Eulerian polynomials and generalizations

Arthur Nunge

LIGM

March 2018
Eulerian polynomials

Algebraic proof

2-ASEP (Work with S. Corteel)

Generalized Eulerian polynomials

- \[
A_3(t) = 1 + 4t + t^2
\]
Eulerian polynomials

Algebraic proof

2-ASEP (Work with S. Corteel)

Generalized Eulerian polynomials

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 4 & 1 \\
1 & 11 & 11 & 1
\end{array}
\]

\[
\mathcal{A}_3(t) = 1 + 4t + t^2
\]

\[
\mathcal{A}_3(t) = tG_{123} + t^2(G_{132} + G_{213} + G_{231} + G_{312}) + t^3G_{321}
\]
• $A_3(t) = 1 + 4t + t^2$

• $A_3(t) = tG_{123} + t^2(G_{132} + G_{213} + G_{231} + G_{312}) + t^3G_{321}$
Eulerian polynomials

Algebraic proof

2-ASEP (Work with S. Corteel)

Generalized Eulerian polynomials

\[ A_3(t) = 1 + 4t + t^2 \]

\[ A_3(t) = tG_{123} + t^2\left(G_{132} + G_{213} + G_{231} + G_{312}\right) + t^3G_{321} \]

\[ \alpha_n(t, q) = \sum_{\sigma \in \mathcal{P}_n} t^{\text{des}(\sigma)} q^{\text{seg}(\sigma)} \]
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$.
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$.
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$.
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$ and $\text{des}(\sigma) = 4$. 
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$ and $\text{des}(\sigma) = 4$.

Eulerian numbers

For any $n$ and $k < n$, define

$$A(n, k) = \#\{\sigma \in S_n \mid \text{des}(\sigma) = k\}$$
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$ and $\text{des}(\sigma) = 4$.

Eulerian numbers

For any $n$ and $k < n$, define

$$A(n, k) = \#\{\sigma \in \mathfrak{S}_n \mid \text{des}(\sigma) = k\}$$

<table>
<thead>
<tr>
<th>$n \setminus k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>11</td>
<td>11</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>26</td>
<td>66</td>
<td>26</td>
<td>1</td>
</tr>
</tbody>
</table>
Descents of a permutation

A position $i$ of a permutation $\sigma$ is a descent if $\sigma_i > \sigma_{i+1}$. We denote by $\text{des}(\sigma)$ the number of descents of $\sigma$.

For $\sigma = 514798263$, the descents are $\text{Des}(\sigma) = \{1, 5, 6, 8\}$ and $\text{des}(\sigma) = 4$.

Eulerian numbers

For any $n$ and $k < n$, define

$$A(n, k) = \# \{ \sigma \in S_n \mid \text{des}(\sigma) = k \}$$

<table>
<thead>
<tr>
<th>$n \setminus k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>11</td>
<td>11</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>26</td>
<td>66</td>
<td>26</td>
<td>1</td>
</tr>
</tbody>
</table>

$$A(n, k) = (n - k)A(n - 1, k - 1) + (k + 1)A(n - 1, k)$$
Eulerian polynomials

For any $n \geq 0$, define the Eulerian polynomials as:

$$A_n(t) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma)} = \sum_{k=0}^{n-1} A(n, k) t^k.$$ 

For example, we have

$$A_3(t) = 1 + 4t + t^2;$$
$$A_4(t) = 1 + 11t + 11t^2 + t^3.$$
Some results about Eulerian numbers and polynomials

- Worpitzky’s identity: For any positive integers $n$ and $k$,

$$k^n = \sum_{i=0}^{k-1} \binom{k + n - i - 1}{n} A(n, i)$$
Some results about Eulerian numbers and polynomials

- Worpitzky’s identity: For any positive integers \( n \) and \( k \),

\[
k^n = \sum_{i=0}^{k-1} \binom{k+n-i-1}{n} A(n, i)
\]

- For any \( n \), we have

\[
\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} k^n t^{k-1}
\]
Some results about Eulerian numbers and polynomials

- Worpitzky’s identity: For any positive integers $n$ and $k$,
  \[ k^n = \sum_{i=0}^{k-1} \binom{k + n - i - 1}{n} A(n, i) \]

- For any $n$, we have
  \[ \frac{A_n(t)}{(1 - t)^{n+1}} = \sum_{k \geq 0} k^n t^{k-1} \]

- For any $n$, we have
  \[ A_n(t) = \sum_{r=0}^{n-1} t^r (1 - t)^{n-1-r} (r + 1)! S(n, r + 1), \]
  where $S(n, k)$ are the Stirling numbers of the second kind.
Some results about Eulerian numbers and polynomials

- Worpitzky’s identity: For any positive integers $n$ and $k$,

$$k^n = \sum_{i=0}^{k-1} \binom{k+n-i-1}{n} A(n,i)$$

- For any $n$, we have

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} k^n t^{k-1}$$

- For any $n$, we have

$$A_n(t) = \sum_{r=0}^{n-1} t^r (1-t)^{n-1-r} (r+1)! S(n,r+1)$$

where $S(n,k)$ are the Stirling numbers of the second kind.

How to prove these?
Generating function

Define the exponential generating function for the Eulerian polynomials as

\[ G(t, x) = \sum_{n \geq 0} A_n(t) \frac{x^n}{n!} . \]
Generating function
Define the exponential generating function for the Eulerian polynomials as

\[ G(t, x) = \sum_{n \geq 0} A_n(t) \frac{x^n}{n!}. \]

Theorem
The expression of the generating function of the Eulerian polynomials is

\[ G(t, x) = \frac{(1 - t) e^x(1-t) }{1 - te^x(1-t)}. \] (1)
Generating function

Define the exponential generating function for the Eulerian polynomials as

\[ G(t, x) = \sum_{n \geq 0} A_n(t) \frac{x^n}{n!}. \]

Theorem

The expression of the generating function of the Eulerian polynomials is

\[ G(t, x) = \frac{(1 - t)e^x(1 - t)}{1 - te^x(1 - t)}. \] (1)

First proof

Using the recurrence of the Eulerian numbers, one proves that \( G \) satisfies the following differential equation

\[ (1 - tx) \frac{\partial}{\partial x} G(t, x) - t(1 - t) \frac{\partial}{\partial t} G(t, x) - G(t, x) = 0. \]

Then prove that (1) is a solution.
Algebraic study of the Eulerian polynomials

- Foata-Schützenberger (1970)
Algebraic study of the Eulerian polynomials

- Foata-Schützenberger (1970)
- Désarménien (1983)
Algebraic study of the Eulerian polynomials

- Foata-Schützenberger (1970)
- Désarménien (1983)
**FQSym**

Consider the free algebra whose bases are indexed by permutations. Let $G$ be the basis whose multiplication rule is as follows. For any permutations $\sigma \in S_n$ and $\tau \in S_p$,

$$G_{\sigma} \cdot G_{\tau} = \sum_{\mu \in \sigma \ast \tau} G_{\mu},$$

where the permutations in the sum are such that $\mu = \mu_1 \mu_2$ and $\text{std}(\mu_1) = \sigma$ and $\text{std}(\mu_2) = \tau$. 
**FQSym**

Consider the free algebra whose bases are indexed by permutations. Let $G$ be the basis whose multiplication rule is as follows. For any permutations $\sigma \in \mathcal{S}_n$ and $\tau \in \mathcal{S}_p$,

$$G_\sigma \cdot G_\tau = \sum_{\mu \in \sigma \ast \tau} G_\mu,$$

where the permutations in the sum are such that $\mu = \mu_1\mu_2$ and $\text{std}(\mu_1) = \sigma$ and $\text{std}(\mu_2) = \tau$. There are $\binom{n+p}{n}$ elements in the sum.
FQSym
Consider the free algebra whose bases are indexed by permutations. Let $G$ be the basis whose multiplication rule is as follows. For any permutations $\sigma \in S_n$ and $\tau \in S_p$, 

$$G_{\sigma} \cdot G_{\tau} = \sum_{\mu \in \sigma \cdot \tau} G_{\mu},$$

where the permutations in the sum are such that $\mu = \mu_1 \mu_2$ and $\text{std}(\mu_1) = \sigma$ and $\text{std}(\mu_2) = \tau$. There are $\binom{n+p}{n}$ elements in the sum. For example,

$$G_{312} \cdot G_{21} = G_{31254} + G_{41253} + G_{41352} + G_{42351} + G_{51243} + \cdots + G_{53421}$$
**FQSym**

Consider the free algebra whose bases are indexed by permutations. Let $\mathbf{G}$ be the basis whose multiplication rule is as follows. For any permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_p$,

$$\mathbf{G}_\sigma \cdot \mathbf{G}_\tau = \sum_{\mu \in \sigma \star \tau} \mathbf{G}_\mu,$$

where the permutations in the sum are such that $\mu = \mu_1 \mu_2$ and $\text{std}(\mu_1) = \sigma$ and $\text{std}(\mu_2) = \tau$. There are $\binom{n+p}{n}$ elements in the sum. For example,

$$\mathbf{G}_{312} \cdot \mathbf{G}_{21} = \mathbf{G}_{31254} + \mathbf{G}_{41253} + \mathbf{G}_{41352} + \mathbf{G}_{42351} + \mathbf{G}_{51243} + \cdots + \mathbf{G}_{53421}$$

There is a natural morphism $\phi$ from this algebra to the algebra of univariate polynomials defined for any $\sigma \in \mathfrak{S}_n$ as $\phi(\mathbf{G}_\sigma) = \frac{x^n}{n!}$.
Noncommutative analog
For any $n \geq 0$, define

$$A_n(t) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma)+1} G_{\sigma}.$$
Noncommutative analog

For any $n \geq 0$, define

$$A_n(t) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma) + 1} G_{\sigma}.$$ 

For example, $A_3(t) = tG_{123} + t^2\left(G_{132} + G_{213} + G_{231} + G_{312}\right) + t^3G_{321}$. 
Noncommutative analog

For any $n \geq 0$, define

$$\mathcal{A}_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)+1} G_\sigma.$$ 

For example, $\mathcal{A}_3(t) = tG_{123} + t^2\left( G_{132} + G_{213} + G_{231} + G_{312} \right) + t^3G_{321}.$

Using $\phi$ we have $\phi(\mathcal{A}_n(t)) = tA_n(t) \frac{x^n}{n!}.$
\[ A_3(t) = tG_{123} + t^2 \left( G_{132} + G_{213} + G_{231} + G_{312} \right) + t^3 G_{321} \]
\[ \quad = tR_3 + t^2 \left( R_{21} + R_{12} \right) + t^3 R_{111} \]
\[ A_3(t) = tG_{123} + t^2 \left( G_{132} + G_{213} + G_{231} + G_{312} \right) + t^3 G_{321} \]

\[ = tR_3 + t^2 \left( R_{21} + R_{12} \right) + t^3 R_{111} \]

And more generally,

\[ A_n = \sum_{I \vdash n} t^{\ell(I)} R_I \]

where \( R \) is the ribbon basis of the noncommutative symmetric functions algebra \( \text{Sym} \) and the sum goes over all composition of size \( n \) (sequences of integers of sum \( n \)).
$A_3(t) = tG_{123} + t^2 \left( G_{132} + G_{213} + G_{231} + G_{312} \right) + t^3 G_{321} \nonumber \\
= tR_3 + t^2 \left( R_{21} + R_{12} \right) + t^3 R_{111} \nonumber$

And more generally,

$$A_n = \sum_{|I|=n} t^\ell(I) R_I$$

where $R$ is the ribbon basis of the noncommutative symmetric functions algebra ($\text{Sym}$) and the sum goes over all composition of size $n$ (sequences of integers of sum $n$).

We use the complete basis $S$ of $\text{Sym}$ with

$$R_I = \sum_{I \succeq J} (-1)^{\ell(I) - \ell(J)} S^J.$$

For example,

$$R_{121} = S^{121} - S^{13} - S^{31} + S_4.$$
Eulerian polynomials

Algebraic proof

2-ASEP (Work with S. Corteel)

Generalized Eulerian polynomials

\[ A_3(t) = tG_{123} + t^2 \left( G_{132} + G_{213} + G_{231} + G_{312} \right) + t^3 G_{321} \]

\[ = tR_3 + t^2 \left( R_{21} + R_{12} \right) + t^3 R_{111} \]

And more generally,

\[ A_n = \sum_{|I|=n} t^{\ell(I)} R_I \]

where \( R \) is the ribbon basis of the noncommutative symmetric functions algebra (\( \text{Sym} \)) and the sum goes over all composition of size \( n \) (sequences of integers of sum \( n \)).

We use the complete basis \( S \) of \( \text{Sym} \) with

\[ R_I = \sum_{I \geq J} (-1)^{\ell(I)-\ell(J)} S_J. \]

For example,

\[ R_{121} = S^{121} - S^{13} - S^{31} + S_4. \]

Then,

\[ A_3(t) = t(1 - t)^2 S_3 + t^2(1 - t) \left( S^{12} + S^{21} \right) + t^3 S^{111}. \]
We have

\[ A_n = \sum_{l|n} t^{\ell(l)} (1 - t)^{n-\ell(l)} S^l; \]

\[ = \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{l|n, \ell(l)=r} S^l. \]
We have

\[
A_n = \sum_{l|n} t^{\ell(l)} (1 - t)^{n-\ell(l)} S^l;
\]

\[
= \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{l|n, \ell(l)=r} S^l.
\]

To apply \( \phi \), we use the fact that \( S \) is a multiplicative basis. For example,

\[
S^{312} = S^{31} \cdot S_2 = S_3 \cdot S^{12} = S_3 \cdot S_1 \cdot S_2.
\]
We have

\[ A_n = \sum_{l | n} t^\ell(l)(1 - t)^{n - \ell(l)} S^l; \]

\[ = \sum_{r=1}^{n} t^r(1 - t)^{n - r} \sum_{l | n, \ell(l) = r} S^l. \]

To apply \( \phi \), we use the fact that \( S \) is a multiplicative basis. For example,

\[ S^{312} = S^{31} \cdot S_2 = S_3 \cdot S^{12} = S_3 \cdot S_1 \cdot S_2. \]

Moreover, \( \phi(S_k) = \frac{x^k}{k!} \) so 

\[ \phi \left( \sum_{l | n, \ell(l) = r} S^l \right) = r! S(n, r) \frac{x^n}{n!}. \]
We have

\[ A_n = \sum_{|I|=n} t^{\ell(I)} (1 - t)^{n-\ell(I)} S^I; \]

\[ = \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{\substack{|I|=n \\ell(I)=r}} S^I. \]

To apply \( \phi \), we use the fact that \( S \) is a multiplicative basis. For example,

\[ S^{312} = S^3 \cdot S_2 = S^3 \cdot S^{12} = S^3 \cdot S_1 \cdot S_2. \]

Moreover, \( \phi(S_k) = \frac{x^k}{k!} \) so \( \phi \left( \sum_{\substack{|I|=n \\ell(I)=r}} S^I \right) = r! S(n, r) \frac{x^n}{n!}. \) Then,

\[ tA_n(t) = \sum_{r=1}^{n} t^r (1 - t)^{n-r} r! S(n, r). \]
\[ A_n = \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{|I|=n} \sum_{\ell(I)=r} S^I. \]
\[ \mathcal{A}_n = \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{l|n, \ell(l)=r} S^l. \]

If we consider the generating function of the $(1 - t)^{-n} \mathcal{A}_n$, we obtain

\[ \sum_{n \geq 0} \frac{\mathcal{A}_n}{(1 - t)^n} = \sum_{r \geq 0} \left( \frac{t}{1 - t} \right)^r (S_1 + S_2 + S_3 + \cdots)^r. \]
\[ A_n = \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{\substack{I|\n元素r \leq n \\ell(I)=r}} S^I. \]

If we consider the generating function of the \((1 - t)^{-n} A_n\), we obtain

\[ \sum_{n \geq 0} \frac{A_n}{(1 - t)^n} = \sum_{r \geq 0} \left( \frac{t}{1 - t} \right)^r (S_1 + S_2 + S_3 + \cdots)^r. \]

We use the fact that

\[ \phi(S_1 + S_2 + S_3 + \cdots) = e^x - 1. \]
\[ A_n = \sum_{r=1}^{n} t^r (1 - t)^{n-r} \sum_{\substack{I \mid n \\ell(I) = r}} S^I. \]

If we consider the generating function of the \((1 - t)^{-n} A_n\), we obtain

\[ \sum_{n \geq 0} \frac{A_n}{(1 - t)^n} = \sum_{r \geq 0} \left( \frac{t}{1 - t} \right)^r (S_1 + S_2 + S_3 + \cdots)^r. \]

We use the fact that

\[ \phi(S_1 + S_2 + S_3 + \cdots) = e^x - 1. \]

Then applying \(\phi\) to the previous equation we obtain

\[ 1 + \sum_{n \geq 0} \frac{t A_n(t)}{(1 - t)^n} \frac{x^n}{n!} = \frac{1 - t}{1 - te^x}. \]
ASEP
The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.

○ ○ ● ○ ○ ● ●
ASEP

The ASEP is a model representing the displacement of particles on a finite one-dimensional lattice.
ASEP
The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.
ASEP

The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.
ASEP
The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.
ASEP
The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.

\[ \alpha \xrightarrow{q} 1 \xrightarrow{1} \beta \]
**ASEP**

The ASEP is a model representing the displacement of particles on a finite one-dimensional lattice.
ASEP
The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.

We associate the word 010011 with the above state of the ASEP.
ASEP
The ASEP is a model representing the displacement of particles on a finite one dimensional lattice.

We associate the word 010011 with the above state of the ASEP.
The steady-state probabilities of the states of the ASEP can be described combinatorialy using some statistics on permutations.
Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$. 
Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$. 
Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$.
Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$. 
Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$. 
Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$. We denote by $\text{PRec}(\sigma)$ the set of the positions of the recoils of $\sigma$ minus one. For $\sigma = 52178643$, $\text{PRec}(\sigma) = \{2, 5, 6, 7\}$. 


Recoils of a permutation

A recoil of a permutation $\sigma$ is a value $\sigma_i$ such that $1 + \sigma_i$ is to its left. For example, the recoils of $\sigma = 52178643$ are the values $\{1, 3, 4, 6\}$. We denote by $\text{PRec}(\sigma)$ the set of the positions of the recoils of $\sigma$ minus one. For $\sigma = 52178643$, $\text{PRec}(\sigma) = \{2, 5, 6, 7\}$.

Theorem

Let $X$ be a state of size $n$ of the ASEP. The steady-state probability of $X$ is proportional to the number of permutations of $\mathfrak{S}_{n+1}$ having their recoils in the same position as the empty spots of $X$.
In fact, sorting the permutations according to their position of recoils and position of descents allow us to compute transitions matrices between two bases of $\text{Sym}$.

<table>
<thead>
<tr>
<th>PRec \ Des</th>
<th>$\emptyset$</th>
<th>${3}$</th>
<th>${2}$</th>
<th>${2, 3}$</th>
<th>${1}$</th>
<th>${1, 3}$</th>
<th>${1, 2}$</th>
<th>${1, 2, 3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>1234</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${3}$</td>
<td>$1243, 1423$</td>
<td>$1342$</td>
<td>$3412$</td>
<td>$2341$</td>
<td>$2413$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${2}$</td>
<td>$1324$</td>
<td>$3124$</td>
<td></td>
<td>$2314$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${2, 3}$</td>
<td>$3142$</td>
<td>$1432, 4132$</td>
<td>$4312$</td>
<td>$2431$</td>
<td>$4231$</td>
<td>$3241$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>${1}$</td>
<td></td>
<td></td>
<td>$2134$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>${1, 3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$2143$</td>
<td>$4213$</td>
<td>$3421$</td>
<td></td>
</tr>
<tr>
<td>${1, 2}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$4213$</td>
<td>$3214$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>${1, 2, 3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$4321$</td>
<td></td>
</tr>
</tbody>
</table>

**Theorem (Hivert - Novelli - Tevlin - Thibon, 2009)**

Let $G^n$ be the transition matrix between the ribbon basis of $\text{Sym}$ and the fundamental one in degree $n$. Let $S$ and $T$ be two subsets of $[n-1]$, then

$$G^n_{S, T} = \# \{ \sigma \in \mathcal{S}_{n+1} | \text{Des}(\sigma) = S, \text{PRec}(\sigma) = T \}.$$
2-ASEP

The 2-ASEP is a generalization of the ASEP with two kinds of particles.
2-ASEP
The 2-ASEP is a generalization of the ASEP with two kinds of particles.
2-ASEP
The 2-ASEP is a generalization of the ASEP with two kinds of particles.

We associate the word 012021 with the above state of the 2-ASEP.
2-ASEP

The 2-ASEP is a generalization of the ASEP with two kinds of particles.

We associate the word 012021 with the above state of the 2-ASEP.

In 2016, O. Mandelshtam and X. Viennot defined a statistic on “Assemblées of permutations” to describe the combinatorics of the 2-ASEP at $q = 1$. Where an assemblée of permutation is a permutation $\sigma$ segmented in blocks where the order of the blocks is not important.

For example, $\sigma = [251][84][637] = [84][251][637]$. 
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$. 

Theorem (Corteel - N., 2018+)

Let $X$ be a state of the 2-ASEP of size $n$. The steady-state probability of $X$ is proportional to the number of segmented permutations $\sigma \in \mathcal{P}_{n+1}$ such that $\text{Seg}(\sigma)$ corresponds to the positions of the particles of type 2 and $\text{PRec}(\sigma)$ corresponds to the positions of the empty spots.
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $PRec(\sigma)$ the set of the position of recoils minus one.

For $\sigma = 52|7138|46$, the segmentations are $Seg(\sigma) = \{2, 6\}$, the positions of recoils minus one are $PRec(\sigma) = \{3, 7\}$.
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $\text{PRec}(\sigma)$ the set of the position of recoils minus one.

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 7\}$.
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $\text{PRec}(\sigma)$ the set of the position of recoils minus one.

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 7\}$
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.
A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $\text{PRec}(\sigma)$ the set of the position of recoils minus one.
A descent is a position $i$ that is not a segmentation and such that $\sigma_i > \sigma_{i+1}$
For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 7\}$, and the descents are $\text{Des}(\sigma) = \{1, 3\}$.
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $\text{PRec}(\sigma)$ the set of the position of recoils minus one.

A descent is a position $i$ that is not a segmentation and such that $\sigma_i > \sigma_{i+1}$.

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 7\}$, and the descents are $\text{Des}(\sigma) = \{1, 3\}$. 

Theorem (Corteel - N., 2018+)

Let $X$ be a state of the 2-ASEP of size $n$. The steady-state probability of $X$ is proportional to the number of segmented permutations $\sigma \in \mathcal{P}_{n+1}$ such that $\text{Seg}(\sigma)$ corresponds to the positions of the particles of type 2 and $\text{PRec}(\sigma)$ corresponds to the positions of the empty spots.
Segmented permutations

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

Some statistics on segmented permutation

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $\text{PRec}(\sigma)$ the set of the position of recoils minus one.

A descent is a position $i$ that is not a segmentation and such that $\sigma_i > \sigma_{i+1}$.

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 7\}$, and the descents are $\text{Des}(\sigma) = \{1, 3\}$. 
**Segmented permutations**

A segmented permutation is a permutation where each values may be separated by bars. We denote by $\mathcal{P}_n$ the set of segmented permutations of size $n$. For example, $\sigma = 52|7138|46 \in \mathcal{P}_8$.

**Some statistics on segmented permutation**

A position $i$ of a permutation $\sigma$ is a segmentation if there is a bar between $\sigma_i$ and $\sigma_{i+1}$.

A recoil of $\sigma$ is a value $\sigma_i$ such that $i - 1$ is not a segmentation and $1 + \sigma_i$ is to the left of $\sigma_i$. Denote by $\text{PRec}(\sigma)$ the set of the position of recoils minus one.

A descent is a position $i$ that is not a segmentation and such that $\sigma_i > \sigma_{i+1}$

For $\sigma = 52|7138|46$, the segmentations are $\text{Seg}(\sigma) = \{2, 6\}$, the positions of recoils minus one are $\text{PRec}(\sigma) = \{3, 7\}$, and the descents are $\text{Des}(\sigma) = \{1, 3\}$.

**Theorem (Corteel - N., 2018+)**

Let $X$ be a state of the 2-ASEP of size $n$. The steady-state probability of $X$ is proportional to the number of segmented permutations $\sigma \in \mathcal{P}_{n+1}$ such that $\text{Seg}(\sigma)$ corresponds to the positions of the particles of type 2 and $\text{PRec}(\sigma)$ corresponds to the positions of the empty spots.
The algebra of segmented compositions

In 2007, Novelli and Thibon defined the algebra of segmented compositions (SCQSym) and its complete and ribbon bases. These bases are indexed by segmented compositions.
The algebra of segmented compositions

In 2007, Novelli and Thibon defined the algebra of segmented compositions (\textit{SCQSym}) and its complete and ribbon bases. These bases are indexed by segmented compositions.

**Complete basis**

\[ S^I \cdot S^J = S^{I \cdot J} \]

For example, \( S_{21|1} \cdot S_{32|21} = S_{21|132|21} \).
The algebra of segmented compositions

In 2007, Novelli and Thibon defined the algebra of segmented compositions (\textsc{SCQSym}) and its complete and ribbon bases. These bases are indexed by segmented compositions.

**Complete basis**

\[ S^I \cdot S^J = S^{I \cdot J} \]

For example, \( S_{21|1} \cdot S_{32|21} = S_{21|132|21} \).

**Ribbon basis**

Again we have

\[ R_I = \sum_{J \preceq I} (-1)^{\operatorname{des}(I) - \operatorname{des}(J)} S^J. \]

For example, \( R_{2|41} = S_{2|41} - S_{2|4|1} - S_{2|5} \).
New bases
Following the idea of Tevlin, it is natural to define a monomial basis ($M_i$) and then a fundamental basis ($L_i$) in SCQSym.
New bases
Following the idea of Tevlin, it is natural to define a monomial basis \( (M_i) \) and then a fundamental basis \( (L_i) \) in \( \text{SCQS} \text{ym} \).

Transition matrix
The transition matrix between the ribbon basis to the fundamental basis is the following for \( n = 3 \).

\[
M_3 = \begin{pmatrix}
1 & . & . & . & . & . & . & . \\
. & 2 & 1 & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . \\
. & . & . & 3 & 1 & . & . & . \\
. & . & . & 2 & . & . & . & . \\
. & . & . & 2 & 1 & 3 & . & . \\
. & . & . & . & . & . & . & 6
\end{pmatrix}
\]
New bases
Following the idea of Tevlin, it is natural to define a monomial basis \((M_i)\) and then a fundamental basis \((L_i)\) in \(SCQSym\).

Transition matrix
The transition matrix between the ribbon basis to the fundamental basis is the following for \(n = 3\).

\[
M_3 = \begin{pmatrix}
1 & \ldots & . & \ldots & . & \ldots & . \\
. & 2 & 1 & \ldots & . & \ldots & . \\
. & . & 1 & \ldots & . & \ldots & . \\
. & . & . & 1 & \ldots & . & \ldots \\
. & . & . & 3 & 1 & \ldots & . \\
. & . & . & 2 & \ldots & \ldots & . \\
. & . & . & . & 2 & \ldots & . \\
. & . & . & . & 1 & 3 & \ldots \\
. & . & . & . & . & . & 6
\end{pmatrix}
\]

Theorem (Corteel - N., 2018+)
For any \(n > 0\), the entries of \(M_n\) are the number of segmented permutations \(\sigma\) with given \((\text{Des}(\sigma), \text{Seg}(\sigma))\) and \((\text{PRec}(\sigma), \text{Seg}(\sigma))\).
Eulerian numbers on segmented permutations

We define the following numbers:

\[ T(n, k) = \# \{ \sigma \in \mathfrak{S}_n | \text{des}(\sigma) = k \} \]

<table>
<thead>
<tr>
<th>(n\backslash k)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>75</td>
<td>91</td>
<td>21</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>541</td>
<td>896</td>
<td>426</td>
<td>56</td>
<td>1</td>
</tr>
</tbody>
</table>
Eulerian numbers on segmented permutations

We define the following numbers:

\[ T(n, k) = \#\{\sigma \in \mathfrak{S}_n | \text{des}(\sigma) = k\} \]

<table>
<thead>
<tr>
<th>( n \setminus k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>75</td>
<td>91</td>
<td>21</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>541</td>
<td>896</td>
<td>426</td>
<td>56</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that the numbers on the first column are the ordered Bell numbers (or Fubini numbers).
Eulerian numbers on segmented permutations

We define the following numbers:

\[ T(n, k) = \#\{\sigma \in \mathfrak{S}_n \mid \text{des}(\sigma) = k\} \]

\[
\begin{array}{c|ccccc}
 n \backslash k & 0 & 1 & 2 & 3 & 4 \\
\hline
 0 & 1 \\
 1 & 1 \\
 2 & 3 & 1 \\
 3 & 13 & 10 & 1 \\
 4 & 75 & 91 & 21 & 1 \\
 5 & 541 & 896 & 426 & 56 & 1 \\
\end{array}
\]

Note that the numbers on the first column are the ordered Bell numbers (or Fubini numbers). We have the recurrence relation

\[ T(n, k) = (n-k)T(n-1, k-1) + (n+1)T(n-1, k) + (k+1)T(n-1, k+1). \]
Eulerian numbers on segmented permutations

We define the following numbers:

\[ T(n, k) = \#\{ \sigma \in \mathfrak{S}_n | \text{des}(\sigma) = k \} \]

<table>
<thead>
<tr>
<th>n \setminus k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>10</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>75</td>
<td>91</td>
<td>21</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>541</td>
<td>896</td>
<td>426</td>
<td>56</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that the numbers on the first column are the ordered Bell numbers (or Fubini numbers). We have the recurrence relation

\[ T(n, k) = (n - k) T(n - 1, k - 1) + (n + 1) T(n - 1, k) + (k + 1) T(n - 1, k + 1). \]

We also have

\[ T(n, n - k - 1) = \#\{ \sigma \in \mathfrak{S}_n | \text{des}(\sigma) + \text{seg}(\sigma) = k \}. \]
Generalized Eulerian numbers with two parameters

We also consider the following refinement

\[ K(n, i, j) = \#\{\sigma \in \mathfrak{P}_n | \text{des}(\sigma) = i, \text{seg}(\sigma) = j\} \]
Generalized Eulerian numbers with two parameters

We also consider the following refinement

\[ K(n, i, j) = \#\{\sigma \in \mathcal{P}_n | \text{des}(\sigma) = i, \text{seg}(\sigma) = j\} \]

<table>
<thead>
<tr>
<th>j \backslash i</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

\( n = 2 : \)

<table>
<thead>
<tr>
<th>j \backslash i</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>44</td>
<td>14</td>
</tr>
</tbody>
</table>

\( n = 3 : \)

<table>
<thead>
<tr>
<th>j \backslash i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>11</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>j \backslash i</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>44</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>36</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>24</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Generalized Eulerian numbers with two parameters

We also consider the following refinement

$$K(n, i, j) = \#\{\sigma \in \mathfrak{S}_n | \text{des}(\sigma) = i, \text{seg}(\sigma) = j\}$$

For any $n$, the numbers on each first row are the usual Eulerian numbers and
the numbers on each first column are the $j! S(n, j)$. 
Generalized Eulerian numbers with two parameters

We also consider the following refinement

\[ K(n, i, j) = \#\{\sigma \in \mathcal{P}_n | \text{des}(\sigma) = i, \text{seg}(\sigma) = j\} \]

\[
\begin{array}{c|cc}
   j \backslash i & 0 & 1 \\
\hline
   0 & 1 & 1 \\
   1 & 2 \\
\end{array}
\quad
\begin{array}{c|ccc}
   j \backslash i & 0 & 1 & 2 \\
\hline
   0 & 1 & 4 & 1 \\
   1 & 6 & 6 \\
   2 & 6 \\
\end{array}
\quad
\begin{array}{c|cccc}
   j \backslash i & 0 & 1 & 2 & 3 \\
\hline
   0 & 1 & 11 & 11 & 1 \\
   1 & 14 & 44 & 14 \\
   2 & 36 & 36 \\
   3 & 24 \\
\end{array}
\]

For any \( n \), the numbers on each first row are the usual Eulerian numbers and the numbers on each first column are the \( j!S(n, j) \). We have

\[
K(n, i, j) = (i + j + 1)\left[ K(n - 1, i, j) + K(n - 1, i, j - 1) \right]
+ (n - i - j)\left[ K(n - 1, i - 1, j) + K(n - 1, i - 1, j - 1) \right].
\]
Generalized Eulerian polynomials

Define our polynomials as

$$\alpha_n(t, q) = \sum_{\sigma \in \mathcal{P}_n} t^{\text{des}(\sigma)} q^{\text{seg}(\sigma)}.$$
Generalized Eulerian polynomials

Define our polynomials as

\[ \alpha_n(t, q) = \sum_{\sigma \in \mathcal{P}_n} t^{\text{des}(\sigma)} q^{\text{seg}(\sigma)}. \]

Some specialization of the variables give the following properties

\[
\begin{align*}
\alpha_n(t, 0) &= A_n(t) \\
\alpha_n(0, q) &= B_n(q) \\
\alpha_n(1, 1) &= 2^{n-1} n! \\
\alpha_n(-1, 1) &= 2^{n-1} \\
\alpha_n(2, 1) &= A050352 \\
\alpha_n(2, 2) &= A050351
\end{align*}
\]
Generating Function

We define the generating function of the generalized Eulerian polynomials as follows:

\[ G(t, q, x) = \sum_{n \geq 0} \alpha_n(t, q) \frac{x^n}{n!}. \]
Generating Function

We define the generating function of the generalized Eulerian polynomials as follows:

\[ G(t, q, x) = \sum_{n \geq 0} \alpha_n(t, q) \frac{x^n}{n!}. \]

The generating function satisfies the following differential equation:

\[(tq - 2q - 1)G(t, q, x) + (1 - tqx - tx) \frac{\partial}{\partial x} G(t, q, x) - (t - t^2)(q + 1) \frac{\partial}{\partial t} G(t, q, x) -
(1 - t)(q^2 + q) \frac{\partial}{\partial q} G(t, q, x) = -2q + tq.\]
Generating Function

We define the generating function of the generalized Eulerian polynomials as follows:

\[ G(t, q, x) = \sum_{n \geq 0} \alpha_n(t, q) \frac{x^n}{n!}. \]

The generating function satisfies the following differential equation:

\[
(tq - 2q - 1)G(t, q, x) + (1 - tqx - tx) \frac{\partial}{\partial x} G(t, q, x) - (t - t^2)(q + 1) \frac{\partial}{\partial t} G(t, q, x) - (1 - t)(q^2 + q) \frac{\partial}{\partial q} G(t, q, x) = -2q + tq.
\]

Theorem (N., 2018+)

We have the following expression of the generating function:

\[
G(t, q, x) = 1 + \frac{e^{x(1-t)} - 1}{1 + q - (t + q)e^{x(1-t)}}.
\]
Some properties of the generalized Eulerian polynomials

- Worpitzky's identity: for any positive integers \( r, k, \) and \( n, \)
  \[
  \binom{k + r - 1}{r} \Delta^{r+1}((k - 1)^n) = \sum_{i=0}^{k-1} \binom{n + k - i}{n - 1} K(n, i, r),
  \]
  where \( \Delta(k^n) = (k + 1)^n - k^n. \)
Some properties of the generalized Eulerian polynomials

- Worpitzky's identity: for any positive integers $r$, $k$, and $n$,

$$
\binom{k + r - 1}{r} \Delta^{r+1}((k - 1)^n) = \sum_{i=0}^{k-1} \binom{n + k - i}{n - 1} K(n, i, r),
$$

where $\Delta(k^n) = (k + 1)^n - k^n$.

- For any $n \geq 0$ we have

$$
\frac{\alpha_n(t, 1)}{(1 - t)^{n+1}} = \sum_{k \geq 0} (1 + t)^{k-1} \frac{k^n}{2^{k-1}}.
$$
Some properties of the generalized Eulerian polynomials

- **Worpitzky’s identity**: for any positive integers \( r, k, \) and \( n, \)

\[
\binom{k + r - 1}{r} \Delta^{r+1}((k - 1)^n) = \sum_{i=0}^{k-1} \binom{n + k - i}{n - 1} K(n, i, r),
\]

where \( \Delta(k^n) = (k + 1)^n - k^n. \)

- For any \( n \geq 0 \) we have

\[
\frac{\alpha_n(t, 1)}{(1 - t)^{n+1}} = \sum_{k \geq 0} (1 + t)^{k-1} \frac{k^n}{2^{k-1}}.
\]

- For any \( n \geq 0 \) we have

\[
\alpha_n(t, q) = \sum_{0 \leq i+j \leq n-1} t^i(q-t)^j(1-t)^{n-i-j-1}2^i(i+j+1)! \binom{i+j}{j} S(n, i+j+1).
\]
Merci de votre attention !