AN EQUIVALENCE OF MULTISTATISTICS ON PERMUTATIONS

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Abstract. We prove a conjecture of J.-C. Novelli, J.-Y. Thibon, and L. K. Williams (2010) about an equivalence of two triples of statistics on permutations. To prove this conjecture, we build a bijection through different combinatorial objects, starting with a Catalan-based object related to the PASEP. As a byproduct of this research, we also provide a new co-sylvester class-preserving bijection on permutations.

1. Introduction

The algebra of noncommutative symmetric functions **Sym** [7] has been studied in algebraic combinatorics during the past twenty years. It is a graded algebra $\mathbf{Sym} = \bigoplus_{n\geq 0} \mathbf{Sym}_n$, where $\mathbf{Sym}_n$ is of dimension 2^{n-1} for $n \geq 1$, so that its bases are indexed by integer compositions, that is, finite sequences of positive integers of sum n. There are many purely combinatorial problems related to this algebra such as, for example, the explicit description of the relations between different bases through their transition matrices. In this paper, we shall be interested in the basis introduced by Tevlin in [11], the so-called monomial basis of **Sym**, for which the transition matrices $\mathfrak{M}^{(n)}$ with the ribbon basis have been described in [6].

In this last paper, the authors prove that the entry $\mathfrak{M}^{(n)}_{I,J}$, indexed by two integer compositions I and J, is equal to the number of permutations satisfying $\mathrm{GC}(\sigma) = I$ and $\mathrm{Rec}(\sigma) = J$, where GC and Rec are two statistics that will be recalled later. Some properties of this basis correspond to properties of the PASEP (Partially Asymmetric Simple Exclusion Process), a physical model in which particles hop back and forth (and in and out) of a one-dimensional lattice. More precisely, in the study of the basis of Tevlin, the sum of the entries of row I of the transition matrix $\mathfrak{M}^{(n)}$ corresponds to the unnormalized steady-state

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probability of the state of the PASEP with $n-1$ sites corresponding to I.

In the combinatorial version of the PASEP, the steady-state probabilities are computed through the enumeration of the so-called permutation tableaux, an object introduced in [3]. These objects are certain fillings of Ferrers diagrams. There is a simple bijection between a state of the PASEP with N sites and a Ferrers diagram of semi-perimeter N such that the steady-state probability of this state of the PASEP corresponds to the sum of the fillings of the Ferrers diagram as permutation tableaux. Moreover, in [10] the authors present a bijection between permutation tableaux and permutations such that the permutation tableaux of a given shape are sent to the permutations of a given GC statistic.

In the PASEP context, there exists a natural q-statistic on permutation tableaux that becomes the number of $31{-}2$ patterns on permutations (denoted by tot) thanks to the bijection of [10]. It is then natural to define a q-analog of the basis of Tevlin as the functions whose transition matrices $\mathfrak{M}^{(n)}(q)$ with the ribbon basis are such that $\mathfrak{M}^{(n)}_{I,J}(q)$ is the sum of the $q^{\mathrm{tot}(\sigma)}$ for all σ satisfying $\mathrm{GC}(\sigma) = I$ and $\mathrm{Rec}(\sigma) = J$.

In [9], the authors studied those matrices in an algebraic way. However, the statistics GC and tot on permutations were not appropriate for an algebraic study. So they built suitable matrices $\widetilde{\mathfrak{M}}^{(n)}(q)$ for their algebraic purpose through other statistics on permutations also recalled later (LC, Rec, and α). Then, the entry $\widetilde{\mathfrak{M}}^{(n)}_{I,J}(q)$ is the sum of the $q^{\alpha(\sigma)}$ for all σ satisfying $\mathrm{LC}(\sigma) = I$ and $\mathrm{Rec}(\sigma) = J$. They conjectured that their matrices are the same as the $\mathfrak{M}^{(n)}(q)$, or equivalently that the triples of statistics $(\mathrm{GC}, \mathrm{Rec}, \mathrm{tot})$ and $(\mathrm{LC}, \mathrm{Rec}, \alpha)$ are equidistributed.

The aim of this paper is to prove this conjecture bijectively. As the triple of statistics $(\mathrm{LC}, \mathrm{Rec}, \alpha)$ has a natural description on subexcedent functions, in order to gain in readability, we build a bijection from permutations to subexcedent functions sending the triple of statistics $(\mathrm{GC}, \mathrm{Rec}, \mathrm{tot})$ to $(\mathrm{LC}, \mathrm{DC}, \alpha)$ where DC is also defined later. The global bijection is described as a sequence of bijections through different combinatorial objects:

$$\mathrm{P} \overset{\psi_{FV}}{\longleftrightarrow} \mathrm{LH} \overset{\psi}{\longleftrightarrow} \mathrm{LLH} \overset{\phi_2}{\longleftrightarrow} \mathrm{DWSF} \overset{\phi_1}{\longleftrightarrow} \mathrm{SF}, \tag{1}$$

where P, LH, LLH, DWSF, and SF are respectively Permutations, Laguerre Histories, Large Laguerre Histories, Decreasing Weighted Subexcedent Functions, and Subexcedent Functions which are all recalled or defined below. The main idea is to send permutations and subexcedent functions to weighted Catalan objects such that the weight corresponds to the third statistics in the triples. To do that we use the well-known Françon-Viennot bijection ψ_{FV} on one side, and we define a new bijection ϕ_1 building a weighted Catalan object from a subexcedent function. Laguerre histories and large Laguerre histories are both weighted Motzkin paths with similar conditions on the weights. In order to avoid confusion and to simplify the description of the map ψ we shall use another common representation on those objects in terms of Weighted Dyck Paths (WDP). Diagram (1) becomes:

$$\mathrm{P} \overset{\psi_{FV}}{\underset{3.1}{\longleftrightarrow}} \mathrm{WDP} \overset{\psi}{\underset{3.2}{\longleftrightarrow}} \mathrm{WDP} \overset{\phi_2}{\underset{4.2}{\longleftrightarrow}} \mathrm{DWSF} \overset{\phi_1}{\underset{4.1}{\longleftrightarrow}} \mathrm{SF}. \tag{2}$$

Under each bijection in the diagram we indicate in which subsection of the paper it is presented.

In the last part of the paper we use the connection between large Laguerre histories and permutations through a variation of ψ_{FV} that we call ψ^0_{FV} to define another triple of statistics close to $(\mathrm{GC}, \mathrm{Rec}, \mathrm{tot})$ also giving a combinatorial interpretation for the entries of $\mathfrak{M}^{(n)}(q)$. By combining ψ^0_{FV} with ψ_{FV} and ψ through the following diagram

$$\mathrm{P} \overset{\psi_{FV}}{\longleftrightarrow} \mathrm{WDP} \overset{\psi}{\longleftrightarrow} \mathrm{WDP} \overset{\psi^0_{FV}}{\longleftrightarrow} \mathrm{P}, \tag{3}$$

we define a new bijection on permutations preserving the co-sylvester classes, classes inherited from a monoid structure related to the combinatorics of binary search trees [5].

In Section 2 we give the background and notations needed in the rest of the paper. In Section 3 we describe the bijections ψ_{FV} and ψ which are natural bijections leading to objects where the statistics GC and tot are naturally defined. The object obtained is a pair consisting of a Catalan object and a weight. In Section 4 we begin by building a Catalan object associated with a weight from subexcedent functions through ϕ_1 and then describe ϕ_2, a Catalan bijection between decreasing subexcedent functions and Dyck paths. In Section 5 we prove the conjectures of [9] and give some properties associated with the global bijection. Finally, in Section 6 we introduce a new co-sylvester class-preserving bijection and a new triple of statistics.

2. Notations and background

2.1. **Permutations, compositions, and subexcedent functions.**

Let us first fix our notations concerning permutations. We represent a permutation σ as a word $\sigma_1\sigma_2\cdots\sigma_n$ such that $\sigma_i = \sigma(i)$. For all the forthcoming examples we fix $\tau = 528713649$. We shall sometimes use the notation $[n] = [1, n]$ to denote the set $\{1, 2, \cdots, n\}$.

A *recoil* of a permutation σ is a value i such that $i+1$ is on the left of i or equivalently such that $\sigma_i^{-1} > \sigma_{i+1}^{-1}$. The recoil set of σ is the set of the values of recoils. For example, the recoil set of τ is $\{1,4,6,7\}$. A 31–2 pattern of σ is a pair (i,j) such that $j > i+1$ and $\sigma_{i+1} < \sigma_j < \sigma_i$. We denote by $\mathrm{tot}(\sigma)$ the number of $31-2$ patterns of σ. For example, $\mathrm{tot}(\tau) = 5$. We shall often need the number of times that a value appears as a 2 in a $31-2$ pattern. We define $\mathrm{tot}_k(\sigma)$ as the number of times k appears as a 2 in a $31-2$ pattern in σ.

A *composition* of an integer n is a sequence $I = (i_1, \ldots, i_r)$ of positive integers of sum n. The integer r is called the *length* of the composition. The *descent set* of I is $\mathrm{Des}(I) = \{i_1,\ i_1+i_2, \ldots, i_1+\cdots+i_{r-1}\}$.

The *major index* $\mathrm{maj}(I)$ of a composition is the sum of the values in the descent set of I. For example, $\mathrm{maj}(1,3,2,1,2) = 18$.

The *recoil composition* $\mathrm{Rec}(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the composition of n whose descent set is the recoil set of σ. For example, $\mathrm{Rec}(\tau) = (1,3,2,1,2)$.

The *Genocchi descent set* [6] of a permutation $\sigma \in \mathfrak{S}_n$, denoted by $\mathrm{GDes}(\sigma)$, is the set of values immediately followed by a smaller value (it is sometimes called the descent tops of σ [10]). The *Genocchi composition of descents* (or G-composition, for short) $\mathrm{GC}(\sigma)$ of a permutation is the integer composition I of n whose descent set is $\{d-1 \mid d \in \mathrm{GDes}(\sigma)\}$. For example, we have $\mathrm{GDes}(\tau) = \{5,6,7,8\}$ and $\mathrm{GC}(\tau) = (4,1,1,1,2)$.

About the statistic LC, the definition on permutations given in [9] is an algorithm and starts by taking the Lehmer code of the inverse permutation, a bijection between permutations and subexcedent functions. We shall directly define it on subexcedent functions as the other two statistics of the triple $(\mathrm{LC}, \mathrm{Rec}, \alpha)$ are easily defined on this object.

A *subexcedent function* of size n is a word u of size n on the alphabet of nonnegative integers such that for each $i \in \{1, \ldots, n\}$ we have $u_i \leq n-i$. We denote by $\mathfrak{SF}_n$ the set of subexcedent functions of size n. They are enumerated by $n!$ as they are in bijection with permutations through the Lehmer code. The *statistic LC* is defined on a subexcedent function u as follows.

- Set $S = \emptyset$ and read u from right to left. At each step, if the entry k is strictly greater than the size of S, add the $(k-|S|)$-th element of the sequence $[1,n]$ not considering the elements of S.
- The set S is the descent set of a composition C, and $\mathrm{LC}(u)$ is the mirror image $\overline{C}$ of C.

For example, with $u = 315503200$, S is $\emptyset$ at first, then the set $\{2\}$ at the third step as the third letter from the right is a 2 and S was empty, then $\{2,3\}$ (fourth step), then $\{2,3,5\}$ (sixth step), then $\{2,3,4,5\}$ (seventh step). Hence C is $(2,1,1,1,4)$, so that $\mathrm{LC}(u) = (4,1,1,1,2)$.

Define the *number of inversions* of a subexcedent function as the sum of its values. We also define *the descent set* of a subexcedent function as

$$\mathrm{Des}(u) = \{i \in \{1,\dots,n-1\} \mid u_i > u_{i+1}\} \tag{4}$$

and the *descent composition* of u (denoted by $\mathrm{DC}(u)$) as the composition of size n whose descent set is $\mathrm{Des}(u)$. One easily checks that $\mathrm{DC}(315503200) = (1,3,2,1,2)$. This definition of the descent composition of a subexcedent function corresponds to the recoil composition of the permutation before taking the Lehmer code of its inverse.

2.2. **Different weighted paths.** Recall that a Dyck path of size n is a path with n increasing steps and n decreasing steps that never goes below the horizontal axis. A Motzkin path of size n is a path with n steps among increasing steps, decreasing steps, and horizontal steps that never goes below the horizontal axis. For any path P we denote by P_i the i-th step of this path. We call the height of a step the distance between the beginning of this step and the horizontal axis. A weight associated with a path is a nonnegative integer vector with the same size as the size of the path.

Definition 2.1. *A Laguerre history L of size n is a weighted Motzkin path of size n with two kinds of horizontal steps ($\longrightarrow$ and $\dashrightarrow$). The weight w is a sequence of integers of size n satisfying for all $i \le n$,*

- $0 \le w_i \le h_i$ *if L_i is $\nearrow$ or $\longrightarrow$;*
- $0 \le w_i \le h_i - 1$ *if L_i is $\searrow$ or $\dashrightarrow$;*

where h_i is the height of the i-th step of L.

A large Laguerre history of size $n-1$ is a weighted Motzkin path of size $n-1$ with two kinds of horizontal steps where for any $i \le n-1$, the weight w satisfies $0 \le w_i \le h_i$.

Figure 1 represents a Laguerre history of size 9 and a large Laguerre history of size 8. For a better readability we only wrote the strictly positive w_i.

Both objects are in bijection and enumerated by $n!$. A common way to build a bijection between these is to represent each by a weighted Dyck path.

Definition 2.2. *A weighted Dyck path of size n is a Dyck path associated with a weight both of size n. For all i, the weight satisfies*

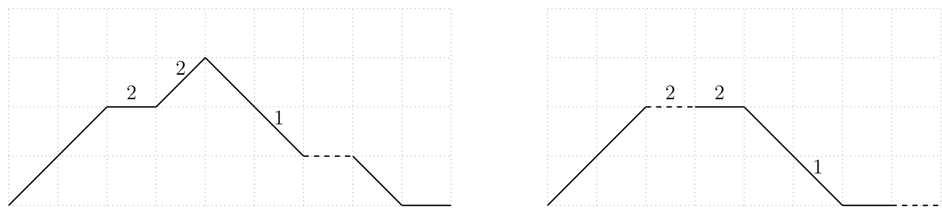


FIGURE 1. A Laguerre history of size 9 and a large Laguerre history of size 8.

$w_i \leq (h_i - 1)/2$ where h_i is the height of the $2i$-th step of the Dyck path.

An example of a weighted Dyck path of size 9 is given in Figure 2. For a better readability we grouped the steps two by two between vertical dashed lines and only wrote the strictly positive w_i above steps D_{2i-1} and D_{2i}. In this representation, h_i is exactly the height of the meeting point of the $(2i-1)$-st and the $2i$-th steps of the Dyck path.

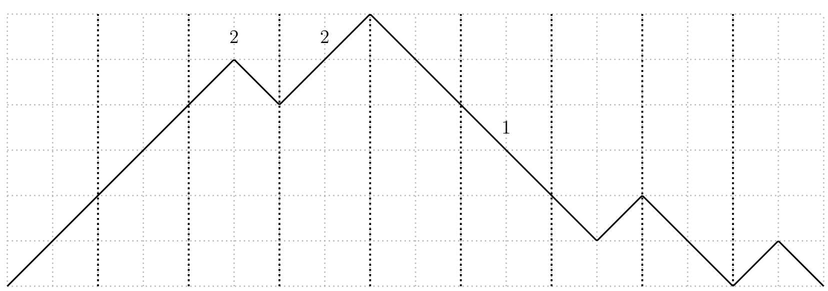


FIGURE 2. A weighted Dyck path of size 9.

The map sending a Laguerre history to a weighted Dyck path is described by the following algorithm.

Algorithm 2.3. *Let L be a Laguerre history of size n and $i \in [n]$. Then the weighted Dyck path D associated with L satisfies*

- $D_{2i-1} = D_{2i} = /$ *if* $L_i = \nearrow$,
- $D_{2i-1}D_{2i} = /\backslash$ *if* $L_i = \longrightarrow$,
- $D_{2i-1}D_{2i} = \backslash/$ *if* $L_i = \dashrightarrow$,
- $D_{2i-1} = D_{2i} = \backslash$ *if* $L_i = \searrow$.

The weight remains the same.

To build a weighted Dyck path from a large Laguerre history L, start by applying Algorithm 2.3 to L then add an extra $/$ step at the beginning and \ step at the end. The weight also remains the same, we just add a 0 at its end to obtain a weight of size n.

Applying the corresponding algorithm to both paths of Figure 1 gives the weighted Dyck path of Figure 2.

The fact that both objects are enumerated by $n!$ comes from the Françon-Viennot bijection.

2.3. **The Françon-Viennot bijection.** The Françon-Viennot bijection, first described in [4], is a bijection between permutations and Laguerre histories. We shall describe a bijection between permutations and weighted Dyck paths (defined below) that can be obtained by applying Algorithm 2.3 to the result of the version of Corteel of the Françon-Viennot bijection described in [2].

This bijection corresponds to the first part of Diagram (2).

$$\mathrm{P} \overset{\psi_{FV}}{\longleftrightarrow} \mathrm{WDP} \overset{\psi}{\longleftrightarrow} \mathrm{WDP} \overset{\phi_2}{\longleftrightarrow} \mathrm{DWSF} \overset{\phi_1}{\longleftrightarrow} \mathrm{SF}$$

In order to compute each step of the Dyck path, the Françon-Viennot bijection compares the values of σ with both of their neighbors. In order to do so with both the first and the last values of σ we use the convention $\sigma_0 = 0$ and $\sigma_{n+1} = n+1$.

Algorithm 2.4 (Françon-Viennot)**.** *Let $\sigma \in \mathfrak{S}_n$, $j \in \{1, \dots, n\}$ and $k = \sigma_j$. Then in the Dyck path of $\psi_{FV}(\sigma)$ we have*

- *$D_{2k-1} = /$ if $\sigma_j < \sigma_{j+1}$ and $D_{2k-1} = \backslash$ otherwise,*
- *$D_{2k} = \backslash$ if $\sigma_{j-1} < \sigma_j$ and $D_{2k} = /$ otherwise.*

The weight is built as follows: w_k is equal to $\mathrm{tot}_k(\sigma)$, the number of $31-2$ patterns where k stands for the 2.

For example, if we consider $\sigma = 528713649$, its image by ψ_{FV} is the weighted Dyck path of Figure 2. One can check that the values 1, 2, 4 and 7 are smaller than their left neighbor in σ and values 1, 2, 4, and 9 are smaller than their right neighbor. The 31–2 patterns are 52-3, 71-3, 52-4, 71-4, and 71-6 so that the weight is indeed $(0, 0, 2, 2, 0, 1, 0, 0, 0)$.

Let us also describe the reverse algorithm building a permutation from a weighted Dyck path (D, w).

Algorithm 2.5. *The permutation σ is built iteratively by*

- *Initialization: $\sigma = \circ$;*
- *At the k-th step of the algorithm, replace the $(w_k + 1)$-st $\circ$ of σ by:*
  - *$\circ k \circ$ if $D_{2k-1} = D_{2k} = /$,*
  - *$k \circ$ if $D_{2k-1}D_{2k} = /\backslash$,*
  - *$\circ k$ if $D_{2k-1}D_{2k} = \backslash /$,*

   – *k if $D_{2k-1} = D_{2k} = \backslash$;*
- *The final permutation is obtained by removing the last $\circ$.*

For the weighted Dyck path of Figure 2, the previous algorithm gives:

$$\sigma = \circ \ \rightarrow\ \circ 1 \circ \ \rightarrow\ \circ 2 \circ 1 \circ \ \rightarrow\ \circ 2 \circ 13 \circ \ \rightarrow\ \circ 2 \circ 13 \circ 4 \circ$$
$$\rightarrow\ 52 \circ 13 \circ 4 \circ \ \rightarrow\ 52 \circ 1364 \circ \ \rightarrow\ 52 \circ 71364 \circ$$
$$\rightarrow\ 52871364 \circ \ \rightarrow\ 5287136 49 \circ \ \rightarrow\ 528713649.$$

3. Permutations to weighted Dyck paths

Let us describe what happens to the triple of statistics on permutations through the Françon-Viennot bijection.

3.1. The statistics through the Françon-Viennot bijection.

Definition 3.1. *Let (D, w) be a weighted Dyck path of size n.*
- *The* total weight *of (D, w) (denoted by $\mathrm{tw}(D, w)$) is the sum of the values of w.*
- *The* descent set *of (D, w) is*

$$\mathrm{Des}(D, w) = \{i \mid w_i > w_{i+1}\} \cup \{i \mid w_i = w_{i+1}, D_{2i} = /\} \tag{5}$$

  and the descent composition $\mathrm{DC}(D, w)$ *is the composition of n whose descent set is $\mathrm{Des}(D, w)$.*
- *The* Genocchi descent set *of (D, w) is*

$$\mathrm{GDes}(D, w) = \{i \in [2, n] \mid D_{2i-1} = \backslash\} \tag{6}$$

  then, as for permutations, the Genocchi composition of descent *of (D, w) (denoted by $\mathrm{GC}(D, w)$) is the composition of n whose descent set is $\{d - 1 \mid d \in \mathrm{GDes}(D, w)\}$.*

On the weighted Dyck path (D, w) of Figure 2 we have $\mathrm{tw}(D, w) = 5$. The positions i such that $D_{2i-1} = \backslash$ are $\mathrm{GDes}(D, w) = \{5, 6, 7, 8\}$ so that $\mathrm{GC}(D, w) = (4, 1, 1, 1, 2)$ and one can check that $\mathrm{DC}(D, w) = (1, 3, 2, 1, 2)$.

Proposition 3.2. *Let σ be a permutation of size n and $(D, w) = \psi_{FV}(\sigma)$. We have the following properties:*
- $\mathrm{tw}(D, w) = \mathrm{tot}(\sigma)$;
- $\mathrm{GC}(D, w) = \mathrm{GC}(\sigma)$;
- $\mathrm{DC}(D, w) = \mathrm{Rec}(\sigma)$.

Proof. The first two assertions come directly from our description of ψ_{FV}. To prove the last one, it is easier to work with ψ_{FV}^{-1} that builds a permutation σ from a weighted Dyck path (D, w), as described in Algorithm 2.5. Let $k \in \{1, \ldots, n-1\}$. If $w_k < w_{k+1}$, then k is on the left of $k+1$ in σ so it is not a recoil for σ. Using the same idea, if $w_k > w_{k+1}$ then k is a recoil for σ. Now, when $w_k = w_{k+1}$, we have a recoil in k if and only if there is a new $\circ$ on the left of k when we place it in σ which happens if and only if $D_{2k} = /$. $\square$

We now have a representation of the triple of statistics on the weighted Dyck paths but the definition of GC on those objects is not very natural. Indeed, the fact that i belongs to $\mathrm{GDes}(D, w)$ or not depends on the $(i+1)$-st group of steps. To simplify the second part of the construction, we transform the Dyck paths to obtain a more suitable statistic.

3.2. **An involution on Dyck paths.**

Definition 3.3. *Let D be a Dyck path of size n. The Dyck path $\psi(D)$ is obtained by sending every pair of steps $/\backslash$ or $\backslash/$ at positions $2i$, $2i+1$ to one another.*

We extend this definition to weighted Dyck paths by carrying the weight. Note that it is an involution. An example is given in Figure 3 where we apply ψ to the weighted Dyck path of Figure 2.

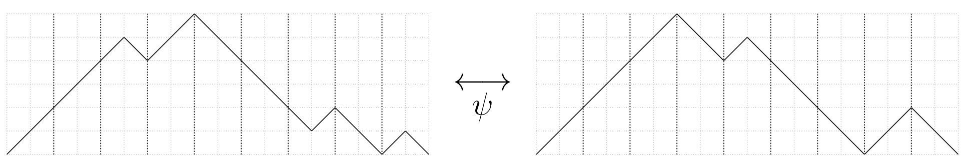


FIGURE 3. An example of ψ.

Definition 3.3 is a visual way of defining ψ, a more formal way would be by using the following lemma.

Lemma 3.4. *Let D be a Dyck path, we have:*

$$\begin{array}{rcl} D_{2i} & = & \psi(D)_{2i+1}; \\ D_{2i+1} & = & \psi(D)_{2i}. \end{array} \tag{7}$$

Proof. If $D_{2i} = D_{2i+1}$ then ψ does not change those step so the result is clear. Otherwise $D_{2i} \neq D_{2i+1}$ and then ψ sends one to the other which ends the proof. $\square$

This involution corresponds to the second part of Diagram (2).

$$\mathrm{P} \overset{\psi_{FV}}{\longleftrightarrow} \mathrm{WDP} \overset{\psi}{\longleftrightarrow} \mathrm{WDP} \overset{\phi_2}{\longleftrightarrow} \mathrm{DWSF} \overset{\phi_1}{\longleftrightarrow} \mathrm{SF}$$

This bijection gives a new bijection between Laguerre histories and large Laguerre histories sending the equivalent of the GC statistic to a very similar one. Unfortunately this bijection is not intuitive on Laguerre objects.

Let us introduce the new statistics on weighted Dyck paths corresponding to DC and GC after applying ψ.

Definition 3.5. *The statistics* GC^0 *and* DC^0 *are defined as follows.*

- *The* Genocchi descent set of type 0 *of a weighted Dyck path of size* n *is*

$$\mathrm{GDes}^0(D, w) = \{i \in [1, n-1] \mid D_{2i} = \backslash\} \tag{8}$$

and the Genocchi composition of descent of type 0 *(denoted by* $\mathrm{GC}^0(D, w)$*) is the composition of* n *whose descent set is* $\mathrm{GDes}^0(D, w)$*.*

- *The* descent set of type 0 *of* (D, w) *is*

$$\mathrm{Des}^0(D, w) = \{i \mid w_i > w_{i+1}\} \cup \{i \mid w_i = w_{i+1}, D_{2i+1} = /\} \tag{9}$$

and the descent composition of type 0 *(denoted by* $\mathrm{DC}^0(D, w)$*) is the composition of* n *whose descent set is* $\mathrm{Des}^0(D, w)$*.*

Proposition 3.6. *Let* (D, w) *be a weighted Dyck path, then*

- $\mathrm{tw}(\psi(D, w)) = \mathrm{tw}(D, w)$*;*
- $\mathrm{GC}^0(\psi(D, w)) = \mathrm{GC}(D, w)$*;*
- $\mathrm{DC}^0(\psi(D, w)) = \mathrm{DC}(D, w)$*.*

Proof. As ψ does not change the weight of the Dyck path, the total weight is carried.

For the two other points, the statistics are really close but just differ on the index of the considered steps, so Lemma 3.4 ends the proof. $\square$

4. Subexcedent functions to weighted Dyck paths

The aim of this section is to build a bijection from subexcedent functions to weighted Dyck paths sending the triple $(\mathrm{LC}, \mathrm{DC}, \mathrm{inv} - \mathrm{maj}(\overline{\mathrm{LC}}))$ to $(\mathrm{GC}^0, \mathrm{DC}^0, \mathrm{tw})$ where the statistic $\mathrm{inv} - \mathrm{maj}(\overline{\mathrm{LC}})$ corresponds to the statistic α in the introduction. To do this, we build a bijection from subexcedent functions to an intermediate object: *decreasing weighted subexcedent functions* which are represented by a Catalan object and a weight, and then build a Catalan bijection between decreasing subexcedent functions and Dyck paths.

4.1. Subexcedent functions to decreasing weighted subexcedent functions.

Definition 4.1. *Let us define a* decreasing subexcedent function *of size* n *and a* weight *for it.*

- *A subexcedent function u of size n is* decreasing *if the word obtained by removing all its zeroes is a strictly decreasing word.*
- *A* weight *associated with a decreasing subexcedent function is a word w of size n such that for all $k \in \{1, \dots, n\}$, w_k is smaller than or equal to the number of $i < k$ such that $0 < u_i \leq n - k$ (*i.e.*, the number of positive values on the left of k that could be at position k).*

For example, the subexcedent function $u = 540300200$ is decreasing. As for the associated weight, the maximum weight W is the weight for which each value is maximal and equal to the number of positive values to its left smaller than n minus its position. The maximum weight of u is 012221000 so the weight 002201000 is correct since it is smaller than the maximum weight component-wise, whereas 000002000 is not.

The decreasing subexcedent functions are indeed Catalan objects since one can build a bijection with nondecreasing parking functions, the nondecreasing words whose i-th value is a positive integer smaller or equal to i.

Algorithm 4.2. *Let u be a decreasing subexcedent function.*

- *Let v be the mirror image of u,*
- *replace each 0 of v by the first nonzero value to its left (if such a value exists),*
- *add one to each value of v.*

For example, if $u = 540300200$, the mirror image is 002003045 and the associated nondecreasing parking function is 113334456.

4.1.1. *Description of the bijection ϕ_1 between subexcedent functions and decreasing weighted subexcedent functions.* This new bijection ϕ_1 can be described as an algorithm that sorts a subexcedent function by successively moving the greatest value to its left.

Algorithm 4.3. *Let u be a subexcedent function of size n. Set the weight w to 0^n.*

- *Step 1: define the* pivot *as the greatest value in u such that one of its occurrences has smaller or equal nonzero values to its left. If the pivot is not defined, the algorithm stops. Otherwise, let k be the position of the rightmost occurrence of the pivot in u.*
- *Step 2: among the values smaller than or equal to the pivot on its left, let i be the position of the rightmost occurrence of the largest one. Modify the subexcedent function by decrementing u_i by 1 and then swapping u_i with u_k. Modify the weight by incrementing w_k. Go back to Step 1.*

Then $\phi_1(u)$ is the resulting pair (u, w) of the algorithm.

Let us give an example with $u = 315503200$. Our algorithm follows the steps:

1) $u = 315503200$, $w = 000000000$, then *pivot* $= 5$, $k = 4$ and $i = 3$;
2) $u = 31\mathbf{54}03200$, $w = 000\mathbf{1}00000$, then *pivot* $= 5$, $k = 3$ and $i = 1$;
3) $u = \mathbf{5}1\mathbf{2}403200$, $w = 00\mathbf{1}100000$, then *pivot* $= 4$, $k = 4$ and $i = 3$;
4) $u = 51\mathbf{41}03200$, $w = 001\mathbf{2}00000$, then *pivot* $= 4$, $k = 3$ and $i = 2$;
5) $u = 5\mathbf{40}103200$, $w = 00\mathbf{2}200000$, then *pivot* $= 3$, $k = 6$ and $i = 4$;
6) $u = 540\mathbf{3}0\mathbf{0}200$, $w = 00220\mathbf{1}000$ and the algorithm stops.

At the end, $\phi_1(315503200) = (540300200, 002201000)$.

This map corresponds to the last part of Diagram (2).

$$\mathrm{P} \xleftrightarrow{\psi_{FV}} \mathrm{WDP} \xleftrightarrow{\psi} \mathrm{WDP} \xleftrightarrow{\phi_2} \mathrm{DWSF} \xleftrightarrow{\phi_1} \mathrm{SF}$$

Proposition 4.4. *The map ϕ_1 is a well-defined function from subexcedent functions to decreasing weighted subexcedent functions.*

Proof. It is clear that ϕ_1 is well-defined for every subexcedent function u and that the algorithm stops (we have at most $n - 1 - \textit{pivot}$ swaps for each pivot).

The result is decreasing in the sense of Definition 4.1 because the algorithm sorts the subexcedent function.

To prove that the result is a decreasing weighted subexcedent function, we need to prove that the weight satisfies the constraints of Definition 4.1. The value of the weight at position j was increased at most once per pivot that ended on its left, so it is smaller than the number of non-zero values on the left of position j at the end of the algorithm. Moreover, it was increased only if the pivot was at position j, which is possible only if the pivot is smaller than or equal to $n - j$. So w satisfies the constraints of the definition. $\square$

Lemma 4.5. *Consider an execution of Step 2 of Algorithm 4.3. Immediately after that we can recover both positions i and k.*

Let j be the rightmost position such that $w_j \neq 0$. The position i of the pivot is the rightmost position on the left of j such that $u_i > u_j$.

The previous position k of the pivot is the nearest position to the right of i such that the associated weight is nonzero, i.e., *k is the smallest $k > i$ such that $w_k \neq 0$.*

Proof. A value u_i is *weighted* if the corresponding weight is nonzero, *i.e.*, $w_i > 0$.

We start by proving the second part of the lemma. Consider a step S where a pivot p in position k should exchange with the value in position i. Then, for all j such that $i < j < k$, we have $w_j = 0$. Indeed, assume there is a j such that $i < j < k$ and $w_j \neq 0$ and let S' be the step corresponding to the previous exchange concerning u_j and a pivot p'. As u_j is to the right of u_i in step S, we must have p' to the right of both u_i and u_j in step S'. Moreover, as p' exchange with u_j, we have either $u_j > u_i$ and after the exchange and the decrementation this inequality becomes $u_j \geq u_i$, or $u_j = u_i$ which implies that u_j is to the right of u_i since the exchange implies u_j, and that p' exchanges with u_i next. In both cases we end up with $u_j \geq u_i$ and as there are no exchange implying u_j between S' and S, only u_i may be decremented so we also have $u_j \geq u_i$ in situation S. But the exchange in situation S is between p and u_i with u_j in the middle which implies $u_j < u_i$ and therefore contradicts $u_j \geq u_i$.

Let us now prove the other part of the lemma. With the notations of the lemma, we necessarily have $u_i > u_j$ so we have to prove that for any s such that $i < s < j$, we have $u_s \leq u_j$. Note that every new pivot starts at the same position than the previous one or to its right. Moreover, as we proved above that there are no weighted values between both exchanged positions, the pivot exchanged with every weighted values to its right up to u_j and u_j was the first one. This implies that u_j is greater than all the values between the pivot u_i and itself. □

The previous lemma shows that the map ϕ_1 is injective by proving that we can find the previous pivot and the value with which it is swapped at each step. We prove in next section that decreasing weighted subexcedent functions are enumerated by $n!$ which proves the following proposition:

Proposition 4.6. *The map ϕ_1 is a bijection.*

Note that the inverse map comes straightforwardly from Lemma 4.5.

4.1.2. *The statistics through ϕ_1.* Let us first define the new statistics on the decreasing weighted subexcedent functions.

Definition 4.7. *Let (u, w) be a decreasing weighted subexcedent function of size n.*

- *The* number of inversions $\mathrm{inv}(u, w)$ *is the sum of the values of u and w.*
- *The* total weight $\mathrm{tw}(u, w)$ *is the sum of the values of w.*

- *The* descent set *of* (u, w) *is*

(10) $$\mathrm{Des}(u, w) = \{i \mid w_i > w_{i+1}\} \cup \{i \mid w_i = w_{i+1}, u_i > u_{i+1}\}$$

and its descent composition $\mathrm{DC}(u, w)$ *is the composition of* n *whose descent set is* $Des(u, w)$*.*
- *The* statistic LC *on* (u, w) *is the composition* $\mathrm{LC}(u)$*.*

Remark 4.8. *We shall make some remarks on the previous definitions.*

(1) The previous definitions are still correct if the weighted subexcedent function is not decreasing. Moreover, if we associate a null weight with a subexcedent function, those definitions give the same statistics as the usual ones on subexcedent functions.

(2) Note that on a decreasing subexcedent function, the mirror composition of the statistic LC *exactly corresponds to the composition whose descent set is the set of nonzero values of* u*. Hence, we have directly* $\mathrm{tw}(u, w) = \mathrm{inv}(u, w) - \mathrm{maj}(\overline{\mathrm{LC}(u, w)})$*.*

Proposition 4.9. *Let* u *be a subexcedent function, then:*

- $\mathrm{tw}(\phi_1(u)) = \mathrm{inv}(u) - \mathrm{maj}(\overline{\mathrm{LC}(u)})$*;*
- $\mathrm{DC}(\phi_1(u)) = \mathrm{DC}(u)$*;*
- $\mathrm{LC}(\phi_1(u)) = \mathrm{LC}(u)$*.*

To prove the last point of this proposition we shall consider the exchanges of Algorithm 4.3 as a succession of elementary exchanges described by the following algorithm, where i and k come from the notations of Algorithm 4.3.

Algorithm 4.10. *Set* j_1 *equal to* i *and* j_2 *equal to* k*.*

- *Step 1: If* $j_1 = k - 1$*, go to step 2. Otherwise, increment* u_{j_1+1} *and then swap* u_{j_1} *and* u_{j_1+1}*. Set* $j_1 = j_1 + 1$ *and redo Step 1.*
- *Step 2: If* $j_2 = i$*, the algorithm stops. Otherwise, decrement* u_{j_2-1} *and then swap* u_{j_2} *and* u_{j_2-1}*. Set* $j_2 = j_2 - 1$ *and redo Step 2.*

For example, if at some point of Algorithm 4.3, we have $u = \dots 4106 \dots$ and we have to exchange 6 with 4, the exchanges of Algorithm 4.10 would be:

- we begin with Step 1:
  - $u = \dots \mathbf{24}06 \dots$;
  - $u = \dots 2\mathbf{14}6 \dots$;
- and then apply Step 2:
  - $u = \dots 21\mathbf{63} \dots$;
  - $u = \dots 2\mathbf{60}3 \dots$;
  - $u = \dots \mathbf{61}03 \dots$

Proof of Proposition 4.9. The proof of this proposition is based on the first point of Remark 4.8 and works by proving that the statistics inv, DC and LC do not change at each exchange of Algorithm 4.3.

The inv statistic is not modified since each decrementation of a value of u is balanced by a incrementation of a value of w.

In order to prove that the descent set of a weighted subexcedent function does not change when we are doing an exchange of Algorithm 4.3, we have to study different cases depending on whether there is a descent at positions $i-1$, i, $k-1$, or k or not. Let $(u^{(1)}, w^{(1)})$ be the weighted subexcedent function before the exchange and $(u^{(2)}, w^{(2)})$ the one after it.

- We start by considering what happens at position k.
  - As $u_k^{(1)}$ is the rightmost occurrence of the *pivot*, we have $k \neq n$ as the pivot is nonzero and $u_k^{(1)} > u_{k+1}^{(1)}$ so the only way not to have a descent at k is that $w_k^{(1)} < w_{k+1}^{(1)}$. But in this case Lemma 4.5 tells us that the *pivot* was at position $k+1$ at the previous step, so $u_i^{(1)} \leq u_{k+1}^{(1)}+1$ and then after the exchange $u_k^{(2)} \leq u_{k+1}^{(2)}$ with $w_k^{(2)} \leq w_{k+1}^{(2)}$ so k is not a descent.
  - In any other case, $(u^{(1)}, w^{(1)})$ has a descent in k, so $w_k^{(1)} \geq w_{k+1}^{(1)}$ and then $w_k^{(2)} > w_{k+1}^{(2)}$ so $(u^{(2)}, w^{(2)})$ also has a descent in k.
- If $i \neq k-1$, Lemma 4.5 implies that $w_{k-1}^{(1)} = 0$. Moreover, $u_{k-1}^{(1)} < u_k^{(1)}$ so $k-1$ is not a descent for $(u^{(1)}, w^{(1)})$ and it is necessarily also the case for $(u^{(2)}, w^{(2)})$.
- If $i = k-1$, it is possible to have a descent or not in i.
  - As $u_i^{(1)} < u_k^{(1)}$, the only way to have a descent in i is to have $w_i^{(1)} > w_k^{(1)}$ and then $w_i^{(2)} \geq w_k^{(2)}$ with $u_i^{(2)} > u_k^{(2)}$.
  - If i is not a descent, $w_i^{(1)} \leq w_k^{(1)}$ and then $w_i^{(2)} < w_k^{(2)}$ so i is not a descent either for $(u^{(2)}, w^{(2)})$.
- For the descent in i when $i < k-1$, we have $w_{i+1}^{(1)} = 0$ and $u_i^{(1)} > u_{i+1}^{(1)}$ so i is always a descent of $(u^{(1)}, w^{(1)})$ in this case. Moreover, we still have $w_{i+1}^{(2)} = 0$ with $u_i^{(2)} > u_{i+1}^{(2)}$ so it is also the case for $(u^{(2)}, w^{(2)})$.
- For the position $i-1$, suppose that $u_{i-1}^{(1)}$ is not strictly greater than the *pivot*. As $w_{i-1}^{(1)} = w_{i-1}^{(2)}$ with $w_i^{(1)} = w_i^{(2)}$ and $u_{i-1}^{(2)} = u_{i-1}^{(1)} \leq u_i^{(1)} < u_i^{(2)}$, $(u^{(1)}, w^{(1)})$ has a descent in $i-1$ if and only

if it is the case for $(u^{(2)}, w^{(2)})$. If $u^{(1)}_{i-1}$ is not strictly greater than the *pivot*, the same argument gives the same result.

We have shown that in any possible situation, the descent set is constant after each exchange of Algorithm 4.3.

To prove that the LC statistic does not change at each step we prove that it is also true after each elementary exchange of Algorithm 4.10. As in the first step we have $u_{j_1} > u_{j_1+1}$ and in the second step we have $u_{j_2-1} \leq u_{j_2}$ we only have to prove that two subexcedent functions $u^{(1)}$ and $u^{(2)}$ which differ only at positions j and $j+1$ such that, if $u^{(1)} = \ldots ab \ldots$ with $a > b$, $u^{(2)} = \ldots b{+}1\ a \ldots$, then $\mathrm{LC}(u^{(1)}) = \mathrm{LC}(u^{(2)})$. There are three different cases. Let S be the set obtained during the computation of the two LC statistics before considering the value at position $j+1$.

- In $u^{(1)}$, if $b \leq |S|$ and $a \leq |S|$, then after considering those two values in $u^{(1)}$, the set used to compute $\mathrm{LC}(u^{(1)})$ is still S. In $u^{(2)}$ we also have $a \leq |S|$ and $b+1 \leq a \leq |S|$ so the set has not changed either.
- In $u^{(1)}$, if $b \leq |S|$ and $a > |S|$, let α be the $(a - |S|)$-th letter of $[1, n] \backslash S$, then after considering those two values, the set is equal to $S \cup \{\alpha\}$. In $u^{(2)}$, after considering a, the set is equal to $S \cup \{\alpha\}$ and then $b + 1 \leq |S| + 1$ so the set does not change.
- In $u^{(1)}$, if $b > |S|$ then the set is changed to $S \cup \{\beta\}$ where β is the $(b - |S|)$-th letter of $[1, n] \backslash S$. Then as $a > b$, we have $a > |S| + 1$ and the set changes again to $S \cup \{\beta, \alpha\}$ where α is the $(a - |S| - 1)$-th letter of $[1, n] \backslash (S \cup \{\beta\})$. For $u^{(2)}$, we still have $a > |S|$ so we add to S the $(a - |S|)$-th value of $[1, n] \backslash S$ which is also α because we necessarily have $\beta < \alpha$. Then when we read $b + 1$, we have $b + 1 > |S| + 1$ so we add to the set the $(b - |S|)$-th value of $[1, n] \backslash (S \cup \{\alpha\})$ which is again β, so the final set is the same in both cases.

This proves the last point of the proposition. $\square$

4.2. **Decreasing weighted subexcedent functions to weighted Dyck paths.** Let us now describe the final bijection between decreasing weighted subexcedent functions and weighted Dyck paths that corresponds to the third part of Diagram (2).

$$\mathrm{P} \overset{\psi_{FV}}{\longleftrightarrow} \mathrm{WDP} \overset{\psi}{\longleftrightarrow} \mathrm{WDP} \overset{\phi_2}{\longleftrightarrow} \mathrm{DWSF} \overset{\phi_1}{\longleftrightarrow} \mathrm{SF}$$

To describe it, we build a Dyck path from a decreasing subexcedent function and carry the weight without modifying it. Then we show that it sends the triple of statistics of the decreasing weighted subexcedent functions to the triple of statistics of the weighted Dyck paths.

4.2.1. *Description of the bijection ϕ_2 between decreasing weighted subexcedent functions and weighted Dyck paths.* The map ϕ_2 on a decreasing weighted subexcedent function (u, w) of size n is defined by carrying the weight w and building the Dyck path D from the decreasing subexcedent function u using the following algorithm.

Algorithm 4.11. *Set $D_1 := /$ and $D_{2n} := \backslash$. Then, for each i in $\{1, \dots, n-1\}$,*

- *set $D_{2i} := \backslash$ if $n-i$ is a value in u and $D_{2i} := /$ otherwise;*
- *set $D_{2i+1} := \backslash$ if $u_i = 0$ and $D_{2i+1} := /$ otherwise.*

An example is given in Figure 4. The positions of the zeroes in the subexcedent function are $\{3, 5, 6, 8, 9\}$ which correspond to the positions i such that $D_{2i+1} = \backslash$. The values correspond to $n - i$ where i are the positions where $D_{2i} = \backslash$.

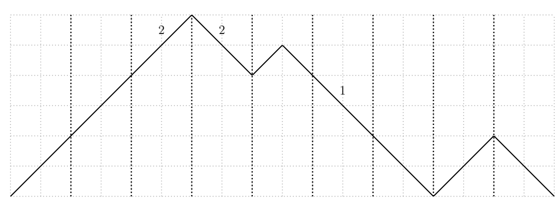
$\phi_2(540300200, 002201000) =$ 

FIGURE 4. An example of ϕ_2.

Lemma 4.12. *In the path D built in Algorithm 4.11, let h_i be the height be the height of the $2i$-th step of D. Then, for all i, we have $(h_i - 1)/2$ equal to the number of nonzero values smaller than or equal to $n - i$ in u to the left of u_i.*

Proof. Let us call W the maximal weight associated with u. Our aim is to prove that for all i we have $(h_i - 1)/2 = W_i$.

We prove this lemma by induction. For $i = 1$, we have $(h_1 - 1)/2 = 0 = W_0$ and the property holds. Assume the property for a given i. We have $W_{i+1} = W_i + 1$ if and only if $n - i$ is not a value in u and $u_i \neq 0$ so that all the values on the left of u_i that count for W_i also count for u_{i+1} and it is also the case for u_i. In this case we have $D_{2i} = D_{2i+1} = /$ and so $h_{i+1} = h_i + 2$ so the property holds. With the same idea, we have $W_{i+1} = W_i - 1$ if and only if $n - i$ is a value in u and $u_i = 0$, in this case we have $h_{i+1} = h_i - 2$. Finally, in the other two cases we have $W_i = W_{i+1}$ and $h_i = h_{i+1}$. □

Proposition 4.13. *The map ϕ_2 is well-defined from decreasing weighted subexcedent functions to weighted Dyck paths and is a bijection.*

Proof. The image of a decreasing weighted subexcedent function is a path. Lemma 4.12 proves that the height of the path is always nonnegative and that $h_n = 1$. As $D_{2n} = \backslash$, the path is a Dyck path. Lemma 4.12 also gives us that the constraints on the weight of a decreasing weighted subexcedent function correspond to the constraints of its image.

As the constraints for the weights correspond exactly from one object to the other, we only need to prove that ϕ_2 is a bijection from decreasing subexcedent functions to Dyck paths to prove that ϕ_2 is a bijection on the weighted objects.

As decreasing subexcedent functions and Dyck paths are both enumerated by the Catalan numbers, we only need to prove that this map is injective. The only way to have the same image from two decreasing subexcedent functions is that the nonzero values and their positions are fixed, but as the subexcedent functions are decreasing, there is no choice in the order of those values. $\square$

Note that by proving that ϕ_2 is a bijection, we proved that decreasing weighted subexcedent functions are enumerated by $n!$, so this finishes the proof of Proposition 4.6.

4.2.2. *The statistics through ϕ_2.*

Proposition 4.14. *Let (u, w) be a decreasing weighted subexcedent function. We have*

- $\mathrm{tw}(u, w) = \mathrm{tw}(\phi_2(u, w))$*;*
- $\mathrm{LC}(u, w) = \mathrm{GC}^0(\phi_2(u, w))$*;*
- $\mathrm{DC}(u, w) = \mathrm{DC}^0(\phi_2(u, w))$*.*

Proof. As the weight is carried without being changed, we have $\mathrm{tw}(u, w) = \mathrm{tw}(\phi_2(u, w))$. For the descent set, as the weight does not change and the positions of the zeroes in u correspond to the $\backslash$ steps at odd positions in D, we also have $\mathrm{DC}(u, w) = \mathrm{DC}(\phi_2(u, w))$. Moreover, $\mathrm{LC}(u, w)$ is the mirror composition of the composition whose descent set is the nonzero values in u and $\mathrm{GC}^0(D, w)$ is related to the $\backslash$ steps at even positions in D, so that we also have $\mathrm{LC}(u, w) = \mathrm{GC}^0(\phi_2(u, w))$. $\square$

5. Main result

5.1. **Proof of previous conjectures.** We now have all the tools we need to prove the conjectures of [9]. To do so, we prove Conjecture 6.2 which is the equidistribution of both triples of statistics.

Let $\Phi = \psi_{FV}^{-1} \circ \psi^{-1} \circ \phi_2 \circ \phi_1$.

Theorem 5.1. *Let I and J be two compositions of n. We have*

$$\sum_{\substack{u \in \mathfrak{S}\mathfrak{F}_n \\ \mathrm{DC}(u)=I \\ \mathrm{LC}(u)=J}} q^{\mathrm{inv}(u)-\mathrm{maj}(\overline{J})} = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{Rec}(\sigma)=I \\ \mathrm{GC}(\sigma)=J}} q^{\mathrm{tot}(\sigma)}. \tag{11}$$

Proof. As a composition of bijections, the map Φ is a bijection between subexcedent function and permutations. Let u be a subexcedent function. Applying Propositions 4.9, 4.14, 3.6, and 3.2 give

$$\begin{array}{rcl} \mathrm{LC}(u) & = & \mathrm{GC}(\Phi(u)) \\ \mathrm{DC}(u) & = & \mathrm{Rec}(\Phi(u)) \\ \mathrm{inv}(u) - \mathrm{maj}\left(\overline{\mathrm{LC}(u)}\right) & = & \mathrm{tot}(\Phi(u)) \end{array} \tag{12}$$

which proves the theorem. □

Remark 5.2. *The other conjectures of [9] are directly obtained from this one.*

Proving those conjectures proves that it is possible to interpret the entries of the transitions matrices of [9] in terms of the triple of statistics $(\mathrm{GC}, \mathrm{Rec}, \mathrm{tot})$. It also gives another q-refinement of the steady-state probabilities of the PASEP with statistics that arise from the combinatorics of the PASEP.

5.2. **Other properties of this bijection.** In addition to the three statistics we have studied, the bijection we defined from subexcedent functions to permutations also carries another statistic.

Definition 5.3. *Define the* left-to-right maxima *of a permutation as the values with only smaller values to their left.*

The same statistic on subexcedent functions is defined as the positions containing zeroes.

For example, with $u = 315503200$ as with $\sigma = \Phi(u) = 528713649$ the right to left maxima are 5, 8, and 9.

Note that the definition of the left-to-right maxima of a subexcedent function corresponds to its definition on permutations after taking the Lehmer code of the inverse of the permutation.

Proposition 5.4. *Let u be a subexcedent function. Its left-to-right maxima corresponds to the left-to-right maxima of $\Phi(u)$.*

To prove this proposition, we need to see what happens to this statistic through the different bijections. First we need the following lemma that gives an equivalent definition of the left-to-right maxima in terms of descents and 31−2 patterns.

Lemma 5.5. *Let σ be a permutation. A value $k = \sigma_j$ is a left-to-right maximum of σ if and only if $\sigma_{j-1} < \sigma_j$ and $\mathrm{tot}_k(\sigma) = 0$.*

Proof of Lemma 5.5. If k is a left-to-right maximum of σ, we necessarily have $\sigma_{j-1} < \sigma_j$ and there are no values greater than k to its left. So $\mathrm{tot}_k(\sigma) = 0$.

Conversely, assume there exists $i < j$ such that $\sigma_i > \sigma_j$ and let i be the rightmost position such that $\sigma_i > \sigma_j$ with $i < j$. Then $\sigma_i > \sigma_{i+1} < \sigma_j$ and this is a 31−2 pattern where k stands for the 2. □

With this lemma we are now able to follow what happens to the left-to-right maxima through the various bijections.

Proof of Proposition 5.4. Let σ be a permutation and k a value of σ. Lemma 5.5 shows that if we apply the Françon-Viennot bijection, the value k is a right to left maximum if and only if $D_{2k} = \setminus$ and $w_k = 0$ where $(D, w) = \psi_{FV}(\sigma)$. Thanks to Lemma 3.4 the value k is a right to left maximum if and only if $D^0_{2k+1} = \setminus$ and $w^0_k = 0$ where $(D^0, w^0) = \psi \circ \psi_{FV}(\sigma)$. Moreover, let (u, w) be the nondecreasing weighted subexcedent function obtained after applying ϕ_2^{-1} to (D^0, w^0). Then k is a right to left maximum of σ if and only if $u_k = 0$ and $w_k = 0$.

Finally, we need to prove that a value at position k in a subexcedent function v is equal to zero if and only if the value at the same position after applying the sorting function ϕ_2 is also zero with a null weight. In Algorithm 4.3 one can see that no pivot shall exchange with a value equal to zero so the result at position k shall also be zero with a null weight. Conversely, if the value at position k in $\phi_1(u)$ has a weight equal to zero, it means that the value never changed during the algorithm so the value is also equal to zero in u. □

6. A variation on the statistics on permutations

6.1. **Another combinatorial interpretation of $\mathfrak{M}^{(n)}(q)$.** In this part of the paper we change the convention on permutations such that $\sigma_0 = \sigma_{n+1} = 0$. In the definition of the Genocchi descent set, we compare each value to its successor so we need to define a new statistic associated with this new convention.

Definition 6.1. *Let $\sigma \in \mathfrak{S}_n$. We define the* Genocchi descent set of type 0 *as*

$$\mathrm{GDes}^0(\sigma) = \{\sigma_i \mid i \in \{1, \ldots, n\},\ \sigma_i > \sigma_{i+1}\}. \tag{13}$$

Note that σ_n *and* n *always belong to* $\mathrm{GDes}^0(\sigma)$. *We also define the* Genocchi composition of descents of type 0 *(denoted by* GC^0*) as the composition of* n *whose descent set is* $\mathrm{GDes}^0(\sigma)\backslash\{n\}$.

For example with $\sigma = 528971364$, we have $\mathrm{GDes}^0(\sigma) = \{4, 5, 6, 7, 9\}$ and $\mathrm{GC}^0(\sigma) = (4, 1, 1, 1, 2)$.

The Françon-Viennot bijection still exists with this convention and builds a large Laguerre history. We call ψ^0_{FV} the application constructing a weighted Dyck path from a permutation with this convention.

Definition 6.2. *Let* $\sigma \in \mathfrak{S}_n$*, define* $\psi^0_{FV}(\sigma)$ *as the path obtained by using Algorithm 2.4 for the values* 1 *to* $n-1$ *in* σ *and then adding an increasing step at the beginning of the path and a decreasing step at the end. The Algorithm builds a weight of size* $n-1$*, we add a* 0 *at the end to obtain a weight of size* n.

Note that we do not need to consider n in the construction as it is always greater than both its neighbors with this convention. An example is given in Figure 5 with $\sigma = 528971364$ alongside with $\psi_{FV}(\sigma)$.

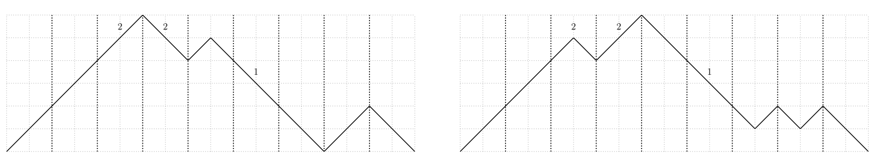


FIGURE 5. An example of ψ^0_{FV} on the left and ψ_{FV} on the right with $\sigma = 528971364$.

The inverse map is built as follows. Apply ψ^{-1}_{FV} to the path obtained after removing the first and last steps. At the end, instead of removing the last $\circ$ in the obtained word, replace it by n.

Proposition 6.3. *Let* $\sigma \in \mathfrak{S}_n$*, we have:*

- $\mathrm{GC}^0(\sigma) = \mathrm{GC}^0(\psi^0_{FV}(\sigma))$*;*
- $\mathrm{Rec}(\sigma) = \mathrm{DC}^0(\psi^0_{FV}(\sigma))$*;*
- $\mathrm{tot}(\sigma) = \mathrm{tw}(\psi^0_{FV}(\sigma))$.

The proof of this statement is the same as the one of Proposition 3.2, the only difference being that we have to add one to the index of the steps in the path as we add an extra step at the beginning when applying the Françon-Viennot bijection with this convention.

From this proposition we derive the following proposition.

Corollary 6.4. *Let I and J be two compositions of n, we have:*

$$\sum_{\substack{\mathrm{Rec}(\sigma)=I \\ \mathrm{GC}^0(\sigma)=J}} q^{\mathrm{tot}(\sigma)} = \sum_{\substack{\mathrm{Rec}(\sigma)=I \\ \mathrm{GC}(\sigma)=J}} q^{\mathrm{tot}(\sigma)}. \tag{14}$$

The proof of this statement is based on the same approach as the proof of Theorem 5.1, using the bijection $\Psi = (\psi^0_{FV})^{-1} \circ \psi \circ \psi_{FV}$.

This result gives another combinatorial interpretation of the entries of the transition matrices described previously. It also implies another way of refining the PASEP in terms of statistics close to the usual one used to describe its combinatorial behavior.

6.2. **Binary search trees.** By construction, Ψ preserves the recoil classes (that is the permutations having the same recoils composition). It happens that Ψ preserves smaller sets known as co-sylvester classes of permutations, which have been described in [5] with sylvester classes. A co-sylvester class is represented by a binary search tree.

Definition 6.5. *A* binary search tree *is a labeled binary tree such that for each node, all the values in the left (resp. right) subtree are smaller (resp. greater) than the value at the node.*

Definition 6.6. *The* restriction *of a binary search tree to an interval is the tree obtained after removing the values that are not in the interval then erasing the subtrees with no nodes and merging the edges with no values between them.*

An example of restriction of a binary tree to an interval is given in Figure 6.

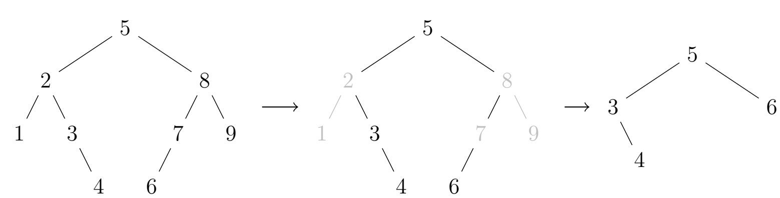


Figure 6. Restriction of a binary search tree to the interval $\{3, 4, 5, 6\}$.

By analogy, the restriction of a permutation to an interval is the word obtained by removing the values that are not in the interval. For example, $528971364_{|\{3,4,5,6\}} = 5364$.

We associate a binary search tree with a permutation through the following algorithm.

Algorithm 6.7. *At the beginning set* $I = \{1, 2, \cdots, n\}$*. The root of the current tree is the value* k *of* I *such that* k *is the first value of* $\sigma_{|_I}$*. Apply this algorithm to the interval of values smaller than* k *in* I *to obtain the left subtree and to the values greater than* k *in* I *for the right subtree.*

Denote by $\mathrm{BST}(\sigma)$ the map associated with this algorithm.

An example of this algorithm is given in Figure 7. In $\sigma = 528971364$ the first value is 5. The left subtree is built with $\sigma_{|_{\{1,2,3,4\}}} = 2134$ and the right subtree is built with $\sigma_{|_{\{6,7,8,9\}}} = 8976$.

Remark 6.8. *The classical way to build the tree is to read the permutation from left to right and add each value to its only possible position as a leaf in the binary search tree. So with* $\sigma = 528971364$*, start with a* 5*, then* 2 *must be to the left of* 5*,* 8 *to the right of* 5*,* 9 *to the right of* 5 *and of* 8*, and so on. In the rest of the paper we shall only need the description of Algorithm 6.7.*

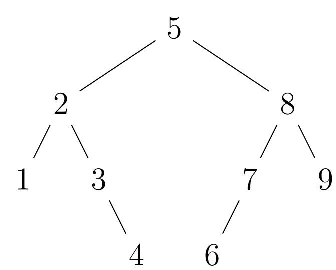


FIGURE 7. The binary search tree associated with $\sigma = 528971364$.

We say that two permutations σ and τ belong to the same co-sylvester class if their associated binary search trees are equal.

To prove the equality of binary search trees, it shall be convenient to use the description of the following lemma.

Lemma 6.9. *Let* A *and* A' *be two binary search trees. We have* $A = A'$ *if and only if they have the same root in their restriction to any interval.*

Proof of Lemma 6.9. If $A = A'$, the property is clearly true. Let A and A' be two binary search trees such that for any interval I they give the same root when restricted to I. In particular A and A' have the same root (if we take I to be the interval of all values in them). The proof of the lemma comes by induction by applying the property on the intervals of values smaller than the root and greater than the root which gives the two subtrees of the root. $\square$

6.3. **A co-sylvester class preserving bijection.** Our aim is to prove the following theorem with $\Psi = (\psi^0_{FV})^{-1} \circ \psi \circ \psi_{FV}$.

Theorem 6.10. *Let σ be a permutation. Then σ and $\Psi(\sigma)$ belong to the same co-sylvester class,* i.e., *have the same binary search tree.*

In order to prove this theorem, we need to build binary search trees associated with weighted Dyck paths corresponding to the one obtained from a permutation before applying ψ_{FV} and ψ^0_{FV}. Let us start with the algorithm associated with ψ_{FV}. Its main idea came after discussing with Aval, Boussicault, and Zubieta [1].

Algorithm 6.11. *Start with $I = \{1, 2, \cdots, n\}$.*

- *Among the $i \in I$ such that w_i is minimal, the root r of the current tree is equal to the smallest i such that $D_{2i} = \backslash$.*
- *If there are no such positions, let r be the maximal value of I among the ones of minimal weight.*

Apply this algorithm to the interval of values smaller than r in I to obtain the left subtree and to the values greater than r for the right subtree.

Denote by $\mathrm{BST}(D, w)$ the map associated with this algorithm.

An example of a Dyck path and its corresponding tree is given in Figure 9. At the first step, the i of minimum weight are $\{1, 2, 5, 7, 8, 9\}$. Among these, 5 is the minimal value such that $D_{2i} = \backslash$. To obtain the left subtree, we apply the algorithm to $I = \{1, 2, 3, 4\}$ on the path. The minimal weights are obtained for i in $\{1, 2\}$ but there are no i in this set such that $D_{2i} = \backslash$ so $r = 2$. Apply this algorithm respectively on $\{1\}$, $\{3, 4\}$, and $I = \{6, 7, 8, 9\}$ to get the other parts of the tree.

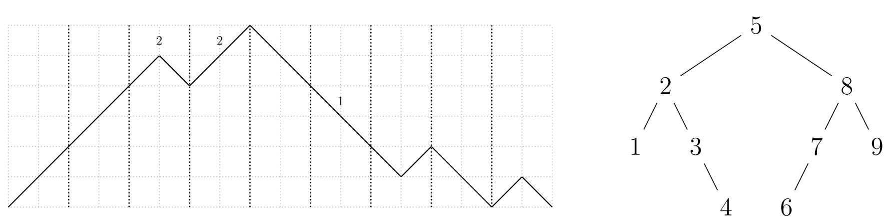


FIGURE 8. A weighted Dyck path and its associated binary search tree.

Remark 6.12. *Note that we could have applied this algorithm on any weighted path with $2n$ steps. It does not need to stay above the horizontal axis nor to end at position $(2n, 0)$.*

Lemma 6.13. *Let σ be a permutation. Then* $\mathrm{BST}(\sigma) = \mathrm{BST}(\psi_{FV}(\sigma))$.

In order to prove this lemma we need the following lemma.

Lemma 6.14. *Let σ be a permutation and $i < j < k$ be such that $\sigma_j < \sigma_k < \sigma_i$ (*i.e.*, they form a 3–1–2 pattern), then σ_k stands for the 2 in a 31−2 pattern where the 1 is to the right of σ_i.*

Proof of Lemma 6.14. Among the values greater than σ_k on the left of σ_j, let σ_r be the rightmost one. Necessarily, $\sigma_r > \sigma_k > \sigma_{r+1}$ and $r \geq i$ so it defines a 31−2 pattern of σ to the right of σ_i where σ_k is the 2. □

Proof of Lemma 6.13. To prove, we use Lemma 6.9. Let $A = \mathrm{BST}(\sigma)$ and $A' = \mathrm{BST}(D, w))$, where $(D, w) = \psi_{FV}(\sigma)$. Let I be an interval of values of σ.

The root of $A_{|_I}$ corresponds to the first value $\sigma_{|_I}$ since one easily sees that $A_{|_I} = \mathrm{BST}(\sigma_{|_I})$ (see an example on Figure 6). We have to prove that this value corresponds to the position returned by Algorithm 6.11 for the same interval on (D, w). In the rest of the proof, consider $\sigma = \cdots\sigma_i\cdots\sigma_j\cdots$ where σ_i is the first value of I in σ and σ_j is another value of I.

In order to prove that we have to search a step of minimal weight, we need to prove that for any $31-2$ pattern where σ_i is the 2, we have a $31-2$ pattern where σ_j is also the 2. Indeed, for any $31-2$ pattern where σ_i is the 2, the value representing the 3 (resp. the 1) is necessarily greater (resp. smaller) than all values in I so σ_j is also between them and so is involved as a 2 in a $31-2$ pattern.

If $\sigma_i > \sigma_{i-1}$ (so $D_{2i} = \backslash$) and if $\sigma_{j-1} < \sigma_j$ with $\sigma_j < \sigma_i$ then applying Lemma 6.14 proves that there is a 31–2 pattern to the right of σ_i where σ_j stands for the 2 and so the weight associated with σ_j in (D, w) is strictly greater than the one associated with σ_i. This proves that if $D_{2i} = \backslash$, it is the leftmost decreasing step in even position of minimal weight in the interval I.

If $\sigma_i < \sigma_{i-1}$ (so $D_{2i} = /$) and if $\sigma_j > \sigma_i$ then $\sigma_{i-1}\sigma_i\cdots\sigma_j$ is a $31-2$ pattern because σ_{i-1} is greater than any value in I. This proves that D_{2i} is the rightmost even step among the ones of minimal weight. Moreover, if $\sigma_i > \sigma_j$ and $\sigma_{j-1} < \sigma_j$, we can again use Lemma 6.14 to prove that there is a $31-2$ pattern where σ_j is a 2 and where the 1 is on the right of σ_i, so $w_i < w_j$ and there are no decreasing steps of minimal weight in position $2k$ for k in I.

We use Lemma 6.13 to prove that $A = A'$. □

Similarly we define the map BST^0 building a binary search tree from a weighted Dyck path equal to the binary search tree associated with a permutation before applying ψ^0_{FV}.

Definition 6.15. *Let (D, w) be a weighted Dyck path. Let D' be the path obtained by removing the first step of D and adding a decreasing step at its end. Define $\mathrm{BST}^0(D, w)$ as the result of $\mathrm{BST}(D', w)$.*

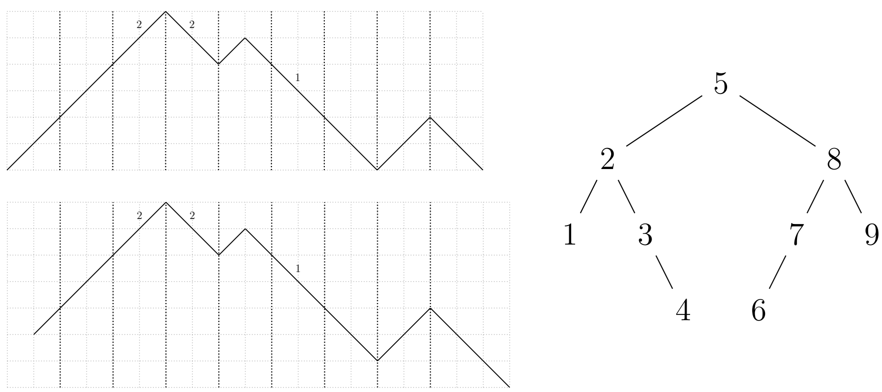


FIGURE 9. A weighted Dyck path with its intermediate path and its binary search tree of type 0.

Similarly to Lemma 6.13, we have the following lemma.

Lemma 6.16. *Let σ be a permutation. Then,* $\mathrm{BST}(\sigma) = \mathrm{BST}^0(\psi^0_{FV}(\sigma))$.

The proof of this lemma is the same as the proof of Lemma 6.13 as ψ^0_{FV} corresponds to the same algorithm as ψ_{FV} (with the different convention) if we remove the first step of the resulting path and add a decreasing step at the end.

The proof of Theorem 6.10 follows directly.

Proof of Theorem 6.10. As it can be seen in Equation (7) in the proof of Lemma 3.6, we have $\psi(D, w)_{2i+1} = D_{2i}$ so for any weighted Dyck path (D, w), we have $\mathrm{BST}(D, w) = \mathrm{BST}^0(\psi(D, w))$. Then, from Lemma 6.13 and Lemma 6.16 we have $\mathrm{BST}(\sigma) = \mathrm{BST}(\Psi(\sigma))$. $\square$

Remark 6.17. *The second fundamental transformation of Foata [8] preserves the sylvester class of a permutation which is the same as the mirror image of its co-sylvester class. Our bijection is really different from Foata's and in particular they are not conjugate to each other. Indeed, they do not have the same number of fixed points over $\mathfrak{S}_n$ for general n. For example, with $n = 5$, the second fundamental transformation of Foata has 26 fixed points whereas ours has 32. The two* *bijections also differ on the number of orbits under the action implied by the successive application of the map.*

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