Eulerian polynomials and generalizations

Arthur Nunge

LIGM

February 2018
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Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

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$$A(n, k) = (n - k)A(n - 1, k - 1) + (k + 1)A(n - 1, k)$$
Eulerian polynomials

For any $n \geq 0$, define the Eulerian polynomials as:

$$A_n(t) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{des}(\sigma)} = \sum_{k=0}^{n-1} A(n, k) t^k.$$ 

For example, we have

$$A_3(t) = 1 + 4t + t^2;$$
$$A_4(t) = 1 + 11t + 11t^2 + t^3.$$
Some results about Eulerian numbers and polynomials

- Worpitzky’s identity: For any positive integers $n$ and $k$,

$$k^n = \sum_{i=0}^{k-1} \binom{k+n-i-1}{n} A(n, i)$$
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where $S(n, k)$ are the Stirling numbers of the second kind.
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How to prove these?
Generating function

Define the exponential generating function for the Eulerian polynomials as

\[ G(t, x) = \sum_{n \geq 0} A_n(t) \frac{x^n}{n!}. \]
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First proof

Using the recurrence of the Eulerian numbers, one proves that \( G \) satisfies the following differential equation

\[ (1 - tx) \frac{\partial}{\partial x} G(t, x) - t(1 - t) \frac{\partial}{\partial t} G(t, x) - G(t, x) = 0. \]

Then prove that (??) is a solution.
Algebraic study of the Eulerian polynomials

- Foata-Schützenberger (1970)
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$$G_\sigma \cdot G_\tau = \sum_{\mu \in \sigma \ast \tau} G_\mu,$$

where the permutations in the sum are in $\mathfrak{S}_{n+p}$. In order to have $\phi$ a morphism we need exactly $\binom{n+p}{n}$ elements in the sum.
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There are two possibilities, we consider the one where we concatenate $\sigma$ and $\tau$ and consider all the possibilities to create a permutation. For example,

$$G_{312} \cdot G_{21} = G_{31254} + G_{41253} + G_{41352} + G_{42351} + G_{51243} + \cdots + G_{53421}$$
Noncommutative analog
For any $n \geq 0$, define

$$A_n(t) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma) + 1} G_\sigma.$$
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Using $\phi$ we have $\phi (A_n(t)) = tA_n(t) \frac{x^n}{n!}.$
\[ A_3(t) = tG_{123} + t^2 \left( G_{132} + G_{213} + G_{231} + G_{312} \right) + t^3 G_{321} \]

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And more generally,

\[ A_n = \sum_{\ell(I) n} t^{\ell(I)} R_I \]

where \( R \) is the ribbon basis of the noncommutative symmetric functions algebra (\( \text{Sym} \)) and the sum goes over all composition of size \( n \) (sequences of integers of sum \( n \)).
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We use the complete basis \( S \) of \( \text{Sym} \) with

\[ R_I = \sum_{I \succeq J} (-1)^{\ell(I) - \ell(J)} S^J. \]

For example,

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Then,

\[ A_3(t) = t(1 - t)^2 S_3 + t^2 (1 - t) \left( S^{12} + S^{21} \right) + t^3 S^{111}. \]
We have

\[ A_n = \sum_{l|n} t^{\ell(l)} (1 - t)^{n-\ell(l)} S^l; \]

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To apply \( \phi \), we use the fact that \( S \) is a multiplicative basis. For example,

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\[ tA_n(t) = \sum_{r=1}^{n} t^r (1 - t)^{n-r} r! S(n, r). \]
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If we consider the generating function of the \((1 - t)^{-n} A_n\), we obtain

\[ \sum_{n \geq 0} \frac{A_n}{(1 - t)^n} = \sum_{r \geq 0} \left( \frac{t}{1 - t} \right)^r (S_1 + S_2 + S_3 + \cdots)^r. \]
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Then applying $\phi$ to the previous equation we obtain

$$1 + \sum_{n \geq 0} \frac{tA_n(t)}{(1 - t)^n} \frac{x^n}{n!} = \frac{1 - t}{1 - te^x}$$
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We associate the word 010011 with the above state of the ASEP. The steady-state probabilities of the states of the ASEP can be described combinatorially using some statistics on permutations.
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Recoils of a permutation
A recoil of a permutation \( \sigma \) is a value \( \sigma_i \) such that \( 1 + \sigma_i \) is to its left. For example, the recoils of \( \sigma = 52178643 \) are the values \( \{1, 3, 4, 6\} \).
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**Theorem**

Let $X$ be a state of size $n$ of the ASEP. The steady-state probability of $X$ is proportional to the number of permutations of $S_{n+1}$ having their recoils in the same position as the empty spots of $X$. 
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In fact, sorting the permutations according to their position of recoils and position of descents allow us to compute transitions matrices between to bases of $\text{Sym}$.

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<tr>
<th>PRec \ Des</th>
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<th>${2}$</th>
<th>${2, 3}$</th>
<th>${1}$</th>
<th>${1, 3}$</th>
<th>${1, 2}$</th>
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<tr>
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<td>$1243, 1423$</td>
<td>1342</td>
<td>2341</td>
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<td></td>
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<tr>
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<td></td>
<td>$1324, 3124$</td>
<td>2314</td>
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<tr>
<td>${2, 3}$</td>
<td>3142</td>
<td>1432</td>
<td>2431</td>
<td>3241</td>
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<tr>
<td>${1}$</td>
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<td>2134</td>
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<tr>
<td>${1, 3}$</td>
<td></td>
<td></td>
<td></td>
<td>$2143, 3421$</td>
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<tr>
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<td></td>
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<td>$4213, 3214$</td>
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<td></td>
<td></td>
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</table>

**Theorem (Hivert, Novelli, Tevlin, Thibon, 2009)**

Let $G^n$ be the transition matrix between the ribbon basis of $\text{Sym}$ and the fundamental one in degree $n$. Let $S$ and $T$ be two subsets of $[n-1]$, then

$$G^n_{S, T} = \#\{\sigma \in \mathfrak{S}_{n+1} | \text{Des}(\sigma) = S, \text{PRec}(\sigma) = T \}.$$
2-ASEP
The 2-ASEP is a generalization of the ASEP with two kinds of particles.
2-ASEP
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We associate the word $012021$ with the above state of the 2-ASEP.
2-ASEP
The 2-ASEP is a generalization of the ASEP with two kinds of particles.

We associate the word 012021 with the above state of the 2-ASEP.
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We associate the word 012021 with the above state of the 2-ASEP.

In 2016, O. Mandelshtam and X. Viennot defined a statistic on “Assemblées of permutations” to describe the combinatorics of the 2-ASEP at $q = 1$. Where an assemblée of permutation is a permutation $\sigma$ segmented in blocks where the order of the blocks is not important.
For example, $\sigma = [251][84][637] = [84][251][637]$. 
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$. 

Theorem (Corteel, N., 2018+)

Let $X$ be a state of the 2-ASEP of size $n$. The steady-state probability of $X$ is proportional to the number of segmented permutations $\sigma \in \mathcal{P}_{n+1}$ such that $\text{Seg}(\sigma)$ corresponds to the position of the particles of type 2 and $\text{PRec}(\sigma)$ corresponds to the position of the empty spots.
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A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

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For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 6, 7\}$.
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Eulerian numbers on segmented permutations

We define the following numbers:

\[ T(n, k) = \# \{ \sigma \in \mathcal{P}_n | \text{des(} \sigma \text{)} = k \} \]

\[
\begin{array}{c|cccccc}
  n \backslash k & 0 & 1 & 2 & 3 & 4 \\
\hline
  0 & 1 \\
  1 & 1 \\
  2 & 3 & 1 \\
  3 & 13 & 10 & 1 \\
  4 & 75 & 91 & 21 & 1 \\
  5 & 541 & 896 & 426 & 56 & 1 \\
\end{array}
\]

Note that the numbers on the first column are the ordered Bell numbers (or Fubini numbers).

We have the recurrence relation

\[ T(n, k) = (n-k)T(n-1, k-1) + (n+1)T(n-1, k) + (k+1)T(n-1, k+1). \]

We also have

\[ T(n, n-k-1) = \# \{ \sigma \in \mathcal{P}_n | \text{des(} \sigma \text{)} + \text{seg(} \sigma \text{)} = k \}. \]
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<table>
<thead>
<tr>
<th>n (\backslash) k</th>
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<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
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<td>426</td>
<td>56</td>
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<th>n \backslash k</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
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<td>3</td>
<td>1</td>
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Generalized Eulerian numbers with two parameters

We also consider the following refinement

\[ K(n, i, j) = \# \{ \sigma \in \mathcal{P}_n | \text{des}(\sigma) = i, \text{seg}(\sigma) = j \} \]
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\[
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  j \backslash i & 0 & 1 \\
  \hline
  0 & 1 & 1 \\
  1 & 2 & \\
\end{array}
\]

\[
\begin{array}{c|ccc}
  j \backslash i & 0 & 1 & 2 \\
  \hline
  0 & 1 & 4 & 1 \\
  1 & 6 & 6 & \\
  2 & 6 & \\
\end{array}
\]

\[
\begin{array}{c|cccc}
  j \backslash i & 0 & 1 & 2 & 3 \\
  \hline
  0 & 1 & 11 & 11 & 1 \\
  1 & 14 & 44 & 14 & \\
  2 & 36 & 36 & \\
  3 & 24 & \\
\end{array}
\]

For any \( n \), the numbers on each first row are the usual Eulerian numbers and the numbers on each first column are the \( j! \mathfrak{S}(n, j) \).

We have

\[
K(n, i, j) = (i + j + 1) [K(n - 1, i, j) + K(n - 1, i, j - 1)] + (n - i - j)
\]

\[
K(n - 1, i - 1, j) + K(n - 1, i, j - 1)
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<td></td>
</tr>
<tr>
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\( n = 2 \) :

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<th>2</th>
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<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
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</table>

\( n = 3 \) :

<table>
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<th>( i )</th>
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<th>3</th>
</tr>
</thead>
<tbody>
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<td>11</td>
<td>1</td>
<td></td>
</tr>
<tr>
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<td>14</td>
<td>44</td>
<td>14</td>
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<tr>
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<tr>
<td>3</td>
<td>24</td>
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\( n = 4 \) :

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Generalized Eulerian numbers with two parameters

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\[
K(n, i, j) = (i + j + 1)\left[ K(n - 1, i, j) + K(n - 1, i, j - 1) \right] \\
+ (n - i - j)\left[ K(n - 1, i - 1, j) + K(n - 1, i - 1, j - 1) \right].
\]
Generalized Eulerian polynomials

Define our polynomials as

$$\alpha_n(t, q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\text{des}(\sigma)} q^{\text{seg}(\sigma)}.$$

Some specialization of the variables give the following properties:

- $$\alpha_n(t, 0) = A_n(t)$$
- $$\alpha_n(0, q) = B_n(q)$$
- $$\alpha_n(1, 1) = 2^n - 1^n$$
- $$\alpha_n(-1, 1) = 2^n - 1^n$$
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\[
\begin{align*}
\alpha_n(t, 0) &= A_n(t) \\
\alpha_n(0, q) &= B_n(q) \\
\alpha_n(1, 1) &= 2^{n-1} n! \\
\alpha_n(-1, 1) &= 2^{n-1} \\
\alpha_n(2, 1) &= A050352 \\
\alpha_n(2, 2) &= A050351
\end{align*}
\]
Generating Function

We define the generating function of the generalized Eulerian polynomials as follows:

\[ G(t, q, x) = \sum_{n \geq 0} \alpha_n(t, q) \frac{x^n}{n!}. \]
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The generating function satisfies the following differential equation:

$$(tq - 2q - 1)G(t, q, x) + (1 - tqx - tx) \frac{\partial}{\partial x} G(t, q, x) - (t - t^2)(q + 1) \frac{\partial}{\partial t} G(t, q, x) - (1 - t)(q^2 + q) \frac{\partial}{\partial q} G(t, q, x) = -2q + tq.$$
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\]

Theorem (N., 2018+)

We have the following expression of the generating function:

\[
G(t, q, x) = 1 + \frac{e^{x(1-t)} - 1}{1 + q - (t + q)e^{x(1-t)}}.
\]
Some properties of the generalized Eulerian polynomials

- Worpitzky’s identity: for any positive integers $r$, $k$, and $n$,

\[
\binom{k + r - 1}{r} \Delta^{r+1}((k - 1)^n) = \sum_{i=0}^{k-1} \binom{n + k - i}{n - 1} K(n, i, r),
\]

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\alpha_n(t, q) = \sum_{0 \leq i+j \leq n-1} t^i(q-t)^j(1-t)^{n-i-j-1} 2^i(i+j+1)! \binom{i+j}{j} S(n, i+j+1).
\]
Merci de votre attention !