From well to better, the space of ideals

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Introduction: Wqo and Bqo

Well quasiorders

A **quasiorder** (qo) is a set Q together with
a *reflexive* and *transitive* binary relation $\leq$.

Definition

A **well quasiorder** (wqo) is a qo that satisfies one of the following equivalent conditions.

1. Q is well founded and has no infinite antichain;
2. there exists no bad sequence, i.e. no $f : \omega \to Q$ s.t.
$$\forall m, n \in \omega \quad m < n \to f(m) \not\leq f(n).$$
3. for every function $f : \omega \to Q$ there exists an infinite $N \subseteq \omega$ s.t.
$$\forall m, n \in N \quad m < n \to f(m) \leq f(n).$$

This is an application of the Ramsey Theorem for $[\omega]^2$.

Wqo? Well, we want better!

For $X, Y \in \mathcal{P}(Q)$ define

$$X \leq Y \quad \longleftrightarrow \quad \forall x \in X \quad \exists y \in Y \quad x \leq y.$$

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Q is a better quasiorder if

- Q is wqo
- $\mathcal{P}(Q)$ is wqo
- $\mathcal{P}(\mathcal{P}(Q))$ is wqo
- $\mathcal{P}^k(Q)$ is wqo
- $\mathcal{P}^\omega(Q)$ is wqo
- $\mathcal{P}^\alpha(Q)$ is wqo

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This amounts to require that there are

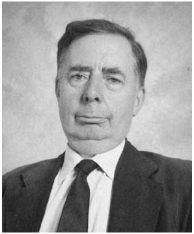
- no bad sequence
- no bad sequence of sequences
- no bad sequence of sequences of . . . of sequences
- no bad ?????
- no bad ?????

We need convenient index sets for "super-sequences".

Wqo? Well, here is better.

Let $X \subseteq \omega$ be infinite. A **front** on X is a family F of finite sets of natural numbers s.t.

1. $\bigcup F = X$ or $F = \{\emptyset\}$;
2. for all $s, t \in F$, $s \sqsubseteq t$ implies $s = t$;
3. every infinite subset of X admits an initial segment in F.



Theorem (Nash-Williams, 1965)

Let F be a front on X. For every $A \subseteq F$ there exists an infinite $Y \subseteq X$ such that

$$\textit{either} \quad F|Y \subseteq A, \qquad \textit{or} \quad F|Y \cap A = \emptyset.$$

where $F|Y = \{s \in F \mid s \subset Y\}$.

Fronts as index sets for super-sequences

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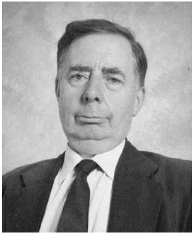
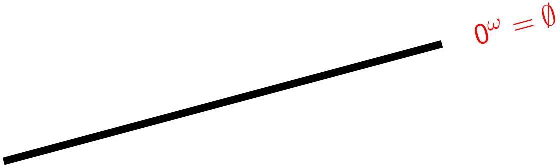


A front F is viewed as a subset of the Cantor space 2^ω via

$$s \subseteq \omega \longmapsto (x_s : \omega \to 2, \quad x_s(n) = 1 \leftrightarrow n \in s)$$
$$\{1, 3\} \longmapsto 0101000000\cdots$$

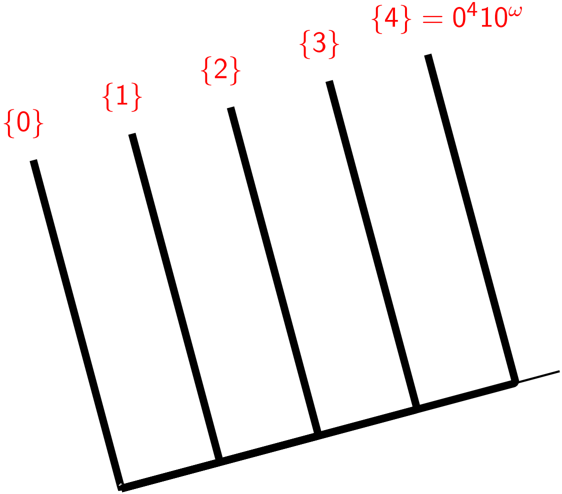
The trivial front $\{\emptyset\}$ is then $\{0^\omega\}$
The front $[\omega]^1$ is then $\{0^n 10^\omega \mid n \in \omega\}$.

Fronts as index sets for super-sequences



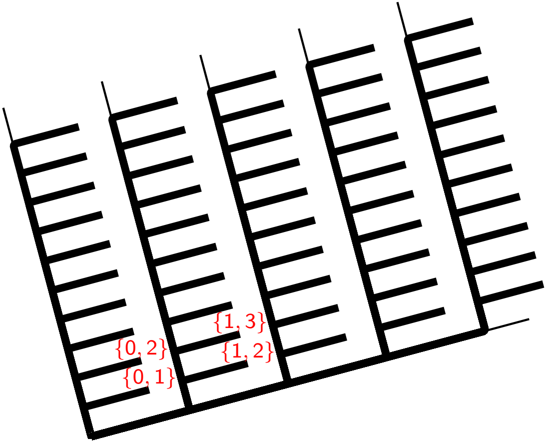
The trivial front: $\{\emptyset\}$

Fronts as index sets for super-sequences



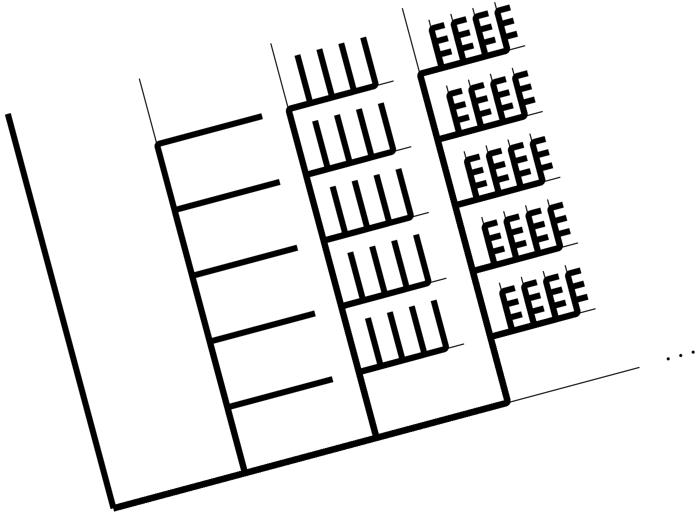
The front $[\omega]^1$

Fronts as index sets for super-sequences



The front $[\omega]^2$

Fronts as index sets for super-sequences



The Schreier barrier $\mathcal{S} = \{s \in [\omega]^{<\infty} \mid 1 + \min s = |s|\}$

Wqo? Well, we want better

For finite sets of natural numbers s and t let

$$s \triangleleft t \quad \text{iff} \quad \begin{array}{c}\text{there exists an infinite set } X \subseteq \omega \text{ s.t.}\\ s \sqsubset X \text{ and } t \sqsubset X \setminus \{\min X\}\end{array}$$

- A **super-sequence** in Q is a map $f : F \to Q$ from a front F.
- A super-sequence $f : F \to Q$ is **bad** if for every $s, t \in F$, $s \triangleleft t$ implies $f(s) \not\leq f(t)$.

Definition (Nash-Williams, 1965)

A qo Q is a **better quasiorder** (bqo) if there is no bad super-sequence in Q.

Well order $\to$ Better quasiorder $\to$ Well quasiorder.

Cauchy super-sequences:
A canonical version of Nash-Williams' Theorem

Cauchy sequences and uniform continuity

Let $\mathcal{X}$ be a compact metric space.
Let $(x_n)_{n\in\omega}$ be a sequence in $\mathcal{X}$. The following are equivalent:

1. $(x_n)_{n\in\omega}$ is a Cauchy sequence

Cauchy sequences and uniform continuity

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1. $(x_n)_{n\in\omega}$ is a Cauchy sequence
2. the map $f : [\omega]^1 \to \mathcal{X}$, $n \mapsto x_n$ is uniformly continuous, when $n \in \omega$ is identified with $0^n10^\omega \in 2^\omega$, i.e.

$$\forall\varepsilon > 0\, \exists\delta > 0\, \forall m, n \quad d_{2^\omega}(0^m10^\omega, 0^n10^\omega) < \delta \to d_{\mathcal{X}}(x_m, x_n) < \varepsilon.$$

3. f admits a continuous extension $\bar{f} : [\omega]^{\leq 1} \to \mathcal{X}$.

Theorem

Any compact Hausdorff space admits a unique uniform structure.

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Theorem

Any compact Hausdorff space admits a unique uniform structure.

Of course, compactness means that:
for every sequence $f : [\omega]^1 \to \mathcal{X}$ there exists an infinite $Y \subseteq \omega$ such that $f : [Y]^1 \to \mathcal{X}$ is Cauchy (or equivalently convergent).

Cauchy super-sequences

Let F be a front on X.

Definition

Let $f : F \to \mathcal{X}$ be a super-sequence. A **sub-super-sequence** of f is a restriction of f to some front $F' \subseteq F$, i.e. to some $F|Y$ for $Y \subseteq X$ infinite.

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A front F is viewed as a subset of 2^{ω}, $\overline{F}$ is its closure.

Definition

We say that a super-sequence $f : F \to \mathcal{X}$ into a metric space is *Cauchy* if it is uniformly continuous. In case $\mathcal{X}$ is compact, this is equivalent to the existence of a continuous extension $\overline{f} : \overline{F} \to \mathcal{X}$.

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Theorem (Carroy, Y.P.)

Every super-sequence in a compact metric space $\mathcal{X}$ admits a Cauchy sub-super-sequence.

An extension of Nash-Williams' theorem

Let F be a front on X.

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Let $A \subseteq F$. There exists an infinite $Y \subseteq X$ such that either $F|Y \subseteq A$, or $F|Y \cap A = \emptyset$.

Let $A_1, \ldots, A_n \subseteq F$. There exists an infinite $Y \subseteq X$ such that

$$A_i \cap F|Y \in \{\emptyset, F|Y\} \quad \text{for every } i = 1, \ldots, n.$$

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Definition

Let $S \subseteq 2^\omega$. A subset $B \subseteq S$ is a *block* of S if there exists a clopen C in 2^ω s.t. $B = C \cap S$.

Theorem (Carroy, Y.P., 2013)

Let $A_1, A_2, A_3, \dots \subseteq F$ there exists an infinite $Y \subseteq X$ such that

$$A_i \cap F|Y \quad \textit{is a block of } F|Y \textit{ for every } i = 1, 2, 3, \dots .$$

Proof of the theorem

Theorem (Carroy, Y.P.)

Every super-sequence in a compact metric space $\mathcal{X}$ admits a Cauchy sub-super-sequence.

Proposition (Folklore)

Let $S \subseteq 2^\omega$ and $f : S \to 2^\omega$ be any function. Then the following are equivalent:

- *f uniformly continuous*
- *for every clopen $C \subseteq 2^\omega$, $f^{-1}(C)$ is a block of S.*

Proof of the theorem.

We can assume $\mathcal{X} = 2^\omega$. Let $f : F \to 2^\omega$ and consider the countable family $\mathcal{B} = \{f^{-1}(C) \mid C \text{ is clopen in } 2^\omega\}$.
Apply the the extension of Nash-Williams theorem to $\mathcal{B}$.
Use the proposition. □

From well to better:

an application to better quasiorder theory

The space of ideals of a wqo

A *non empty* subset I of a qo Q is an **ideal** if

- I is a downset;
- I is directed, i.e. for all $p, q \in I$ there is $r \in I$ with $p \leq r$ and $q \leq r$.

Let $\mathrm{Idl}(Q)$ be the po of ideals of Q under inclusion.

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Let 2^Q be the generalised Cantor space of subsets of Q.

Any qo Q is naturally mapped into 2^Q via

$$Q \longrightarrow 2^Q$$
$$q \mapsto \downarrow q.$$

We identify Q (the po quotient of Q) with its image in 2^Q.

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Proposition (M. Pouzet and N. Sauer, 2005)

If Q is wqo then the closure of Q in 2^Q equals $\mathrm{Idl}(Q)$.

- $\mathrm{Idl}(Q)$ *is compact;*
- $\mathrm{Idl}(Q)$ *is scattered;*
- *the set of isolated points of* $\mathrm{Idl}(Q)$ *equals* Q*.*

Application: a result on bqo

Let $\mathrm{Idl}^*(Q)$ denote the po of non principal ideals of Q under inclusion. We have $\mathrm{Idl}(Q) = \mathrm{Idl}^*(Q) \cup Q$.

Theorem (Carroy, Y.P., 2013)

Let Q be wqo. If $\mathrm{Idl}^*(Q)$ is bqo, then Q is bqo.

This result was conjectured by M. Pouzet in 1978.

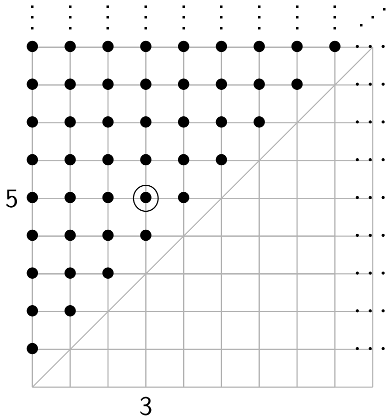
Actually more is true:

Scholium

Let Q be wqo. If $\mathrm{Idl}^\omega(Q)$ is bqo, then Q is bqo.

Where $\mathrm{Idl}^\omega(Q)$ is the po of ideals of cofinality ω.
$I \in \mathrm{Idl}(Q)$ has cofinality ω if there exists strictly increasing sequence $(q_n)_{n\in\omega}$ such that $I = \{p \in Q \mid \exists n\ p \leq q_n\}$.

Rado's counterexample



Rado's poset $\mathcal{R}$

Richard Rado, 1954.

A wqo not bqo on $[\omega]^2$:

$$\{m_0, m_1\} \leq \{n_0, n_1\}$$

iff

$$\begin{cases} m_0 = n_0 \text{ and } m_1 \leq n_1, \text{ or} \\ m_0 < n_0 < m_1 < n_1 \end{cases}$$

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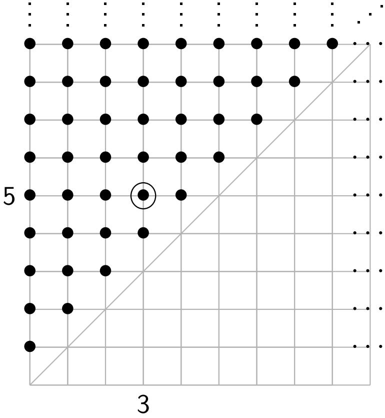

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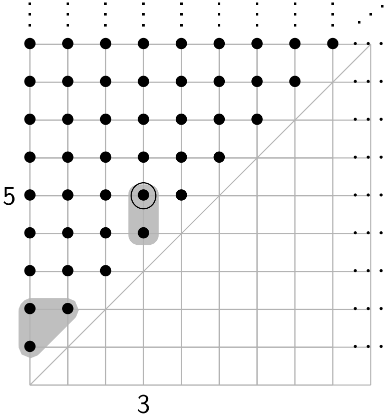

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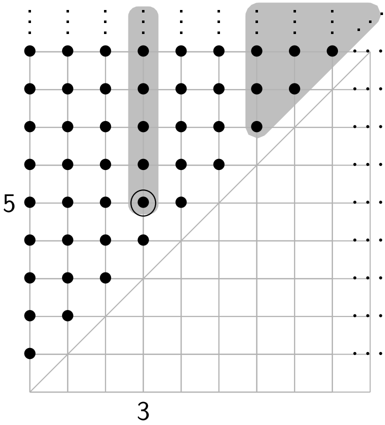

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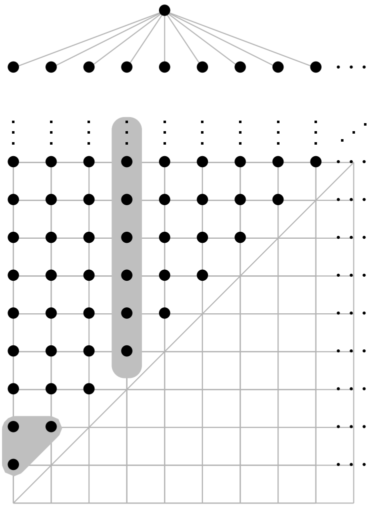

Ideals of Rado's poset $\mathcal{R}$

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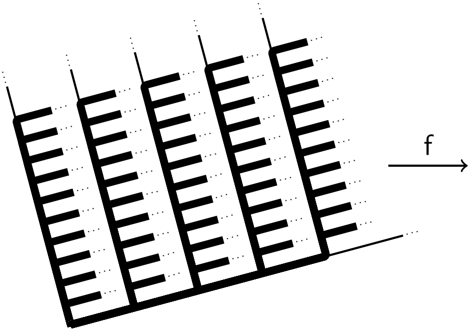
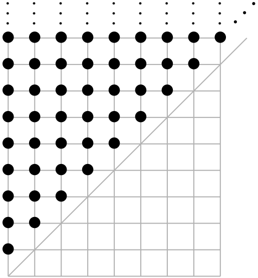
Proof by Rado's example



The barrier $[\omega]^2$



Rado's poset $\mathcal{R}$

$$f : [\omega]^2 \longrightarrow \mathcal{R}$$
$$\{m, n\} \longmapsto \{m, n\}$$

the map f is bad:

$\{m, n\} \lhd \{n, k\} \quad \rightarrow \quad \{m, n\} \not\leq \{n, k\}$

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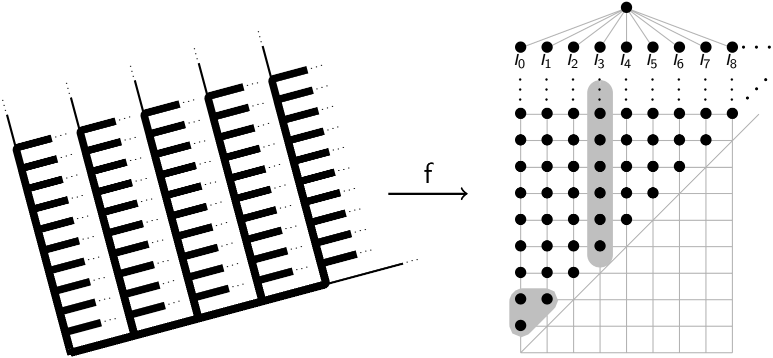

The barrier $[\omega]^2$ Rado's ideals $\mathrm{Idl}(\mathcal{R})$

$$\mathrm{f} : [\omega]^2 \longrightarrow \mathcal{R}$$
$$\{m, n\} \longmapsto \{m, n\}$$

the map f is Cauchy!

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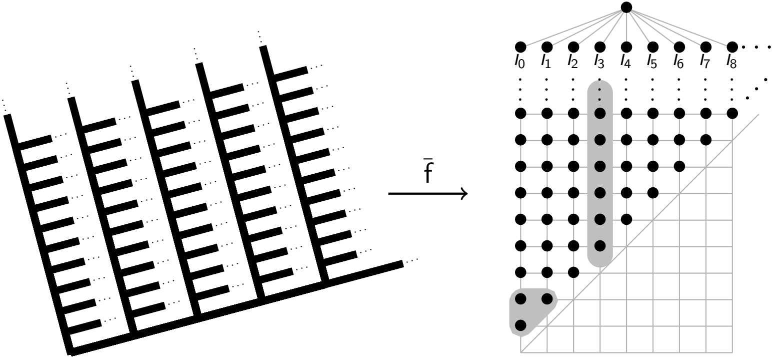

The closed set $[\omega]^{\leq 2}$ — Rado's ideals $\mathrm{Idl}(\mathcal{R})$

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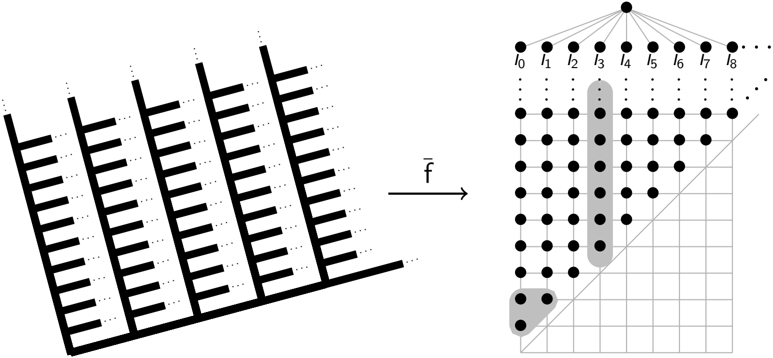

The closed set $[\omega]^{\leq 2}$ Rado's ideals $\mathrm{Idl}(\mathcal{R})$

$$\begin{aligned}\bar{f} : [\omega]^{\leq 2} &\longrightarrow \mathrm{Idl}(\mathcal{R}) \\ \{m, n\} &\longmapsto \{m, n\} \\ \{n\} &\longmapsto I_n \\ \emptyset &\longmapsto \top\end{aligned}$$

It yields the bad map

$$\bar{f} : [\omega]^1 \to \mathrm{Idl}^*(\mathcal{R})$$

inside $\mathrm{Idl}^*(\mathcal{R})$.

A result in bqo theory

Theorem (Carroy, Y.P.)

Let Q be wqo. If $\mathrm{Idl}^*(Q)$ *is bqo, then Q is bqo.*

Rough sketch of the proof.

- Assume Q wqo and $\mathrm{Idl}^*(Q)$ is bqo.
- Let $f : F \to Q$ be a super-sequence (to see: f is not bad);
- Go to a Cauchy sub-super-sequence $g : F' \to Q$;
- Extend it continuously to $\overline{g} : \overline{F'} \to \mathrm{Idl}(Q)$;
- Then

either $\overline{g}$ is locally constant and so g can't be bad.

or there exists a (non trivial) front $G \subseteq \overline{F'}$ such that $\overline{g} : G \to \mathrm{Idl}^*(Q)$.
Then use the fact that $\mathrm{Idl}^*(Q)$ is bqo to show that f is not bad. □

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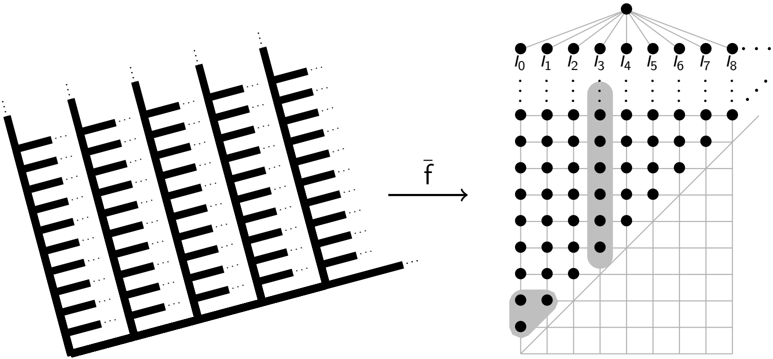

The closed set $[\omega]^{\leq 2}$ Rado's ideals $\mathrm{Idl}(\mathcal{R})$

Thanks for listening!