Variations on a theme of Doeblin

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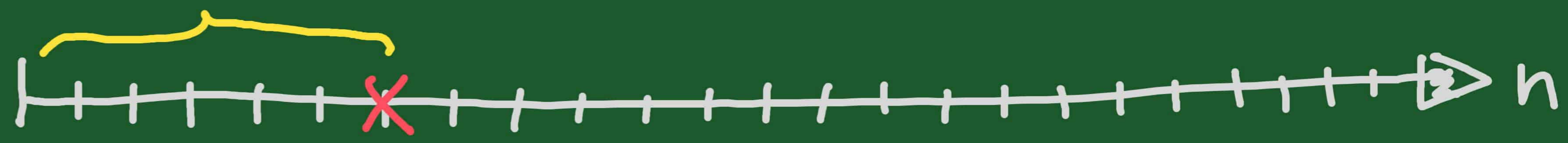
Doeblin's observation

(large digits in CF-expansions)

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}}$$

digit process $x \mapsto a_0, a_1, a_2, \dots$

$\varphi_\ell = \varphi_\ell^{(1)} :=$ position n of first digit $a_n \geq \ell$



Thm (Poisson Limit Theorem, Baby version)

As $\ell \to \infty$: $\quad P\left[\frac{1}{\ell \cdot \log 2} \varphi_\ell > s \right] \longrightarrow e^{-s}$

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$\varphi_\ell = \varphi_\ell^{(1)} :=$ position n of first digit $a_n \geq \ell$

$\varphi_\ell^{(j+1)} :=$ distance to next digit $\geq \ell$

Thm (Poisson Limit Theorem)

As $\ell \to \infty$: $\mathbb{P}\left[\bigcap_{j=1}^{d} \left\{ \frac{1}{\ell \cdot \log 2} \varphi_\ell^{(j)} > s_j \right\}\right] \longrightarrow \prod_{j=1}^{d} e^{-s_j}$

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digit process $x \mapsto a_0, a_1, a_2, \ldots$

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$\varphi_\ell^{(j+1)} :=$ distance to next digit $\geq \ell$

Thm (Poisson Limit Theorem, spatiotemporal version)

As $\ell \to \infty$: $$P\left[\bigcap_{j=1}^{d} \left\{ \frac{1}{\ell \cdot \log 2} \varphi_\ell^{(j)} > s_j, \Theta_\ell^{(j)} \in B_\ell^{(j)} \right\}\right] \longrightarrow \prod_{j=1}^{d} \frac{e^{-s_j}}{2}$$

here $\Theta_\ell^{(j)} := j^{\text{th}}$ digit $\geq \ell$, $B_\ell^{(j)} := 2\mathbb{N}$ or $2\mathbb{N}+1$ [Pene & Saussol], [RZ]

Doeblin's observation

(large digits in CF-expansions)

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}}$$

digit process $x \mapsto a_0, a_1, a_2, \ldots$

$\varphi_\ell = \varphi_\ell^{(1)} :=$ position n of first digit $a_n \geq \ell$

$\varphi_\ell^{(j+1)} :=$ distance to next digit $\geq \ell$

Thm (Poisson Limit Theorem, a local spatiotemporal version)

As $\ell \to \infty$: $$P\left[\bigcap_{j=1}^{d} \left\{ \varphi_\ell^{(j)} = k_\ell^{(j)}, \Theta_\ell^{(j)} = d_\ell(j) \right\}\right] \sim \prod_{j=1}^{d} \frac{e^{-s_j}}{d_\ell(j)^2 \cdot \log 2}$$

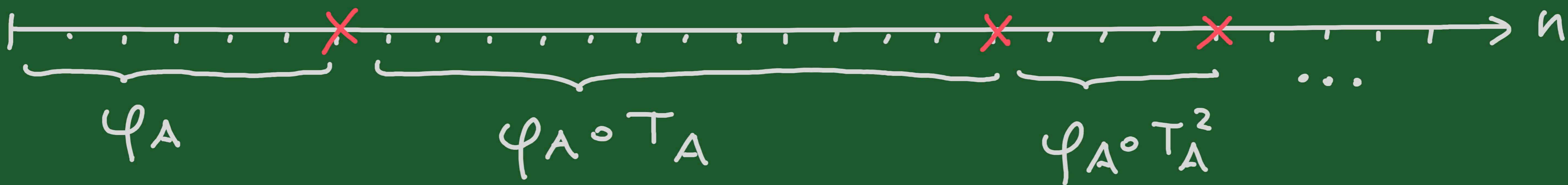
where $k_\ell^{(j)} \sim s_j \cdot \ell \cdot \log 2$ and $d_\ell(j) \geq \ell$. [M. Auer & RZ]

General Setup: $(X,\mathcal{A},\mu)$ proba space, $T: X \to X$ ergodic, μ-preserving

small set A, $\mu(A)>0$, first hitting time

$$\varphi_A(x) := \inf\{n \geq 1 : T^n x \in A\}, \quad x \in X$$

regard as random variable under μ



$T_A x := T^{\varphi_A(x)} x$, $x \in X$ $\rightsquigarrow$ consecutive waiting times $\varphi_A \circ T_A^j$, $j \geq 0$

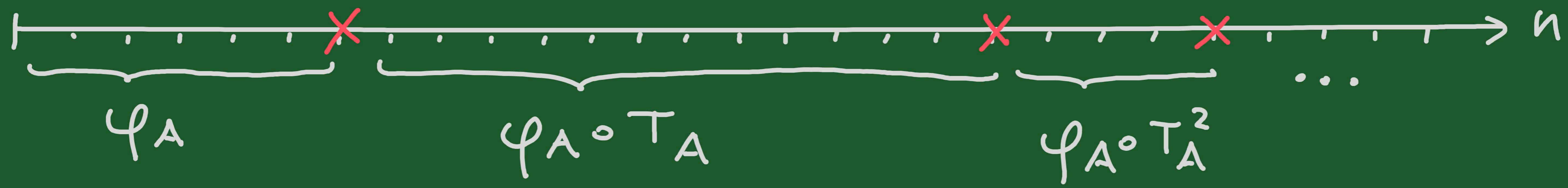
$\Phi_A := (\varphi_A \circ T_A^j)_{j \geq 0}$ hitting-time process under μ

General Setup: $(X, \mathcal{A}, \mu)$ proba space, $T: X \to X$ ergodic, μ-preserving

small set A, $\mu(A) > 0$, first hitting time [return time]

$\varphi_A(x) := \inf\{n \geq 1 : T^n x \in A\}$, $x \in X$ [$x \in A$]

regard as random variable under μ [under μ_A]



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[return-time process under μ_A]

normalized versions: $\mu(A) \cdot \varphi_A$ and $\mu(A) \cdot \Phi_A$ (Kac)

Poisson asymptotics

(A) hitting times of rare events $(A_\ell)_{\ell \geq 1}$, $0 < \mu(A_\ell) \to 0$

$R_\ell := \mu(A_\ell) \cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_\xi := (\xi_1, \xi_2, \xi_3, \dots)$ iid exponential random vars

Poisson asymptotics

(A) hitting times of rare events $(A_\ell)_{\ell \geq 1}$, $0 < \mu(A_\ell) \to 0$

$$R_\ell := \mu(A_\ell) \cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_\mathcal{E} \qquad \text{i.e.} \quad \mu\left[\bigcap_{j=1}^{d} \left\{\mu(A_\ell)\varphi_{A_\ell} \circ T_{A_\ell}^{j-1} > s_j\right\}\right] \to \prod_{j=1}^{d} e^{-s_j} \qquad (\heartsuit)$$

Poisson asymptotics (A) hitting times of rare events $(A_\ell)_{\ell \geq 1}$, $0 < \mu(A_\ell) \to 0$

$R_\ell := \mu(A_\ell) \cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_{\mathcal{E}}$ i.e. $\mu\left[\bigcap_{j=1}^{d} \left\{\mu(A_\ell)\varphi_{A_\ell} \circ T_{A_\ell}^{j-1} > s_j\right\}\right] \to \prod_{j=1}^{d} e^{-s_j}$ (♥)

$\leadsto$ distributional convergence of random elements of $[0,\infty]^{\mathbb{N}}$:

$\int \psi(R_\ell)\, d\mu \longrightarrow \mathbb{E}[\psi(\Phi_{\mathcal{E}})]$

$\forall\, \psi : [0,\infty]^{\mathbb{N}} \to \mathbb{R}$ bdd & Lipschitz

Poisson asymptotics

A hitting times of rare events $(A_\ell)_{\ell\geq 1}$, $0<\mu(A_\ell)\to 0$

$$R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_{\mathcal{E}} \qquad \text{i.e. } \mu\left[\bigcap_{j=1}^{d}\left\{\mu(A_\ell)\,\varphi_{A_\ell}\circ T_{A_\ell}^{j-1} > s_j\right\}\right] \to \prod_{j=1}^{d} e^{-s_j} \qquad (\heartsuit)$$

$\llcorner\!\!\to$ distributional convergence of random elements of $[0,\infty]^{\mathbb{N}}$:

$$\int \psi(R_\ell)\,d\mu \to \mathbb{E}[\psi(\Phi_{\mathcal{E}})]$$

$\forall\, \psi : [0,\infty]^{\mathbb{N}} \to \mathbb{R}$ bdd & Lipschitz

B return times of rare events (A_ℓ)

$$R_\ell = \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu_{A_\ell}}{\Longrightarrow} \Phi_{\mathcal{E}} \qquad \text{i.e. } \mu_{A_\ell}\left[\bigcap_{j=1}^{d}\left\{\mu(A_\ell)\,\varphi_{A_\ell}\circ T_{A_\ell}^{j-1} > s_j\right\}\right] \to \prod_{j=1}^{d} e^{-s_j} \qquad (\heartsuit\heartsuit)$$

or $\int \psi(R_\ell)\,d\mu_{A_\ell} \to \mathbb{E}[\psi(\Phi_{\mathcal{E}})]$ $\forall \psi \ldots$

- goal: understand when $\heartsuit$ and $\heartsuit\heartsuit$ hold

Poisson asymptotics

(A) hitting times of rare events $(A_\ell)_{\ell\geq 1}$, $0<\mu(A_\ell)\to 0$

$$R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_{\mathcal{E}}$$

i.e. $$\mu\left[\bigcap_{j=1}^{d}\left\{\mu(A_\ell)\,\varphi_{A_\ell}\circ T_{A_\ell}^{j-1} > s_j\right\}\right] \to \prod_{j=1}^{d} e^{-s_j} \qquad (\heartsuit)$$

$\llcorner\!\!\to$ distributional convergence of random elements of $[0,\infty]^{\mathbb{N}}$:

$$\int \psi(R_\ell)\,d\mu \to \mathbb{E}[\psi(\Phi_{\mathcal{E}})]$$

$\forall\, \psi : [0,\infty]^{\mathbb{N}} \to \mathbb{R}$ bdd & Lipschitz

(B) return times of rare events (A_ℓ)

$$R_\ell = \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu_{A_\ell}}{\Longrightarrow} \Phi_{\mathcal{E}}$$

i.e. $$\mu_{A_\ell}\left[\bigcap_{j=1}^{d}\left\{\mu(A_\ell)\,\varphi_{A_\ell}\circ T_{A_\ell}^{j-1} > s_j\right\}\right] \to \prod_{j=1}^{d} e^{-s_j} \qquad (\heartsuit\heartsuit)$$

or $$\int \psi(R_\ell)\,d\mu_{A_\ell} \to \mathbb{E}[\psi(\Phi_{\mathcal{E}})] \quad \forall \psi \ldots$$

- necessary:

R_ℓ under μ $\sim$ R_ℓ under μ_{A_ℓ} : $\Longleftrightarrow$ $$\int \psi(R_\ell)\,d\mu - \int \psi(R_\ell)\,d\mu_{A_\ell} \to 0 \quad \forall \psi \ldots$$

Poisson asymptotics

Ⓐ hitting times $R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi\,\mathcal{E}$ $\quad(\heartsuit)$

Ⓑ return times $R_\ell = \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu_{A_\ell}}{\Longrightarrow} \Phi\,\mathcal{E}$ $\quad(\heartsuit\heartsuit)$

Remark: $\heartsuit + \heartsuit\heartsuit$ always holds for some, but never for all $(A_\ell)_{\ell\geq 1}$

Poisson asymptotics

Ⓐ hitting times $R_\ell := \mu(A_\ell) \cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_{\mathcal{E}}$ (♥)

Ⓑ return times $R_\ell = \mu(A_\ell) \cdot \Phi_{A_\ell} \overset{\mu_{A_\ell}}{\Longrightarrow} \Phi_{\mathcal{E}}$ (♥♥)

[Thm ♥ iff ♥♥ iff R_ℓ under $\mu \sim R_\ell$ under μ_{A_ℓ}] [HSV] [Z]

(means $\int \psi(R_\ell)\,d\mu - \int \psi(R_\ell)\,d\mu_{A_\ell} \to 0 \quad \forall \psi \ldots$)

...

Poisson asymptotics (A) hitting times $R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_{\mathcal{E}}$ (♥)

(B) return times $R_\ell = \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu_{A_\ell}}{\Longrightarrow} \Phi_{\mathcal{E}}$ (♥♥)

[**Thm** ♥ iff ♥♥ iff R_ℓ under $\mu \sim R_\ell$ under μ_{A_ℓ}]

Rem: neither of ♥ and ♥♥ can hold if parts of A_ℓ return quickly:

if $\exists\, \tau_\ell \geq 1$ with (i) $\mu(A_\ell)\cdot \tau_\ell \xrightarrow{\mu_{A_\ell}} 0$

(ii) $\mu_{A_\ell}(\varphi_{A_\ell} > \tau_\ell) \not\longrightarrow 1$

then ♥ and ♥♥ fail (return-time limit has atom at zero).

Exple: A_ℓ small nbd of fixed point

...

Poisson asymptotics

(A) hitting times $R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu}{\Longrightarrow} \Phi_{\mathcal{E}}$ (♥)

(B) return times $R_\ell = \mu(A_\ell)\cdot \Phi_{A_\ell} \overset{\mu_{A_\ell}}{\Longrightarrow} \Phi_{\mathcal{E}}$ (♥♥)

[**Thm** ♥ iff ♥♥ iff R_ℓ under $\mu \sim R_\ell$ under μ_{A_ℓ}]

$\rightsquigarrow$ strategy for proving ♥: consider R_ℓ under μ and μ_{A_ℓ}

(†) note: as $\ell \to \infty$, the μ_{A_ℓ} get more and more singular

(‡) but: if T mixes efficiently, $\mu_{A_\ell}\circ T^{-\tau}$ should approach μ $(\tau \to \infty)$

$\rightsquigarrow$ want: (‡) beats (†): small number τ_ℓ of steps turns μ_{A_ℓ} into nicer measure $\mu_{A_\ell}\circ (T^{\tau_\ell})^{-1} =: \nu_\ell \in \mathcal{V}$ which allows for good control like

$$\sup_{\nu\in\mathcal{V}} |||\nu\circ T^{-n} - \mu||| \le \kappa_n \to 0 .$$

...

Poisson asymptotics – sufficient conditions old & new

- **Thm** (standard argument): $\heartsuit$ holds if $\exists\, \tau_\ell : A_\ell \to \mathbb{N}$ s.t.

  (i) $\mu(A_\ell)\cdot \tau_\ell \xrightarrow{\mu_{A_\ell}} 0$

  (ii) $\mu_{A_\ell}(\varphi_{A_\ell} > \tau_\ell) \longrightarrow 1$ (fast enough)

  (iii) $\mu_{A_\ell} \circ (T^{\tau_\ell})^{-1} \in \mathcal{V}$, where $\sup_{\nu\in\mathcal{V}} \| \nu\circ T^{-n} - \mu \| \le \kappa_n \to 0$.

- **Sketch of proof** – **Step 1 (easy)**: let $R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell}$, by (i) & (ii),

  R_ℓ under μ_{A_ℓ} $\sim$ $R_\ell \circ T^{\tau_\ell}$ under μ_{A_ℓ} $\sim$ R_ℓ under $\underbrace{\mu_{A_\ell}\circ (T^{\tau_\ell})^{-1}}_{=:\,\nu_\ell}$ $\blacktriangledown$

- **Step 2**: use (iii) to show that

$$R_\ell \text{ under } \nu_\ell \sim R_\ell \text{ under } \mu$$

$\blacksquare$

- **Remark**: for (τ_ℓ) as in (i), condition (ii) is necessary for $\heartsuit$.

Poisson asymptotics – sufficient conditions old & new

- Thm (new Poisson Limit Thm): $\heartsuit$ holds if $\exists\, \tau_\ell : A_\ell \to \mathbb{N}$ s.t.

  (i) $\mu(A_\ell)\cdot \tau_\ell \xrightarrow{\mu_{A_\ell}} 0$

  (ii) $\mu_{A_\ell}(\varphi_{A_\ell} > \tau_\ell) \longrightarrow 1$

  (iii) $\mu_{A_\ell} \circ (T^{\tau_\ell})^{-1} \in \mathcal{V}$, where $\mathcal{V}$ is compact in TV-norm.

- note: here T need not be mixing
- note: the assumptions only concern times of order $o(\mu(A_\ell)^{-1})$
- Remark: $\exists$ more flexible versions of this thm for
  - compound Poisson processes (A_ℓ nbd of periodic point)
  - spatiotemporal Poisson limits
  - non-ergodic systems

Poisson asymptotics - sufficient conditions old & new

- **Thm** (new Poisson Limit Thm): ♥ holds if $\exists\, \tau_\ell : A_\ell \to \mathbb{N}$ s.t.
  (i) $\mu(A_\ell)\cdot\tau_\ell \xrightarrow{\mu_{A_\ell}} 0$
  (ii) $\mu_{A_\ell}(\varphi_{A_\ell} > \tau_\ell) \longrightarrow 1$
  (iii) $\mu_{A_\ell}\circ(T^{\tau_\ell})^{-1} \in \mathcal{V}$, where $\mathcal{V}$ is compact in TV-norm.

- Sketch of proof - Step 1 (easy): let $R_\ell := \mu(A_\ell)\cdot\Phi_{A_\ell}$, by (i) & (ii),

  R_ℓ under $\mu_{A_\ell} \sim R_\ell\circ T^{\tau_\ell}$ under $\mu_{A_\ell} \sim R_\ell$ under $\underbrace{\mu_{A_\ell}\circ(T^{\tau_\ell})^{-1}}_{=:\,\nu_\ell}$ ▽

- Step 2: use (iii) to show that

$$R_\ell \text{ under } \nu_\ell \sim R_\ell \text{ under } \mu$$ ▨

- Now: How does step 2 work without mixing?

The interesting step in the proof

$$R_\ell := \mu(A_\ell)\cdot \Phi_{A_\ell} \in [0,\infty]^{\mathbb{N}}$$

- claim: $\nu_\ell \in \mathcal{V}\ \forall \ell \geq 1 \implies R_\ell$ under $\nu_\ell \sim R_\ell$ under μ
- ingredient I: asymptotic T-invariance in measure: $\operatorname{dist}(R_\ell \circ T, R_\ell) \xrightarrow{\mu} 0$
- ingredient II: Mean Ergodic Thm: $\forall \nu \ll \mu : \frac{1}{n}\sum_{k=0}^{n-1} \nu \circ T^{-k} \xrightarrow{TV} \mu$ uniform in $\nu \in \mathcal{V}$
- need $\left| \int \psi(R_\ell)\, d\nu_\ell - \int \psi(R_\ell)\, d\mu \right| \to 0 \quad \forall \psi : [0,\infty]^{\mathbb{N}} \to \mathbb{R}$ bdd & Lipschitz

By II: $\exists n$: $\left| \int \psi(R_\ell)\, d\mu - \psi(R_\ell)\, d\left(\frac{1}{n}\sum_{k=0}^{n-1} \nu_\ell \circ T^{-k}\right)\right| < \frac{\varepsilon}{2}\ \forall \ell \geq 1$. Otoh,

$$\left| \psi(R_\ell)\, d\left(\frac{1}{n}\sum_{k=0}^{n-1} \nu_\ell \circ T^{-k}\right) - \int \psi(R_\ell)\, d\nu_\ell \right| \leq \frac{1}{n}\sum_{k=0}^{n-1} \left| \int \psi(R_\ell)\, d\nu_\ell - \int \psi(R_\ell)\, d\nu_\ell \circ T^{-k} \right|$$

$$\leq \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=1}^{k-1} \int \underbrace{\left| \psi(R_\ell) - \psi(R_\ell \circ T) \right|}_{\leq \operatorname{Lip}(\psi)\cdot \operatorname{dist}(R_\ell \circ T, R_\ell)} d\nu_\ell \circ T^{-k+1}$$

(use compactness of $\mathcal{V}$ again) $\quad < \frac{\varepsilon}{2} \quad \forall \ell \geq \ell_0$