transformation like if there is overlap?

Transformations for $1 < \beta \leq 3/2$

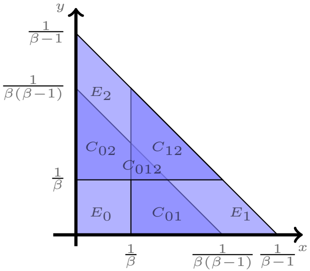


Figure: $f_{\vec{q}_i}(\Delta)$ for $1 < \beta \leq \frac{3}{2}$

The greedy transformation T_β from S_β into S_β is given by

$$T_\beta(\vec{z}) = \begin{cases} f_{\vec{q}_0}^{-1}\vec{z}, & \text{if } \vec{z} \in E_0, \\ f_{\vec{q}_1}^{-1}\vec{z}, & \text{if } \vec{z} \in C_{01} \cup E_1, \\ f_{\vec{q}_2}^{-1}\vec{z}, & \text{if } \vec{z} \in C_{012} \cup C_{12} \cup C_{02} \cup E_2. \end{cases}$$

Transformations for $1 < \beta \le 3/2$

The lazy transformation L_β from S_β into S_β is given by

$$L_\beta(\vec{z}) = \begin{cases} f_{\vec{q}_0}^{-1}\vec{z}, & \text{if } \vec{z} \in C_{012} \cup C_{01} \cup C_{02} \cup E_0, \\ f_{\vec{q}_1}^{-1}\vec{z}, & \text{if } \vec{z} \in C_{12} \cup E_1, \\ f_{\vec{q}_2}^{-1}\vec{z}, & \text{if } \vec{z} \in E_2. \end{cases}$$

Let μ be an arbitrary T_β-invariant probability measure on $(S_\beta, \mathcal{S})$.

Proposition

For $\beta \in (1, 3/2]$, the systems $(S_\beta, \mathcal{S}, \mu, T_\beta)$ and $(S_\beta, \mathcal{S}, \nu, L_\beta)$ are isomorphic, where $\nu = \mu \circ \psi^{-1}$.

Transformations for $1 < \beta \le 3/2$

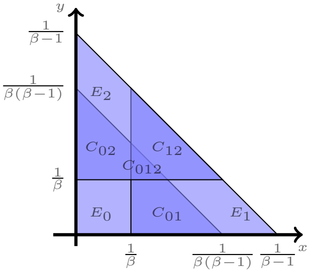


Figure: $f_{\vec{q}_i}(\Delta)$ for $1 < \beta \le \frac{3}{2}$

- To define a random transformation, we need to randomly select the map used in the switch regions C_{ij} and C_{012}.
- Inspired by the method in [Dajani & Kraaikamp, 2003], we introduce two symbolic spaces: $\Omega = \{0,1\}^{\mathbb{N}}$ and $\Upsilon = \{0,1,2\}^{\mathbb{N}}$.

Transformations for $1 < \beta \leq 3/2$

Define random transformation K_β on $\Omega \times \Upsilon \times S_\beta$

$$K_\beta(\omega, \upsilon, \vec{z}) = \begin{cases} (\omega, \upsilon, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ (\sigma\omega, \upsilon, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \omega_1 = 0,\ \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}, \\ (\sigma\omega, \upsilon, f_{\vec{q}_j}^{-1}\vec{z}), & \text{if } \omega_1 = 1,\ \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}, \\ (\omega, \sigma'\upsilon, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \vec{z} \in C_{012},\ \upsilon_1 = i \in \{0, 1, 2\}. \end{cases}$$

For these transformations, we consider

- the existence of acim;
- the existence of the measure of maximal entropy.

For $1 < \beta < 3/2$

Theorem (Main results-1)

T_β admits an acim. And so does L_β.

Theorem (Main results-2)

Let $\beta \in (1, 3/2)$. Then K_β has an invariant measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where μ_β is absolutely continuous with respect to λ_2.

Theorem (Main results-3)

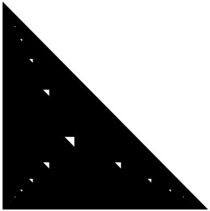
Let $\beta \in (1, \beta_*)$, where $\beta_* \approx 1.4656$ is the root of $x^3 - x^2 - 1 = 0$. The measure $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$ is the unique K_β-invariant measure of maximal entropy.

For $3/2 < \beta \leq \beta^*$

Broomhead, Montaldi and Sidorov(2004) gave a special class of Sierpinski gasket with positive Hausdorff measure. By a simple affine map, we have this proposition.

Proposition

Let $\beta^* \approx 1.5437$ be the root of $x^3 - 2x^2 + 2x = 2$. Then S_β has a non-empty interior if $\beta \in (\frac{3}{2}, \beta^*]$, and each hole has the form $f^n_{\vec{q}_i}(H)$, see the following figure.



For $3/2 < \beta \leq \beta^*$

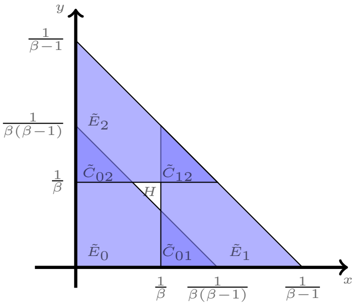


Figure: $f_{\vec{q}_i}(\Delta)$ for $\frac{3}{2} < \beta \leq \beta^*$

- The triple overlapping region C_{012} disappears.
- S_β has holes and duplex overlapping areas.

For $3/2 < \beta \le \beta^*$

Define the random transformation $K_\beta : \Omega \times S_\beta \to \Omega \times S_\beta$:

$$K_\beta(\omega, \vec{z}) = \begin{cases} (\omega, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ (\sigma\omega, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \omega_1 = 0, \text{and } \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}, \\ (\sigma\omega, f_{\vec{q}_j}^{-1}\vec{z}), & \text{if } \omega_1 = 1, \text{and } \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}. \end{cases}$$

Theorem (Main results-4)

Let $3/2 < \beta \le \beta^*$. The dynamical systems $(\Omega \times S_\beta, \mathcal{A} \times \mathcal{S}, \nu_\beta, K_\beta)$ and $(\Upsilon, \mathcal{B}, \mathbb{P}, \sigma')$ are isomorphic. Moreover, the measure ν_β is the unique K_β-invariant measure of maximal entropy.

Piecewise C^2 and expanding maps

Let S be a bounded region in $\mathbb{R}^N$.
Let $\mathcal{P} = \{S_1, S_2, \ldots, S_m\}$ be a finite partition of S.
Let τ be a transformation from S into S.
We say τ is piecewise C^2 and expanding with respect to $\mathcal{P}$ if:

- each S_i is a bounded closed domain having a piecewise C^2 boundary of finite $(N-1)$-dimensional measure;
- $\tau_i = \tau|_{S_i}$ is a C^2, 1-1 transformation from $\mathrm{int}(S_i)$ onto its image and can be extended as a C^2 transformation onto S_i;
- there exists $0 < c < 1$ such that for any $i = 1, 2, \ldots, m$,

$$\|D\tau_i^{-1}\| < c,$$

  where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\|\cdot\|$ is the euclidean matrix norm, i.e., $\|A\| = (\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2)^{\frac{1}{2}}$ for $m \times n$ order matrix $A = (a_{ij})$.

Assume that the faces of ∂S_i meet at angles bounded uniformly away from 0.

Theorem (Boyarsky & Góra 1989, Corollary 1)

Let $\tau : S \to S, S \subset \mathbb{R}^N$, be piecewise C^2 and such that some iterate τ^k satisfies $c(1 + 1/a) < 1$(c and a corresponds to τ^k), then τ admits an absolutely continuous invariant measure (acim).

Then we can prove the first main result.

Theorem (Main results-1)

T_β admits an acim. And so does L_β.

Theorem (Main results-2)

Let $\beta \in (1, 3/2)$. Then K_β has an invariant measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where μ_β is absolutely continuous with respect to λ_2.

- Idea: Bahsoun and Góra (2005), gave a sufficient condition for the existence of an acim for a random map with position dependent probabilities on a bounded domain of $\mathbb{R}^N$.
- Steps:
  - step 1: a random map R with position dependent probabilities on S_β;
  - step 2: for the skew product transformation R' on $S_\beta \times [0, 1)$;
  - step 3: for the skew product transformation R_β on $\Omega \times \Upsilon \times S_\beta$;
  - step 4: for the random transformation K_β on $\Omega \times \Upsilon \times S_\beta$.

The position dependent random map R

Let $\mathcal{P}$ be a partition of $S_\beta : \mathcal{P} = \{S_1, \dots, S_q\}$.
For $k = 1, \dots, K$, let $\tau_k : S_\beta \to S_\beta$ be piecewise one-to-one and C^2 , non-singular maps.
Let $p_k : S_\beta \to [0, 1]$ be piecewise C^1 functions such that $\sum_{k=1}^{K} p_k = 1$.

Denote the position dependent random map by

$$R = \{\tau_1, \dots, \tau_K; p_1(\vec{z}), \dots, p_K(\vec{z})\},$$

which means $R(\vec{z}) = \tau_k(\vec{z})$ with probability $p_k(\vec{z})$.

The Perron-Frobenius operator P_R

The transition function for R:

$$\mathbf{P}(\vec{z}, A) = \sum_{k=1}^{K} p_k(\vec{z}) \mathbb{1}_A(\tau_k(\vec{z})) \leftrightsquigarrow T^{-1}(A)$$

$\dashrightarrow$ An operator $\mathbf{P}_*$ on the set of probability measure on $(S_\beta, \mathcal{S})$:

$$\mathbf{P}_*\mu(A) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(\vec{z}) d\mu(\vec{z}) \leftrightsquigarrow \mu(T^{-1}(A))$$

$\dashrightarrow$ The Perron-Frobenius operator of the random map R:

$$\int_A P_R f(\vec{z}) d\lambda_2(\vec{z}) = \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(\vec{z}) f(\vec{z}) d\lambda_2(\vec{z}).$$

We call P_R the Perron-Frobenius operator of the random map R.

The Perron-Frobenius operator P_R

The properties of P_R:

- P_R is linear;
- P_R is non-negative;
- $P_R f = f \iff \mu = f \cdot \lambda_2$ is R-invariant;
- $\|P_R f\|_1 \leq \|f\|_1$, where $\|\cdot\|_1$ denotes the L^1 norm;
- $P_{R\circ T} = P_R \circ P_T$. In particular, $P_R^N = P_{R^N}$.

Assume:

- the faces of ∂S_i meet at angles bounded uniformly away from 0;
- the probabilities $p_k(\vec{z})$ are piecewise C^1 functions on the partition $\mathcal{P}$;
- **Condition(A):**

$$\max_{1\leq i\leq q} \sum_{k=1}^{K} p_k(\vec{z}) \|D\tau_{k,i}^{-1}(\tau_{k,i}(\vec{z}))\| < c < 1,$$

where $D\tau_{k,i}^{-1}(\vec{z})$ is the derivative matrix of $\tau_{k,i}^{-1}$ at $\vec{z}$.

Using the multidimensional notion of variation [Giusti1984]:

$$V(f) = \int_{\mathbb{R}^N} \|Df\| d\lambda_N$$
$$= \sup \left\{ \int_{\mathbb{R}^N} f \mathrm{div}(g) d\lambda_N : g = (g_1, \ldots, g_N) \in C_0^1(\mathbb{R}^N, \mathbb{R}^N) \right\}$$

Let $\vec{z}$ be a point in ∂S_i and $J_{k,i}$ the Jacobian of $\tau_{k|S_i}$ at $\vec{z}$.
We recall the following theorem.

Theorem (Bahsoun & Góra 2005, Theorem 6.3)

If R is a random map which satisfies Condition (A), then

$$V(P_R f) \leq c(1 + 1/a)V(f) + (M + \frac{c}{a\delta})\|f\|_1,$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0, \delta = \min\{\delta(S_i) : i = 1 \ldots, q\} > 0, M_{k,i} = \sup_{\vec{z} \in S_i}(Dp_k(\vec{z}) - \frac{DJ_{k,i}}{J_{k,i}} p_k(\vec{z}))$ *and* $M = \sum_{k=1}^{K} \max_{1 \leq i \leq q} M_{k,i}$.

Consider the Banach space $BV(S_\beta)$ with the norm $\|f\|_{BV} = \|f\|_{L_1} + V(f)$.

Step 1: a random map R with position dependent probabilities on S_β

Define the piecewise one-to-one and C^2, non-singular transformations

$$\tau_1(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, \\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, \\ \beta\vec{z}, & \text{if } \vec{z} \in C_{012}, \end{cases} \qquad \tau_4(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, \\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, \\ \beta\vec{z}, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_2(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, \\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, \\ \beta\vec{z} - \vec{q}_1, & \text{if } \vec{z} \in C_{012}, \end{cases} \qquad \tau_5(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, \\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, \\ \beta\vec{z} - \vec{q}_1, & \text{if } \vec{z} \in C_{012}, \end{cases}$$

$$\tau_3(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, \\ \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in C_{ij}, \\ \beta\vec{z} - \vec{q}_2, & \text{if } \vec{z} \in C_{012}, \end{cases} \qquad \tau_6(\vec{z}) = \begin{cases} \beta\vec{z} - \vec{q}_i, & \text{if } \vec{z} \in E_i, \\ \beta\vec{z} - \vec{q}_j, & \text{if } \vec{z} \in C_{ij}, \\ \beta\vec{z} - \vec{q}_2, & \text{if } \vec{z} \in C_{012}. \end{cases}$$

Define the probabilities

$$\begin{aligned} p_1(\vec{z}) &= p \cdot s, & p_4(\vec{z}) &= (1-p) \cdot s, \\ p_2(\vec{z}) &= p \cdot t, & p_5(\vec{z}) &= (1-p) \cdot t, \\ p_3(\vec{z}) &= p \cdot (1-s-t), & p_6(\vec{z}) &= (1-p) \cdot (1-s-t). \end{aligned}$$

Lemma

Let R be a random map which is given by $\{\tau_1, \ldots, \tau_6; p_1(\vec{z}), \ldots, p_6(\vec{z})\}$. Then R admits an acim μ_β.

Step 2: for the skew product transformation R' on $S_\beta \times [0,1)$

Let $J_k = \{(\vec{z}, w) : \sum_{i<k} p_i(\vec{z}) \le w < \sum_{i\le k} p_i(\vec{z})\}$.
Define $\varphi_k : J_k \to I$ as follows

$$\varphi_k(\vec{z}, w) = \frac{1}{p_k(\vec{z})} w - \frac{\sum_{r=1}^{k-1} p_r(\vec{z})}{p_k(\vec{z})}.$$

Define the skew product transformation R' on $S_\beta \times [0,1)$

$$R'(\vec{z}, w) = (\tau_k(x), \varphi_k(\vec{z}, w)), \quad (\vec{z}, w) \in J_k.$$

Lemma (Bahsoun, Bose & Quas, 2012)

μ_β is invariant for the random map R if and only if $\mu_\beta \otimes \lambda_1$ is invariant for the skew product R'.

Step 3: for the skew product transformation R_β on $\Omega \times \Upsilon \times S_\beta$

Define

$$R_\beta(\omega, \upsilon, \vec{z}) = \begin{cases} (\sigma\omega, \sigma'\upsilon, \beta\vec{z} - \vec{q}_i), & \vec{z} \in E_i, i \in \{0,1,2\}, \\ (\sigma\omega, \sigma'\upsilon, \beta\vec{z} - \vec{q}_i), & \vec{z} \in C_{ij}, ij \in \{01, 02, 12\}, \quad \omega_1 = 0, \\ (\sigma\omega, \sigma'\upsilon, \beta\vec{z} - \vec{q}_j), & \vec{z} \in C_{ij}, ij \in \{01, 02, 12\}, \quad \omega_1 = 1, \\ (\sigma\omega, \sigma'\upsilon, \beta\vec{z} - \vec{q}_i), & \vec{z} \in C_{012}, \upsilon_1 = i, i \in \{0,1,2\}. \end{cases}$$

Lemma

$(S_\beta \times I, \mathcal{S} \times \mathcal{B}(I), \mu_\beta \otimes \lambda_1, R')$ *and* $(\Omega \times \Upsilon \times S_\beta, \mathcal{A} \times \mathcal{B} \times \mathcal{S}, m_1 \otimes m_2 \otimes \mu_\beta, R_\beta)$ *are isomorphic.*

Idea: $I \to \{0,1,2,3,4,5\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1,2\}^{\mathbb{N}}$

Step 4: for the random transformation K_β on $\Omega \times \Upsilon \times S_\beta$

Lemma

Let μ be an arbitrary probability measure on S_β. Then $m_1 \otimes m_2 \otimes \mu \circ K_\beta^{-1} = m_1 \otimes m_2 \otimes \mu \circ R_\beta^{-1}$.

Idea: It suffices to verify the measures on sets of the form $C_1 \times C_2 \times S$.

The above lemma implies that any product measure of the form $m_1 \otimes m_2 \otimes \mu$ is K_β-invariant if and only if it is R_β-invariant.
The second main result follows.

Theorem (Main results-2)

Let $\beta \in (1, 3/2)$. Then K_β has an invariant measure of the form $m_1 \otimes m_2 \otimes \mu_\beta$, where μ_β is absolutely continuous with respect to λ_2.

Basic properties of K_β for $1 < \beta \leq 3/2$

$$K_\beta(\omega, \upsilon, \vec{z}) = \begin{cases} (\omega, \upsilon, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ (\sigma\omega, \upsilon, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \omega_1 = 0,\ \vec{z} \in C_{ij},\ \ ij \in \{01, 12, 02\}, \\ (\sigma\omega, \upsilon, f_{\vec{q}_j}^{-1}\vec{z}), & \text{if } \omega_1 = 1,\ \vec{z} \in C_{ij},\ \ ij \in \{01, 12, 02\}, \\ (\omega, \sigma'\upsilon, f_{\vec{q}_i}^{-1}\vec{z}), & \text{if } \vec{z} \in C_{012},\ \ \upsilon_1 = i \in \{0, 1, 2\}. \end{cases}$$

Let

$$d_1(\omega, \upsilon, \vec{z}) = \begin{cases} \vec{q}_i, & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ & \text{or } (\omega, \upsilon, \vec{z}) \in \Omega \times \{\upsilon_1 = i\} \times C_{012}, \\ & \text{or } (\omega, \upsilon, \vec{z}) \in \{\omega_1 = 0\} \times \Upsilon \times C_{ij},\ ij \in \{01, 12, 02\}, \\ \vec{q}_j, & \text{if } (\omega, \upsilon, \vec{z}) \in \{\omega_1 = 1\} \times \Upsilon \times C_{ij},\ ij \in \{01, 12, 02\}. \end{cases}$$

Set $d_n = d_n(\omega, \upsilon, \vec{z}) = d_1(K_\beta^{n-1}(\omega, \upsilon, \vec{z}))$.

- $(d_1, d_2, d_3, \ldots)$ is a coding of $\vec{z}$.

Basic properties of K_β for $1 < \beta \leq 3/2$

Proposition

Suppose $\omega, \omega' \in \Omega, \upsilon, \upsilon' \in \Upsilon$ *are such that* $\omega \prec \omega'$ *and* $\upsilon \prec \upsilon'$. *Then for* $\vec{z} \in S_\beta$,

$$(d_1(\omega, \upsilon, \vec{z}), d_2(\omega, \upsilon, \vec{z}), \ldots) \preceq (d_1(\omega', \upsilon', \vec{z}), d_2(\omega', \upsilon', \vec{z}), \ldots).$$

Proposition

For $\beta \in (1, 3/2]$, *let* $\vec{z} \in S_\beta$ *and* $\vec{z} = \sum_{i=1}^{\infty} a_i \beta^{-i}$ *with* $a_i \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}$ *be a representation of* $\vec{z}$ *in base* β. *Then there exists an* $\omega \in \Omega$ *and an* $\upsilon \in \Upsilon$ *such that* $a_i = d_i(\omega, \upsilon, \vec{z})$.

Theorem (Main results-3)

Let $\beta \in (1, \beta_*)$. The measure $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$ is the unique K_β-invariant measure of maximal entropy.

Idea:

- $\Omega \times \Upsilon \times S_\beta \to \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}^{\mathbb{N}} \to \{0, 1, 2\}^{\mathbb{N}}$;
- $(\Upsilon, \mathcal{B}, \mathbb{P}, \sigma')$ has the maximal entropy;
- Entropy is isomorphism invariant.

Isomorphism

Two dynamical systems $(X, \mathcal{F}, \mu, Q)$ and $(Y, \mathcal{G}, \nu, U)$ are isomorphic:

- there exist measurable sets $N \subset X$ and $M \subset Y$ with
  (i)$\mu(N) = \nu(M) = 0$,
  (ii)$Q(X \setminus N) \subset X \setminus N, U(Y \setminus M) \subset Y \setminus M$,
- there exists a measurable map $\psi : X \setminus N \to Y \setminus M$ such that
  (i) ψ is one-to-one and onto,
  (ii)ψ is measurable, i.e., $\psi^{-1}(G) \in \mathcal{F}$ for all $G \in \mathcal{G}$,
  (iii)ψ preserves the measures, i.e., $\nu(G) = \mu(\psi^{-1}(G))$,
  (iv)ψ preserves the dynamics of Q and U, i.e., $\psi \circ Q = U \circ \psi$,

The map ψ is called an isomorphism.

- Define a map

$$\varphi : \Omega \times \Upsilon \times S_\beta \to \{\vec{q}_1, \vec{q}_2, \vec{q}_3\}^{\mathbb{N}} \quad \to \Upsilon$$
$$(\omega, \upsilon, \vec{z}) \to (d_1, d_2, d_3, \ldots) \to (b_1, b_2, b_3, \ldots)$$

- Let $Z = Z_1 \cap Z_2, D = D_1 \cap D_2$, where

$$Z_1 = \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ infinitely often}\},$$
$$Z_2 = \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ infinitely often}\}.$$
$$D_1 = \{(b_1, b_2, \ldots) \in \Upsilon : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C \text{ for infinitely many } j\text{'s}\},$$
$$D_2 = \{(b_1, b_2, \ldots) \in \Upsilon : \sum_{i=1}^{\infty} \frac{\vec{q}_{b_{j+i-1}}}{\beta^i} \in C_{012} \text{ for infinitely many } j\text{'s}\}.$$

- Let $\varphi' = \varphi|_Z$. The map $\varphi' : Z \to D$ is a bimeasurable bijection.
- If $1 < \beta < \beta_*$, then $\mathbb{P}(D) = 1$.

Lemma

For $\beta \in (1, \beta_*]$, the dynamical systems $(\Omega \times \Upsilon \times S_\beta, \mathcal{A} \times \mathcal{B} \times \mathcal{S}, \nu_\beta, K_\beta)$ and $(\Upsilon, \mathcal{B}, \mathbb{P}, \sigma')$ are isomorphic, where $\nu_\beta(A) = \mathbb{P}(\varphi(Z \cap A))$.

Remark

The above lemma implies that any other K_β-invariant measure with support Z has entropy strictly less than $\log 3$. We need investigate the entropy of K_β-invariant measure μ for which $\mu(Z^c) > 0$.

Divide Z^c into three K_β-invariant Borel sets: $Z^c = Z_3 \cup Z_4 \cup Z_5$, where

$$\begin{aligned} Z_3 =& \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ for finitely many } n\text{'s} \\ & \text{and } K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ infinitely often}\}, \\ Z_4 =& \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ for finitely many } n\text{'s}, \\ & \text{and } K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ infinitely often}\}, \\ Z_5 =& \{(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times S_\beta : K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C \text{ for finitely many } n\text{'s}, \\ & \text{and } K_\beta^n(\omega, \upsilon, \vec{z}) \in \Omega \times \Upsilon \times C_{012} \text{ for finitely many } n\text{'s}\}. \end{aligned}$$

Lemma

Let $\beta \in (1, 3/2]$. Let μ be a K_β-invariant measure for which $\mu(Z^c) > 0$. Then $h_\mu(K_\beta) < \log 3$.

Proof

There exist K_β-invariant probability measures μ_{12}, μ_3, μ_4 and μ_5 concentrated on Z, Z_3, Z_4 and Z_5, respectively, such that

$$\mu = (1 - \alpha_3 - \alpha_4 - \alpha_5)\mu_{12} + \alpha_3\mu_3 + \alpha_4\mu_4 + \alpha_5\mu_5,$$

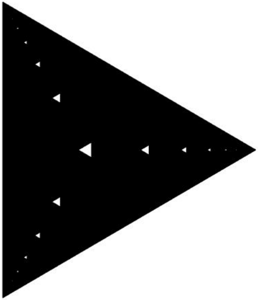
where $0 \leq \alpha_3, \alpha_4, \alpha_5 \leq 1$ and $0 < \alpha_3 + \alpha_4 + \alpha_5 \leq 1$. Then

$$h_\mu(K_\beta) = (1-\alpha_3-\alpha_4-\alpha_5)h_{\mu_{12}}(K_\beta)+\alpha_3 h_{\mu_3}(K_\beta)+\alpha_4 h_{\mu_4}(K_\beta)+\alpha_5 h_{\mu_5}(K_\beta).$$

Since $h_{\mu_{12}}(K_\beta) \leq \log 3$, and $h_{\mu_3}(K_\beta), h_{\mu_4}(K_\beta), h_{\mu_5}(K_\beta) < \log 3$, then we have the result.

For $3/2 < \beta < 2$

- In this case, we should notice that there are holes in the attractor.
- A special structure of those holes: they are all centered on three radial lines originating from the center of the attractor and extending to the three vertices.[Broomhead et al., 2004]



For $3/2 < \beta \leq \beta^*$

Proposition

Let $\beta^* \approx 1.5437$ be the root of $x^3 - 2x^2 + 2x = 2$. Then S_β has a non-empty interior if $\beta \in (\frac{3}{2}, \beta^*]$, and each hole has the form $f^n_{\vec{q}_i}(H)$, see the following figure.

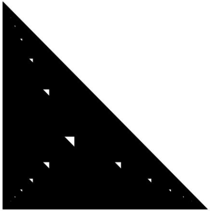
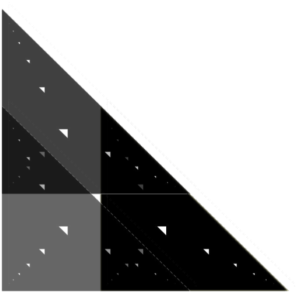


Figure: S_β for $\frac{3}{2} < \beta \leq \beta^*$

Overlaps and non-overlaps



- All holes $f^n_{\vec{q}_i}(H)$ in the attractor S_β are in 'nonoverlapping areas'.
- The 'holes' $f_{\vec{q}_i} f^n_{\vec{q}_j}(H)(i \neq j)$ are covered by $f_{\vec{q}_j}(S_\beta)$;
- $f_{\vec{q}_i} f^n_{\vec{q}_j}(H)(i \neq j)$ belongs to the nonoverlapping area.

Basic properties of K_β for $3/2 < \beta \leq \beta^*$

$$K_\beta(\omega, \vec{z}) = \begin{cases} (\omega, \beta\vec{z} - \vec{q}_i), & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ (\sigma\omega, \beta\vec{z} - \vec{q}_i), & \text{if } \omega_1 = 0, \text{and } \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}, \\ (\sigma\omega, \beta\vec{z} - \vec{q}_j), & \text{if } \omega_1 = 1, \text{and } \vec{z} \in C_{ij},\ ij \in \{01, 12, 02\}. \end{cases}$$

Let

$$d_1 = d_1(w, \vec{z}) = \begin{cases} \vec{q}_i, & \text{if } \vec{z} \in E_i, i = 0, 1, 2, \\ & \text{if } (\omega, \vec{z}) \in \{\omega_1 = 0\} \times C_{ij},\ ij \in \{01, 12, 02\}, \\ \vec{q}_j, & \text{if } (\omega, \vec{z}) \in \{\omega_1 = 1\} \times C_{ij},\ ij \in \{01, 12, 02\}. \end{cases}$$

Set $d_n = d_n(\omega, \vec{z}) = d_1(K_\beta^{n-1}(\omega, \vec{z}))$.

- $(d_1, d_2, d_3, \ldots)$ is a coding of $\vec{z}$.

Basic properties of K_β for $3/2 < \beta \le \beta^*$

Proposition

Suppose $\omega, \omega' \in \Omega$ *are such that* $\omega \prec \omega'$*. Then*

$$(d_1(\omega, \vec{z}), d_2(\omega, \vec{z}), \ldots) \preceq (d_1(\omega', \vec{z}), d_2(\omega', \vec{z}), \ldots).$$

Proposition

For $\beta \in (3/2, \beta^*]$*. Let* $\vec{z} \in S_\beta$ *and* $\vec{z} = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i}$ *with* $a_i \in \{\vec{q}_0, \vec{q}_1, \vec{q}_2\}$ *be a representation of* $\vec{z}$ *in base* β*. Then there exists an* $\omega \in \Omega$ *such that* $a_i = d_i(\omega, \vec{z})$*.*

Theorem (Main results-4)

Let $3/2 < \beta \leq \beta^*$. *The dynamical systems* $(\Omega \times S_\beta, \mathcal{A} \times \mathcal{S}, \nu_\beta, K_\beta)$ *and* $(\Upsilon, \mathcal{B}, \mathbb{P}, \sigma')$ *are isomorphic. Moreover, the measure* ν_β *is the unique* K_β*-invariant measure of maximal entropy.*

Idea: The proof is similar as that of the third main result.

[1] Bahsoun W, Bose C and Quas A.
Deterministic representation for position dependent random maps.
Discrete Conti. Dyn. Syst. 22(3):529-540, 2012.

[2] Boyarsky A and Góra P.
Absolutely continuous invariant measures for piecewise expanding C^2 transformations in $\mathbb{R}^N$.
Israel Jour. Math. 67(3): 272-286, 1989.

[3] Bahsoun W and Góra P.
Position dependent random maps in one and higher dimensions.
Studia Mathematica, 166(3): 271-286, 2005.

[4] Broomhead D, Montaldi J and Sidorov N.
Golden Gasket: Variations on the Sierpiński Sieve.
Nonlinearity, 17: 1455-1480, 2004.

[5] Dajani K and Kraaikamp C.
On random β-expansions.
Ergodic Theory Dynam. Systems, 23: 461479, 2003.

[6] Dajani K and de Vries M.
Measures of maximal entropy for random β-expansions.
J. Eur. Math. Soc. 7: 51-68, 2005.

[7] Dajani K and de Vries M.
Invariant densities for random β-expansions.
J. Eur. Math. Soc. 9: 157-176, 2007.

[8] Giusti E.
Minimal Surfaces and Functions of Bounded Variation.
Birkhauser, 1984.

[9] Hofbauer F.
The topological entropy of the transformation $x \mapsto ax(1-x)$.
Mh. Math. 90: 117-141, 1980.

[10] Parry W.
On the β-expansions of real numbers.

Thank you!