Eulerian polynomials on segmented permutations

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Motivations

In the combinatorics of the 2-ASEP, the authors of [1] considered segmented permutations on which they defined a descent statistic. Our aim here is to study the number of segmented permutations according to this statistic, generalizing the Eulerian numbers.

$n\backslash r$	0	1	2	3	4	5
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Figure 1: Triangle of the usual Eulerian numbers.

$n\backslash r$	0	1	2	3	4	5
2	3	1				
3	13	10	1			
4	75	91	25	1		
5	541	896	426	56	1	
6	4683	9829	6734	1674	119	1

Figure 2: Triangle of the generalized Eulerian numbers.

Segmented permutations

A segmented permutation is a permutation where the values can be separated by bars. We denote by $\mathfrak{P}_n$ the set of segmented permutations of size n. There are $2^{n-1}n!$ segmented permutations of size n.

Let $\sigma \in \mathfrak{P}_n$.

- $\mathrm{seg}(\sigma)$ is the number of bars in σ. For example,
$$\mathrm{seg}(3|7156|24) = 2.$$
- A descent is a position i such that $\sigma_i > \sigma_{i+1}$ and there is no bar between them. $\mathrm{des}(\sigma)$ counts the number of descents of σ. For example,
$$\mathrm{des}(3|7156|24) = 1.$$

Generalized Eulerian numbers

For any $0 \leq k \leq n-1$, define
$$T(n,k) := \#\{\sigma \in \mathfrak{P}_n \mid \mathrm{des}(\sigma) = k\}. \qquad \text{(see Figure 2)}$$

For any $0 \leq k \leq n-1$,

- $T(n,0) =$ Fubini numbers (A008277);
- $T(n, n-k-1) = \#\{\sigma \in \mathfrak{P}_n \mid \mathrm{seg}(\sigma) + \mathrm{des}(\sigma) = k\}$.
- $T(n,k) = (n-k)T(n-1,k-1) + (n+1)T(n-1,k) + (k+1)T(n-1,k+1)$.

Refined generalized Eulerian numbers

For any $0 \leq i+j \leq n-1$, define
$$K(n,i,j) := \#\{\sigma \in \mathfrak{P}_n \mid \mathrm{des}(\sigma) = i,\ \mathrm{seg}(\sigma) = j\}.$$

$n = 2$:

$j\backslash i$	0	1
0	1	1
1	2	

$n = 3$:

$j\backslash i$	0	1	2
0	1	4	1
1	6	6	
2	6		

$n = 4$:

$j\backslash i$	0	1	2	3
0	1	11	11	1
1	14	44	14	
2	36	36		
3	24			

$n = 5$:

$j\backslash i$	0	1	2	3	4
0	1	26	66	26	1
1	30	210	210	30	
2	150	420	150		
3	240	240			
4	120				

n:

$j\backslash i$	0	$\cdots$	$n-1$
0		$\cdots$	
$\vdots$	$\vdots$	$K(n,i,j)$	
$n-1$			

Row 0 ← Eulerian numbers; column 0 ↑ Ordered Bell numbers

For any $0 \leq i+j \leq n-1$,

- $K(n,i,j) = K(n, n-j-i, j)$;
- $K(n,0,j) = (j+1)!S(n,j+1)$;
- $K(n,i,j) = (i+j+1)\Big[K(n-1,i,j) + K(n-1,i,j-1)\Big] + (n-i-j)\Big[K(n-1,i-1,j) + K(n-1,i-1,j-1)\Big]$.

Generalized Eulerian polynomials

Definitions and first properties

Define
$$P_n(t) := \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)}; \qquad \alpha_n(t,q) := \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{seg}(\sigma)}.$$

Let $n \geq 0$ then

- $\alpha_n(t,0) = A_n(t)$,
- $\alpha_n(0,q) = B_n(q)$,
- $\alpha_n(t,t) = t^{n-1}P_n\left(\frac{1}{t}\right)$,
- $\alpha_n(-1,1) = 2^{n-1}$,
- $\alpha_n(2,1) =$ A050351,
- $\alpha_n(2,2) =$ A050352.

Define the generating function $G(t,q) := \sum_{n \geq 0} \alpha_n(t,q) \frac{x^n}{n!}$. Using the recurrence on $K(n,i,j)$, we obtain a differential equation on the G.
$$(tq - 2q - 1)G(t,q,x) + (1 - tqx - tx)\frac{\partial}{\partial x}G(t,q,x) -$$
$$(t - t^2)(q+1)\frac{\partial}{\partial t}G(t,q,x) - (1-t)(q^2+q)\frac{\partial}{\partial q}G(t,q,x) = -2q + tq.$$

Theorem

We have the following generating function:
$$G(t,q,x) = 1 + \frac{e^{x(1-t)} - 1}{1 + q - (t+q)e^{x(1-t)}}. \qquad (1)$$

Corollaries

- Worpitzky's identity: for any positive integers r, k, and n,
$$\binom{k+r-1}{r}\Delta^{r+1}((k-1)^n) = \sum_{i=0}^{k-1}\binom{n+k-i}{n-1}K(n,i,r),$$
where $\Delta(k^n) = (k+1)^n - k^n$.
- For any $n \geq 0$, we have
$$\frac{\alpha_n(t,1)}{(1-t)^{n+1}} = \sum_{k \geq 0}(1+t)^{k-1}\frac{k^n}{2^{k-1}}.$$
- For any $n \geq 0$, we have
$$\alpha_n(t,q) = \sum_{0 \leq i+j \leq n-1} t^i (q-t)^j (1-t)^{n-i-j-1} 2^i (i+j+1)! \binom{i+j}{j} S(n, i+j+1).$$

Algebraic study

An algebra on segmented permutation

SPQSym is a graded algebra whose bases are indexed by segmented permutations. The product on the basis G_σ is given by the convolution
$$\mathrm{G}_\sigma \cdot \mathrm{G}_\tau = \sum_{\mu \in \sigma * \tau} \mathrm{G}_\mu.$$

For example,
$$\mathrm{G}_{2|13} \cdot \mathrm{G}_{12} = \mathrm{G}_{2|1345} + \mathrm{G}_{2|13|45} + \mathrm{G}_{2|1435} + \mathrm{G}_{2|14|35} + \cdots + \mathrm{G}_{4|3512} + \mathrm{G}_{4|35|12}$$

Noncommutative generalized Eulerian polynomials

For any $n \geq 0$, define
$$\mathcal{A}_n(t,q) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)+1} q^{\mathrm{seg}(\sigma)} \mathrm{G}_\sigma.$$

For example,
$$\mathcal{A}_2(t,q) = t\mathrm{G}_{12} + t^2\mathrm{G}_{21} + tq(\mathrm{G}_{1|2} + \mathrm{G}_{2|1}).$$

The subalgebra **SCQSym**

Consider
$$R_I = \sum_{\mathrm{Des}(\sigma) = I} \mathrm{G}_\sigma.$$

For example, $R_{11|1} = \mathrm{G}_{21|3} + \mathrm{G}_{32|1} + \mathrm{G}_{31|2}$.

The $(R_I)_I$ form a linear basis of a subalgebra **SCQSym** [2] of **SPQSym**. By analogy with the usual case, consider
$$S^I = \sum_{I \succeq J} (-1)^{\mathrm{des}(I) - \mathrm{des}(J)} S^J.$$

We have
$$\mathcal{A}_n(t,q) = (1-t)^n \sum_{I \models n} \left(\frac{t}{1-t}\right)^{\mathrm{des}(I)} \left(\frac{q-t}{1-t}\right)^{\mathrm{seg}(I)} S^I.$$

(1) then comes directly using the algebra morphism $S_k \mapsto \frac{x^k}{2^k k!}$

References

[1] S. Corteel and A. Nunge, *"2-Species Exclusion Processes and Combinatorial Algebra," FPSAC2017 proceedings*, 2017.

[2] J.-C. Novelli and J.-Y. Thibon, *"Hopf algebras and dendriform structures arising from parking functions," Fund. Math.*, 2007.