2-species exclusion processes and combinatorial algebras

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Complete basis (analog of h_λ)

For all n, define

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$$S^I = S_{i_1} S_{i_2} \cdots S_{i_r}.$$

For example, $S_2(a_1, a_2, a_3) = a_1^2 + a_1 a_2 + a_1 a_3 + a_2^2 + a_2 a_3 + a_3^2$.

Ribbon basis

$$R_I = \sum_{J \preceq I} (-1)^{l(J)-l(I)} S^J .$$

For example, $R_{221} = S^{221} - S^{41} - S^{23} + S^5$.

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Polynomial realization

$$R_I = \sum_{\mathrm{Des}(w)=I} w.$$

For example, $R_{221}(a_1, a_2) = a_1 a_2 a_1 a_2 a_1 + a_2 a_2 a_1 a_2 a_1$.

Tevlin's bases

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Transition matrices

The transition matrices between the ribbon basis and the fundamental basis of size 3 and 4 are:

$$\mathfrak{M}_3 = \begin{pmatrix} 1 & . & . & . \\ . & 2 & 1 & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix}$$

$$\mathfrak{M}_4 = \begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 3 & 2 & . & 1 & 1 & . & . \\ . & . & 2 & . & 1 & . & . & . \\ . & . & 1 & 3 & . & 2 & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 2 & 1 & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix}$$

Statistics on permutations

- $\mathrm{Rec}(\sigma)$ is the composition associated with the values of recoils (*i.e.*, the values k such that $k+1$ is on the left).
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- $\mathsf{GC}(\sigma)$ is the composition associated with the values of descents (*i.e.*, the values $k = \sigma_i$ such that $\sigma_i > \sigma_{i+1}$) minus one.
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Combinatorial interpretation (F. Hivert, J.-C. Novelli, L. Tevlin, J.-Y. Thibon, 2009)

$$\begin{pmatrix} 1 & . & . & . & . & . & . & . \\ . & 3 & 2 & . & 1 & 1 & . & . \\ . & . & 2 & . & 1 & . & . & . \\ . & . & 1 & 3 & . & 2 & 1 & . \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 2 & 1 & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{pmatrix}$$

GC \ Rec	4	31	22	211	13	121	112	1111
4	1234							
31		1243, 1423 4123	1342 3412		2341	2413		
22			1324 3124		2314			
211			3142	1432, 4132 4312		2431 4231	3241	
13					2134			
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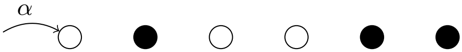

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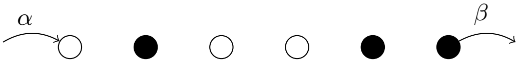

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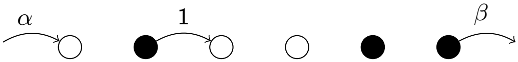

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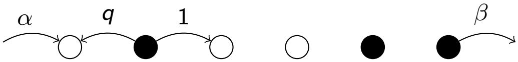

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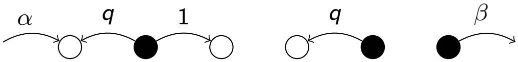

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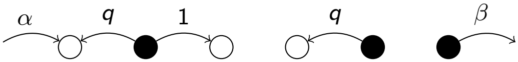

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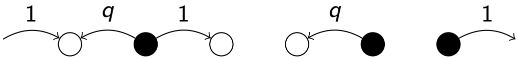

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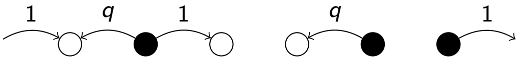

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Combinatorial study of the ASEP

Let I be a composition associated with a state of the ASEP, the un-normalized steady-state probability of this state is given by:

Permutations

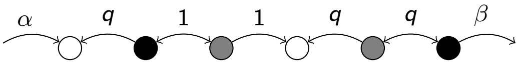
$$\sum_{\mathrm{GC}(\sigma)=I} q^{\#31\text{–}2(\sigma)}$$

Laguerre histories

$$\sum_{\mathrm{type}(P)=I} q^{\mathrm{weight}(P)}$$

2-ASEP

The 2-ASEP is a generalization of the ASEP with two kinds of particles.



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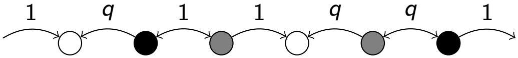

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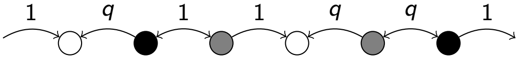

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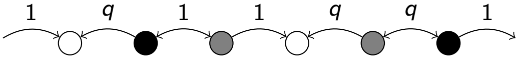

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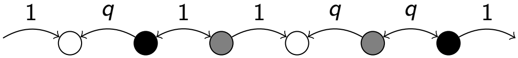

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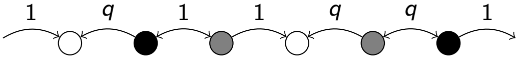

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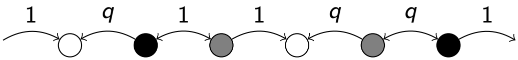

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Algebraic study

The algebra of segmented compositions

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Complete basis

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For example, $S_{21|1} \cdot S_{32|21} = S_{21|132|21} + S_{21|1|32|21}$.

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Ribbon basis

Again we have

$$R_I = \sum_{J \preceq I} (-1)^{l(J)-l(I)} S_J.$$

For example, $R_{22|41} = S_{22|41} - S_{4|41} - S_{22|5} + S_{4|5}$.

Number of generators

Let G be the generating series of an algebraic basis of **SCQSym** and B be the generating series of the vectorial basis associated with it.

$$B = 1 + \sum_{n \geq 1} G^n = \frac{1}{1 - G}$$

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Here $B = 1 + \sum_{n \geq 1} 3^{n-1} t^n$ so we have

$$G = \frac{t}{1-2t} = \sum_{n \geq 1} 2^{n-1} t^n.$$

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We index our generating family by segmented compositions without bars.

Analogue of Tevlin's bases

Let I be a non-segmented composition. Define M_I with the same transition matrix as for the case of **Sym**.

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We use a generalization of the structure coefficients. We obtain the following transition matrix between the complete basis and this monomial basis.

$$\mathcal{A}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & . & . & . & . & . \\ 1 & 3 & 2 & 4 & . & . & . & . & . \\ 1 & 1 & 2 & 2 & . & . & . & . & . \\ 1 & 3 & 3 & 6 & . & . & . & . & . \\ . & . & . & . & 3 & 4 & . & . & . \\ . & . & . & . & 3 & 6 & . & . & . \\ . & . & . & . & . & . & 2 & 2 & . \\ . & . & . & . & . & . & 3 & 6 & . \\ . & . & . & . & . & . & . & . & 6 \end{pmatrix}$$

Fundamental basis

Define the fundamental basis from the monomial one as follows.

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Transition matrix

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$$\mathcal{M}_3 = \left(\begin{array}{cccc|cc|cc|c}
1 & . & . & . & . & . & . & . & . \\
. & 2 & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
\hline
. & . & . & . & 3 & 1 & . & . & . \\
. & . & . & . & . & 2 & . & . & . \\
\hline
. & . & . & . & . & . & 2 & . & . \\
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The transition matrix between the ribbon basis to the fundamental basis is the following for $n = 3$.

$$\mathcal{M}_3 = \left(\begin{array}{cccc|cc|cc|c}
1 & . & . & . & . & . & . & . & . \\
. & 2 & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
\hline
. & . & . & . & 3 & 1 & . & . & . \\
. & . & . & . & . & 2 & . & . & . \\
\hline
. & . & . & . & . & . & 2 & . & . \\
. & . & . & . & . & . & 1 & 3 & . \\
\hline
. & . & . & . & . & . & . & . & 6
\end{array}\right)$$

$$(\mathcal{M}_n)_{I,J} = \#\{\sigma \mid \mathsf{GC}(\sigma) = I, \mathrm{Rec}(\sigma) = J\}$$

q-analogs

We define q-analogs of $S_\mathbf{I}$ and $M_\mathbf{I}$ and we obtain the following transition matrices.

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$$\begin{pmatrix}
1 & 1 & 1 & 1 & . & . & . & . & . \\
1 & q^2+q+1 & q+1 & q^2+2q+1 & . & . & . & . & . \\
1 & 1 & q+1 & q+1 & . & . & . & . & . \\
1 & q^2+q+1 & q^2+q+1 & q^3+2q^2+2q+1 & . & . & . & . & . \\
. & . & . & . & q^2+q+1 & q^2+2q+1 & . & . & . \\
. & . & . & . & q^2+q+1 & q^3+2q^2+2q+1 & . & . & . \\
. & . & . & . & . & . & q+1 & q+1 & . \\
. & . & . & . & . & . & q^2+q+1 & q^3+2q^2+2q+1 & . \\
. & . & . & . & . & . & . & . & q^3+2q^2+2q+1
\end{pmatrix}$$

$$S_{\mathbf{I}} \to M_{\mathbf{J}}$$

q-analogs

We define q-analogs of $S_{\mathbf{I}}$ and $M_{\mathbf{I}}$ and we obtain the following transition matrices.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & . & . & . & . & . \\
. & q+1 & 1 & q+2 & . & . & . & . & . \\
. & . & 1 & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
. & . & . & . & q^2+q+1 & q^2+2q+1 & . & . & . \\
. & . & . & . & . & q+1 & . & . & . \\
. & . & . & . & . & . & q+1 & q+1 & . \\
. & . & . & . & . & . & q & q^2+2q+1 & . \\
. & . & . & . & . & . & . & . & q^3+2q^2+2q+1
\end{pmatrix}$$

$$S_{\mathbf{I}} \to L_{\mathbf{J}}$$

q-analogs

We define q-analogs of $S_{\mathbf{I}}$ and $M_{\mathbf{I}}$ and we obtain the following transition matrices.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & . & . & . & . & . \\
. & q+1 & 1 & q+2 & . & . & . & . & . \\
. & . & 1 & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
. & . & . & . & q^2+q+1 & q^2+2q+1 & . & . & . \\
. & . & . & . & . & q+1 & . & . & . \\
. & . & . & . & . & . & q+1 & q+1 & . \\
. & . & . & . & . & . & q & q^2+2q+1 & . \\
. & . & . & . & . & . & . & . & q^3+2q^2+2q+1
\end{pmatrix}$$

$$S_{\mathbf{I}} \to L_{\mathbf{J}}$$

$$\begin{pmatrix}
1 & . & . & . & . & . & . & . & . \\
. & q+1 & 1 & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . \\
. & . & . & . & q^2+q+1 & q & . & . & . \\
. & . & . & . & . & q+1 & . & . & . \\
. & . & . & . & . & . & q+1 & . & . \\
. & . & . & . & . & . & q & q^2+q+1 & . \\
. & . & . & . & . & . & . & . & q^3+2q^2+2q+1
\end{pmatrix}$$

$$R_{\mathbf{I}} \to L_{\mathbf{J}}$$

Enumerative result

We obtain an enumerative formula for the probabilities of the 2-ASEP. Let I be a segmented composition, the steady-state probability of the state of the 2-ASEP encoded by I is given by the following formula.

$$[r+1]_q! \sum_{I \succeq J} \left(\frac{-1}{q}\right)^{l(I)-l(J)} q^{-\mathrm{st}(I,J)} c_J(q)$$

Where $c_J(q) = [k]_q^{j_1} [k-1]_q^{j_2} \cdots [2]_q^{j_{k-1}} [1]_q^{j_k}$.

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- find α and β statistics on partially signed permutations.
- Understand the refinement $(\mathsf{GC}, \mathrm{Rec})$ on the 2-ASEP.