An equivalence of multistatistics on permutations

Arthur Nunge

Laboratoire IGM

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\[ \alpha \quad \bullet \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bullet \quad \beta \]
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\[ \begin{align*} &\alpha \\
&\overset{1}{\circ} \\
&\circ \\
&\circ \\
&\bullet \\
&\beta \end{align*} \]
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\[
\begin{array}{c}
\alpha \quad q \quad 1 \quad \beta \\
\end{array}
\]
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We associate the composition \((2, 3, 1, 1)\) to the above step of the PASEP.
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### PASEP

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![PASEP Step](image)

We associate the composition $(2, 3, 1, 1)$ to the above step of the PASEP.

### Combinatorial study of the PASEP

The PASEP is closely related with permutations. Let $\lambda$ be a composition associated to a state of the PASEP, the steady-state probability of this state is given by $\sum_{\text{GC}(\sigma) = \lambda} q^{\text{tot}(\sigma)}$ renormalized to make it a probability.

- $\text{GC}(\sigma)$ (*Genocchi composition*) is the descent composition of the values of $\sigma$
- $\text{tot}(\sigma)$ is the number of $31$-$2$ patterns in $\sigma$. 

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Tevlin’s basis (2007)

Tevlin defined a “monomial basis” $L_I$ of the non commutative symmetric functions algebra (NCSF). He conjectured that the expansion of the ribbon basis on the $L_I$ has nonnegative integer coefficients.
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Combinatorial interpretation of Tevlin’s basis

<table>
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<th>GC \ Rec</th>
<th>4</th>
<th>31</th>
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<td>3142, 1432, 4132</td>
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</table>

Theorem (Hivert, Novelli, Tevlin, Thibon, 2009)

*For* $I$ *a composition of* $n$, *we have* $R_I = \sum_J G_{IJ}L_J$ *where* $G_{IJ}$ *is equal to the number of permutations* $\sigma$ *satisfying* $\text{Rec}(\sigma) = I$ *and* $\text{GC}(\sigma) = J$. 
Novelli, Thibon, and Williams defined a $q$-analog of NCSF where the transition matrix from $L_I(q)$ to $R_J(q)$ is given by the following matrix:

\[
\begin{pmatrix}
1 & . & . & . \\
. & 1+q & 1 & . \\
. & . & 1 & . \\
. & . & . & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . \\
. & 1+q + q^2 & 1+q & . & 1 & q & . & . & . & . \\
. & . & 1+q & 1+q & . & 1 & . & . & . & . \\
. & . & . & q & 1+q + q^2 & 1+q & 1 & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & 1+q & 1 & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
\]
**q-analog of Tevlin’s basis (2010)**

Novelli, Thibon, and Williams defined a $q$-analog of $\text{NCSF}$ where the transition matrix from $L_I(q)$ to $R_J(q)$ is given by the following matrix:

$$
\begin{pmatrix}
1 & 1+q & 1 \\
1 & 1+q & 1 \\
& 1 & 1 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 1+q+q^2 & 1+q & 1+q & 1+q & 1+q \\
1 & 1+q & 1+q & 1 & 1 & 1 \\
& 1 & 1+q & 1+q & 1 \\
& & 1 & 1+q & 1 \\
& & & 1 & 1 \\
& & & & 1 \\
\end{pmatrix}
$$

**Theorem (Novelli, Thibon, Williams, 2010)**

For $I$ a composition of $n$, we have $R_I(q) = \sum_J F_{IJ}(q)L_J(q)$ where:

$$F_{IJ}(q) = \sum_{\text{Rec}(\sigma)=I, \text{LC}(\sigma)=J} q^{\alpha(\sigma)}$$
Remark

*PASEP theory implies that the previous matrix should also be described with the statistics* $\text{Rec}$, $\text{GC}$, and $\text{tot}$.

Two ways of grouping the permutations

Conjecture (Novelli, Thibon, Williams, 2010)

*Sending permutations of the left table to* $q^{\alpha(\sigma)}$ *gives the same matrix than sending the permutations of the right table to* $q^{\text{tot}(\sigma)}$. 
**Involved combinatorial objects**

- Permutations;
- Weighted Dyck Paths;
- Subexceedent Functions;
- Decreasing Weighted Subexceedent Functions.

**Steps of the bijection**

\[ P \leftrightarrow \phi \leftrightarrow FV \leftrightarrow WDP \leftrightarrow \psi \leftrightarrow WDP \leftrightarrow DWSF \leftrightarrow SF \leftrightarrow Lh \leftrightarrow P \]

**An equivalence of multistatistics on permutations**
Sketch of proof: let’s make some bijections

Involved combinatorial objects

- Permutations;
- Weighted Dyck Paths;
- Subexceedent Functions;
- Decreasing Weighted Subexceedent Functions.

Steps of the bijection

\[ P \xleftarrow{\phi_{FV}} WDP \xrightarrow{\psi_1} DWSF \xrightarrow{\psi_2} SF \xrightarrow{Lh} R \xrightarrow{\phi_{FV}^{-1}} P \]

Catalan
Sketch of proof: let’s make some bijections

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Steps of the bijection
\[
P \xleftrightarrow{\phi_F} \text{WDP} \xleftrightarrow{\phi_1} \text{WDP}
\]
- Catalan
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An equivalence of multistatistics on permutations
### Sketch of proof: let's make some bijections

#### Involved combinatorial objects
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#### Steps of the bijection

<table>
<thead>
<tr>
<th>P $\leftrightarrow_{\phi_{FV}}$ WDP $\leftrightarrow_{\phi_1}$ WDP</th>
<th>SF $\leftrightarrow_{Lh}$ P</th>
</tr>
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<tbody>
<tr>
<td>Catalan</td>
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Sketch of proof: let’s make some bijections

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Steps of the bijection

\[ P \overset{\phi_{FV}}{\leftrightarrow} WDP \overset{\phi_1}{\leftrightarrow} WDP \overset{\psi_1}{\leftrightarrow} DWSF \overset{Lh}{\leftrightarrow} P \]

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Sketch of proof: let’s make some bijections

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Steps of the bijection
\[ P \xleftarrow{\phi_{FV}} WDP \xleftarrow{\phi_1} WDP \xrightarrow{\psi_2} DWSF \xrightarrow{\psi_1} SF \xleftarrow{Lh} P \]

\[ \text{Catalan} \quad \text{Catalan} \quad \text{Catalan} \]
Weighted Dyck paths

A weight for a Dyck path is a word $w$ satisfying for all $i$, $w_i \leq (h_i - 1)/2$ where $h_i$ is the height of the Dyck path between the $(2i - 1)$-th and $2i$-th steps.
The Françon-Viennot bijection: $P \rightarrow WDP$

Let $\sigma \in S_n$ we construct $\psi_{FV}(\sigma)$ as follows:

- The $(2k - 1)$-th is $/$ iff $k = \sigma_i < \sigma_{i+1}$,
- The $(2k)$-th is $/$ iff $\sigma_{i-1} > \sigma_i = k$.

Moreover, $w_k$ is equal to the number of 31-2 patterns such that $k$ plays the rôle of 2.

Example

$$\phi_{FV}(0.528713649.\infty) =$$
The Françon-Viennot bijection: $P \rightarrow WDP$

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![Diagram showing an example of the bijection](image)
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![Diagram showing the Françon-Viennot bijection]
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**Example**

$$\phi_{FV}(0.528713649.\infty) = \text{Diagram}$$
The Françon-Viennot bijection: $P \rightarrow WDP$

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Example

$\phi_{FV}(0.528713649.\infty) =$

![Graph of \( \phi_{FV}(0.528713649.\infty) \)]
The Françon-Viennot bijection: $\sigma \rightarrow WDP$

Let $\sigma \in \mathcal{S}_n$ we construct $\psi_{FV}(\sigma)$ as follows:

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Example

$$\phi_{FV}(0.528713649.\infty) =$$

![Diagram](attachment:image.png)
\[ \phi_1 : \text{WDP} \rightarrow \text{WDP} \]

\( \phi_1 \) is the involution exchanging \( \bigtriangleup \) with \( \bigtriangledown \).

**Example**

![Example Diagram]
\( \phi_1 : \text{WDP} \rightarrow \text{WDP} \)

\( \phi_1 \) is the involution exchanging \( \rhd \) with \( \rhd \).

**Example**

\[
\begin{array}{cccccccc}
1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
\( \phi_1 \): WDP \( \rightarrow \) WDP

\( \phi_1 \) is the involution exchanging \( \uparrow \downarrow \) with \( \downarrow \uparrow \).

Example
Summary

\[ P \xrightleftharpoons{\phi_{FV}} WDP \xrightleftharpoons{\phi_1} WDP \xrightleftharpoons{\psi_2} \text{WDSF} \xrightleftharpoons{\psi_1} \text{SF} \xrightleftharpoons{Lh} P \]
Subexceedent functions

A subexceedent function of size $n$ is a word of nonnegative integers $f$ such that for all $i \leq n$, we have $f_i \leq n - i$. 
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A subexceedent function of size $n$ is a word of nonnegative integers $f$ such that for all $i \leq n$, we have $f_i \leq n - i$.

Bijection with permutations

We use the Lehmer code of the inverse of a permutation $\sigma$ to construct a subexceedent function $f$ as follows: $f_{\sigma_j} = \#\{i < j|\sigma_i > \sigma_j\}$. For instance,

$$\sigma = 528197634, \ Lh(\sigma) =$$
**Subexceedent functions**

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**Bijection with permutations**

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$$\sigma = 528197634, \ Lh(\sigma) = 3$$
Subexceedent functions

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Bijection with permutations

We use the Lehmer code of the inverse of a permutation $\sigma$ to construct a subexceedent function $f$ as follows: $f_{\sigma_j} = \# \{i < j | \sigma_i > \sigma_j \}$. For instance,

$$\sigma = 528197634, \ Lh(\sigma) = 31$$
Subexceedent functions

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Bijection with permutations

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$$\sigma = 528197634, \ Lh(\sigma) = 315$$
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Bijection with permutations

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$$\sigma = 528197634, \quad Lh(\sigma) = 315503200$$
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A subexceedent function of size $n$ is a word of nonnegative integers $f$ such that for all $i \leq n$, we have $f_i \leq n - i$.

Bijection with permutations

We use the Lehmer code of the inverse of a permutation $\sigma$ to construct a subexceedent function $f$ as follows: $f_{\sigma_j} = \# \{ i < j \mid \sigma_i > \sigma_j \}$. For instance,

$$\sigma = 528197634, \quad \text{Lh}(\sigma) = 315503200$$

Decreasing subexceedent functions

A subexceedent function is decreasing if the word obtained by removing all the zeros is strictly decreasing.
For example, $L = 540300200$. 
**$\psi_1$: SF $\rightarrow$ DWSF**

- $L = 315503200$, $P = 000000000$
$\psi_1: \text{SF} \rightarrow \text{DWSF}$

- $L = 315503200, P = 000000000$, then \textit{pivot} = 5;
\[ \psi_1 : \text{SF} \rightarrow \text{DWSF} \]

- \( L = 315503200, P = 000000000, \) then \( \text{pivot} = 5; \)
- \( L = 314503200, P = 000000000 \)
\( \psi_1 : \text{SF} \rightarrow \text{DWSF} \)

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- \( L = 315403200, P = 000000000, \)
- \( L = 315403200, P = 000100000 \)
\[ \psi_1 : SF \rightarrow DWSF \]

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\( \psi_1 : \text{SF} \to \text{DWSF} \)

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\( \psi_1: \text{SF} \to \text{DWSF} \)

- \( L = 315503200, \ P = 000000000, \) then pivot = 5;
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- \( L = 512403200, \ P = 001100000, \) then pivot = 4;
\[ \psi_1 : \text{SF} \rightarrow \text{DWSF} \]

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- \( L = 315403200, P = 000100000, \text{then } \text{pivot} = 5; \)
- \( L = 512403200, P = 001100000, \text{then } \text{pivot} = 4; \)
- \( L = 514103200, P = 001200000 \)
<table>
<thead>
<tr>
<th>Function</th>
<th>SF</th>
<th>DWSF</th>
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<td>( \psi_1 ): SF \rightarrow DWSF</td>
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<td></td>
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<td>( L = 315503200, P = 000000000, ) then pivot = 5;</td>
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<td>( L = 512403200, P = 001100000, ) then pivot = 4;</td>
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<tr>
<td>( L = 514103200, P = 001200000, ) then pivot = 4;</td>
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\( \phi_1: \text{SF} \rightarrow \text{DWSF} \)

- \( L = 315503200, P = 000000000, \text{then pivot} = 5; \)
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- \( L = 514103200, P = 001200000, \text{then} \ pivot = 4; \)
- \( L = 540103200, P = 002200000, \text{then} \ pivot = 3; \)
Introduction

Permutations to weighted Dyck paths
Subexceedent functions to weighted Dyck paths
Conclusion

$\psi_1: \text{SF} \rightarrow \text{DWSF}$

- $L = 315503200$, $P = 000000000$, then $pivot = 5$;
- $L = 315403200$, $P = 000100000$, then $pivot = 5$;
- $L = 512403200$, $P = 001100000$, then $pivot = 4$;
- $L = 514103200$, $P = 001200000$, then $pivot = 4$;
- $L = 540103200$, $P = 002200000$, then $pivot = 3$;
- $L = 54030200$, $P = 002201000$
The function $\psi_1$ maps standard set partitions to Dyck words. Let $L = 315503200, P = 000000000$, then $pivot = 5$; let $L = 31\mathbf{54}03200, P = 000\mathbf{1}00000$, then $pivot = 5$; let $L = \mathbf{51}2403200, P = 001100000$, then $pivot = 4$; let $L = 51\mathbf{41}03200, P = 001200000$, then $pivot = 4$; let $L = 5\mathbf{40}103200, P = 002200000$, then $pivot = 3$; let $L = 540\mathbf{30}0200, P = 00220\mathbf{1}000$ the algorithm stops.
Let \( \sigma \in S_n \) we construct \( \psi_{FV}(\sigma) \) as follows:

- The \((2k)\)-th step is \( \backslash \) iff \( n - k \) is a value of \( f \),
- The \((2k + 1)\)-th step is \( \backslash \) iff \( f_k = 0 \).

Example

\[ \psi_2(540300200) = \]

\[ \]

Arthur Nunge

An equivalence of multistatistics on permutations
\[ \psi_2: \text{DSF} \rightarrow \text{DP} \]

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**Example**

\[ \psi_2(540300200) = \]

![Diagram](image)
\[ \psi_2: \text{DSF} \rightarrow \text{DP} \]

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Example

\[ \psi_2(540300200) = \]
Let $\sigma \in S_n$ we construct $\psi_F(\sigma)$ as follows:

- The $(2k)$-th step is $\downarrow$ iff $n - k$ is a value of $f$,
- The $(2k+1)$-th step is $\uparrow$ iff $f_k = 0$.

Example

$\psi_2(540300200) =$

![Graph showing the transformed permutation]
Theorem

The map \( \phi = Lh^{-1} \circ \psi_1^{-1} \circ \psi_2^{-1} \circ \phi_1 \circ \phi_{FV} \) is a bijection satisfying

- \( \text{Rec}(\phi(\sigma)) = \text{Rec}(\sigma); \)
- \( \text{LC}(\phi(\sigma)) = \text{GC}(\sigma); \)
- \( \alpha(\phi(\sigma)) = \text{tot}(\sigma). \)
Perspectives

- Generalisation of the bijection for a larger type of PASEP.
- Study of a variant of $\phi_{FV}$ applied after the involution on weighted Dyck paths implying a third combinatorial interpretation and a new bijection preserving sylvester classes on permutations.
Thank you!