Eulerian polynomials and generalizations

Arthur Nunge

IRIF

Mach 2019

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1				
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1	4	1		$A_3(t) = 1 + 4t + t^2$
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- $\alpha_n(t, q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{seg}(\sigma)}$

Historical definition (1749)

In an attempt to compute the evaluation of the zeta function on negative integers, Euler defined the polynomials $A_n(t)$ as follow :

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- $A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4.$

Descents of a permutation

A position i of a permutation σ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\mathrm{des}(\sigma)$ the number of descents of σ.

For $\sigma = 514798263$, the descents are $\mathrm{Des}(\sigma) = \{1, 5, 6, 8\}$

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For any n and $k < n$, define

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$$A_n(t) = \sum_{k=0}^{n-1} A(n,k) t^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}.$$

Some results about Eulerian numbers and polynomials

- We have

$$\sum_{n\geq 0} A_n(t)\frac{x^n}{n!} = \frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}.$$

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$$A_n(t) = \sum_{r=0}^{n-1} t^r(1-t)^{n-1-r}(r+1)!S(n,r+1),$$
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How to prove these ?

Proposition

For any $k < n$, we have

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k).$$

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Corollary

Let G denotes $A_n(t)$ generating function, it satisfies the following differential equation

$$(1-tx)\frac{\partial}{\partial x}G(t,x) - t(1-t)\frac{\partial}{\partial t}G(t,x) - G(t,x) = 0.$$

Algebraic study of the Eulerian polynomials

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$$\mathbb{G}_\sigma \cdot \mathbb{G}_\tau = \sum_{\mu \in \sigma * \tau} \mathbb{G}_\mu,$$

where the permutations in the sum are in $\mathfrak{S}_{n+p}$. In order to have ϕ a morphism we need exactly $\binom{n+p}{n}$ elements in the sum.

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There are two possibilities, we consider the one where we concatenate σ and τ and consider all the possibilities to create a permutation. For example,

$$\mathbb{G}_{312} \cdot \mathbb{G}_{21} = \mathbb{G}_{31254} + \mathbb{G}_{41253} + \mathbb{G}_{41352} + \mathbb{G}_{42351} + \mathbb{G}_{51243} + \cdots + \mathbb{G}_{53421}$$

Noncommutative analog

For any $n \geq 0$, define

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Using ϕ we have $\phi(\mathcal{A}_n(t)) = tA_n(t)\dfrac{x^n}{n!}.$

$$\begin{aligned}\mathcal{A}_3(t) &= t\mathbb{G}_{123} + t^2\left(\mathbb{G}_{132} + \mathbb{G}_{213} + \mathbb{G}_{231} + \mathbb{G}_{312}\right) + t^3\mathbb{G}_{321} \\ &= t\mathbf{R}_3 + t^2\left(\mathbf{R}_{21} + \mathbf{R}_{12}\right) + t^3\mathbf{R}_{111}\end{aligned}$$

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$$= t\mathbf{R}_3 + t^2\left(\mathbf{R}_{21} + \mathbf{R}_{12} \right) + t^3\mathbf{R}_{111}$$

And more generally,

$$\mathcal{A}_n(t) = \sum_{I \models n} t^{\ell(I)}\mathbf{R}_I$$

where **R** is the ribbon basis of the noncommutative symmetric functions algebra (**Sym**) and the sum goes over all composition of size n (sequences of integers of sum n).

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We use the complete basis **S** of **Sym** with

$$\mathbf{R}_I = \sum_{I \succeq J} (-1)^{\ell(I)-\ell(J)}\mathbf{S}^J.$$

For example,

$$\mathbf{R}_{121} = \mathbf{S}^{121} - \mathbf{S}^{13} - \mathbf{S}^{31} + \mathbf{S}_4.$$

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Then,

$$\mathcal{A}_3(t) = t(1-t)^2\mathbf{S}_3 + t^2(1-t)\left(\mathbf{S}^{12} + \mathbf{S}^{21}\right) + t^3S^{111}.$$

We have

$$\mathcal{A}_n(t) = \sum_{I \models n} t^{\ell(I)}(1-t)^{n-\ell(I)} \mathbf{S}^I;$$

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To apply ϕ, we use the fact that $\mathbf{S}$ is a multiplicative basis. For example,

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Moreover, $\phi\left(S_k\right) = \dfrac{x^k}{k!}$ so $\phi\left(\sum_{\substack{I \models n \\ \ell(I)=r}} \mathbf{S}^I\right) = r! S(n,r) \dfrac{x^n}{n!}$.

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$$tA_n(t) = \sum_{r=1}^{n} t^r (1-t)^{n-r} r! S(n,r).$$

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If we consider the generating function of the $(1-t)^{-n}\mathcal{A}_n$, we obtain

$$\sum_{n\geq 0} \frac{\mathcal{A}_n}{(1-t)^n} = \sum_{r\geq 0} \left(\frac{t}{1-t}\right)^r (\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \cdots)^r.$$

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We use the fact that

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Then applying ϕ to the previous equation we obtain

$$1 + \sum_{n\geq 0} \frac{tA_n(t)}{(1-t)^n} \frac{x^n}{n!} = \frac{1-t}{1-te^x}$$

Segmented permutations

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In size 3

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132	1|32	13|2	1|3|2
213	2|13	21|3	2|1|3
231	2|31	23|1	2|3|1
312	3|12	31|2	3|1|2
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In general, there are $2^{n-1}n!$ segmented permutations of size n.

Some statistics on segmented permutation

- A position i of a permutation σ is a segmentation if there is a bar between σ_i and σ_{i+1}. We denote by $\mathrm{seg}(\sigma)$ the number of descents of σ.
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- A descent is a position i that is not a segmentation and such that $\sigma_i > \sigma_{i+1}$. We denote by $\mathrm{des}(\sigma)$ the number of descents of σ.
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Eulerian numbers on segmented permutations

We define the following numbers:

$$T(n,k) = \#\{\sigma \in \mathfrak{P}_n \,|\, \mathrm{des}(\sigma) = k\}$$

$n \backslash k$	0	1	2	3	4
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We also have

$$T(n,n-k-1) = \#\{\sigma \in \mathfrak{P}_n | \operatorname{des}(\sigma) + \operatorname{seg}(\sigma) = k\}.$$

Generalized Eulerian numbers with two parameters

We also consider the following refinement

$$K(n,i,j) = \#\{\sigma \in \mathfrak{P}_n \mid \mathrm{des}(\sigma) = i, \mathrm{seg}(\sigma) = j\}$$

Generalized Eulerian numbers with two parameters

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$$K(n,i,j) = \#\{\sigma \in \mathfrak{P}_n \mid \mathrm{des}(\sigma) = i, \mathrm{seg}(\sigma) = j\}$$

$n = 2$:

$j \backslash i$	0	1
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$n = 3$:

$j \backslash i$	0	1	2
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$$\begin{aligned} K(n,i,j) = (i+j+1)\Big[K(n-1,i,j) + K(n-1,i,j-1)\Big] \\ +(n-i-j)\Big[K(n-1,i-1,j) + K(n-1,i-1,j-1)\Big]. \end{aligned}$$

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Some specialization of the variables give the following properties

$$\begin{cases} \alpha_n(t,0) &= A_n(t) \\ \alpha_n(0,q) &= B_n(q) \\ \alpha_n(1,1) &= 2^{n-1} n! \\ \alpha_n(-1,1) &= 2^{n-1} \\ \alpha_n(2,1) &= A050352 \\ \alpha_n(2,2) &= A050351 \end{cases}$$

Generating Function

We define the generating function of the generalized Eulerian polynomials as follows:

$$F(t,q,x) = \sum_{n\geq 0} \alpha_n(t,q)\frac{x^n}{n!}.$$

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The generating function satisfies the following differential equation:

$$(tq-2q-1)F(t,q,x)+(1-tqx-tx)\frac{\partial}{\partial x}F(t,q,x)-(t-t^2)(q+1)\frac{\partial}{\partial t}F(t,q,x)-$$
$$(1-t)(q^2+q)\frac{\partial}{\partial q}F(t,q,x) = -2q+tq.$$

Theorem (N., 2018+)

We have the following expression of the generating function:

$$F(t,q,x) = 1 + \frac{e^{x(1-t)} - 1}{1 + q - (t+q)e^{x(1-t)}}.$$

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$$F(t,q,x) = \frac{G(t,x)}{1 - qG(t,x)}.$$
- Use a noncommutative analog.

Some properties of the generalized Eulerian polynomials

- For any $n \geq 0$ we have

$$\frac{\alpha_n(t,1)}{(1-t)^{n+1}} = \sum_{k\geq 0} k^n \frac{(1+t)}{2^{k+1}}^{k-1}.$$

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- Worpitzky's identity: for any positive integers r, k, and n,

$$\binom{k+r-1}{r} \Delta^{r+1}((k-1)^n) = \sum_{i=0}^{k-1} \binom{n+k-i}{n-1} K(n,i,r),$$

where $\Delta(k^n) = (k+1)^n - k^n$.

FQSym (permutations)

$\uparrow$

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SCQSym (segmented compositions)

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$$\mathbf{Sym} \hookrightarrow \mathbf{FQSym}$$

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We have the morphism $\phi(\mathbf{G}_\sigma) = \frac{x^n}{2^{n-1} n!}$ for any $\sigma \in \mathfrak{P}_n$.

Noncommutative analog

For any $n \geq 0$, define

$$\widetilde{\mathcal{A}}_n(t) = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)+1} q^{\mathrm{seg}(\sigma)} \mathbf{G}_{\sigma}.$$

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$$\begin{aligned}\widetilde{\mathcal{A}}_n(t) =& \sum_{I \Vdash n} t^{\mathrm{des}(I)} q^{\mathrm{seg}(I)} R_I \\ =& (1-t)^n \sum_{I \Vdash n} \left(\frac{t}{1-t}\right)^{\mathrm{des}(I)} \left(\frac{q-t}{1-t}\right)^{\mathrm{seg}(I)} S_I\end{aligned}$$

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- Unimodality of $T(n,k)$ and $K(n,i,j)$:

$n \backslash k$	0	1	2	3	4
0	1				
1	1				
2	3	1			
3	13	10	1		
4	75	91	25	1	
5	541	896	426	56	1

$j \backslash i$	0	1
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- Consider $K(n,i,j)/j!$;
- algebraic understanding of $\widetilde{\mathcal{A}_n}(t)$.

Merci de votre attention !